ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (2), 735–753
doi:10.15330/cmp.17.2.735-753

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №2, С.735–753

Further results on deferred statistical convergence in neutrosophic n-normed spaces

Kişi Ö.[1], Gürdal M.[2,✉]

This paper conducts an in-depth analysis of strong deferred summability and deferred statistical convergence, offering key findings within the context of neutrosophic n-normed spaces. Additionally, we thoroughly examine the concept of deferred statistical Cauchy sequences, demonstrating that every neutrosophic n-normed space is deferred statistically complete under this framework.

Key words and phrases: neutrosophic n-normed space, deferred statistical convergence, deferred statistical Cauchy sequence, deferred statistical completeness.

[1] Department of Mathematics, Bartin University, 74100, Bartın, Türkiye
[2] Department of Mathematics, Süleyman Demirel University, 32260, Isparta, Türkiye
✉ Corresponding author
E-mail: `okisi@bartin.edu.tr` (Kişi Ö.), `gurdalmehmet@sdu.edu.tr` (Gürdal M.)

1 Introduction

Approximately 60 years ago, L.A. Zadeh [24] introduced fuzzy sets to address problems beyond the reach of traditional crisp set theory. Since then, fuzzy sets have become essential in fields like artificial intelligence, robotics, control engineering, and decision-making. Despite their impact, fuzzy sets have sometimes fallen short, driving ongoing research to overcome these limitations.

To address these challenges, K. Atanassov [4] introduced intuitionistic fuzzy sets as a more nuanced approach. Recognizing that decision-making often involves more than binary choices, F. Smarandache [23] introduced the neutrosophic set (NS) in 2005, expanding on fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). An NS considers all possible outcomes, characterizing each element by a triplet: truth-membership (T), indeterminacy-membership (I), and falsity-membership (F). The concept of neutrosophy first entered academic literature in 1998 [22].

O. Kaleva and S. Seikkala [11] extended fuzzy set theory by introducing fuzzy metric spaces (FMS), where the distance between two points is a non-negative fuzzy number. Building on this, J.H. Park [19] generalized FMS by defining intuitionistic fuzzy metric spaces (IFMS) using t-norms and t-conorms, as initially applied to FMS by A. George and P. Veeramani [7]. A.K. Katsaras [12] recognized the difficulty in determining exact vector norms in some cases, proposing fuzzy norms as a better alternative.

H. Fast [6] conducted the earliest research on statistical convergence. Later on, in an effort to improve knowledge of summability theory, M. Mursaleen and H.H.E. Edely [16] expanded this idea to double sequences.

УДК 519.23
2020 *Mathematics Subject Classification:* 40A35, 46S40.

Recently, M. Kirişci and N. Şimşek [13] introduced a broader concept of fuzzy normed spaces known as neutrosophic normed linear spaces (NNLS) and investigated statistical convergence in these spaces. Following their pioneering work, numerous studies have been published on NNLS and their connections to summability theory. V. Kumar et al. [15] defined N-n-NLS, explored Cauchy sequences, and studied completeness within N-n-NLS. They also established the relationships between these concepts.

In [3], R.P. Agnew introduced the deferred Cesàro mean, enhancing existing systems. I. Dağadur and Ş. Sezgek [5] further expanded on this concept in the context of double sequences, thereby making a significant contribution to the field. For more details on deferred statistical convergence and its extensions, see [8–10, 14, 18, 20, 21].

This study examines deferred statistical convergence and the deferred Cesàro mean for double sequences in N-n-NLS, deepening our understanding of these concepts. By introducing deferred statistical Cauchy sequences and proving their equivalence to deferred statistical convergence, we offer key insights into sequence behavior within this framework. Our findings enhance the understanding of N-n-NLS and contribute to the broader field of mathematical analysis.

2 Definitions and Preliminaries

In this section, we provide an overview of basic definitions and terminology which will be useful to describe our main results. Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ represent the set of all natural numbers and the set of all real numbers, respectively.

Definition 1. *A binary operation $\otimes : J \times J \to J$, where $J = [0,1]$, is named to be a continuous t-norm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in J$, the below conditions hold:*

($\mathfrak{a}$) $\otimes$ is associative and commutative;

($\mathfrak{b}$) $\otimes$ is continuous;

($\mathfrak{c}$) $\nu_1 \otimes 1 = \nu_1$ for all $\nu_1 \in J$;

($\mathfrak{d}$) $\nu_1 \otimes \nu_2 \leq \nu_3 \otimes \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Definition 2. *A binary operation $\circledast : J \times J \to J$, where $J = [0,1]$ is named to be a continuous t-conorm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in J$, supplies the following requirements:*

($\mathfrak{a}$) $\circledast$ is associative and commutative;

($\mathfrak{b}$) $\circledast$ is continuous;

($\mathfrak{c}$) $\nu_1 \circledast 0 = \nu_1$ for all $\nu_1 \in J$;

($\mathfrak{d}$) $\nu_1 \circledast \nu_2 \leq \nu_3 \circledast \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Lastly, we review the idea of the n-norm as follows.

Definition 3. *Let W be a real space of dimension $m \geq n$ (m is finite and infinite, $n \in \mathbb{N}$). The real valued function $\|\cdot\|_n$ on $W^n := W \times \cdots \times W$ is called n-norm on W if and only if it satisfies the axioms listed below:*

($\mathfrak{a}$) $\|\tau_1, \ldots, \tau_n\|_n = 0$ if and only if $\tau_1, \ldots, \tau_n \in W$ are linearly dependent;

($\mathfrak{b}$) $\|\tau_1, \ldots, \tau_n\|_n$ remains invariant for $1 \leq i \leq n$;

($\mathfrak{c}$) $\|\tau_1, \ldots, \alpha\tau_n\|_n = |\alpha| \, \|\tau_1, \ldots, \tau_n\|_n$ for any $\alpha \in \mathbb{R}$;

($\mathfrak{d}$) $\|\tau_1, \ldots, \tau_{n-1}, u + v\|_n \leq \|\tau_1, \ldots, \tau_{n-1}, u\|_n + \|\tau_1, \ldots, \tau_{n-1}, v\|_n$.

The pair $(W, \|\cdot\|_n)$ is known as n-normed linear space.

Definition 4. *Let W be a linear space over F and $\otimes$, $\circledast$ denote t-norm and t-conorm, respectively. Let $\Im, \Re, \wp$ be functions from $W^n \times (0,\infty)$ to $[0,1]$. A six tuple $(W, \Im, \Re, \wp, \otimes, \circledast)$ is called a neutrosophic n-normed linear space (N-n-NLS for short), if the below properties are satisfied for any $(\tau_1, \ldots, \tau_n; \zeta) \in W^n \times (0,\infty)$:*

(1) $\Im(\tau_1, \ldots, \tau_n; \zeta) + \Re(\tau_1, \ldots, \tau_n; \zeta) + \wp(\tau_1, \ldots, \tau_n; \zeta) \leq 3$;
(2) $\Im(\tau_1, \ldots, \tau_n; \zeta) > 0$;
(3) $\Im(\tau_1, \ldots, \tau_n; \zeta) = 1$ *if and only if* τ_i *are linearly dependent for* $1 \leq i \leq n$;
(4) $\Im(\tau_1, \ldots, \tau_n; \zeta)$ *remains invariant for* $1 \leq i \leq n$;
(5) $\Im(\tau_1, \ldots, \tau_{n-1}, \alpha\tau_n; \zeta) = \Im\left(\tau_1, \ldots, \tau_{n-1}, \tau_n; \frac{\zeta}{|\alpha|}\right)$ *for* $\alpha \neq 0$, $\alpha \in F$;
(6) $\Im(\tau_1, \ldots, \tau_{n-1}, \tau_n; \zeta_1) \otimes \Im(\tau_1, \ldots, \tau_{n-1}, \tau'_n; \zeta_2) \geq \Im(\tau_1, \ldots, \tau_{n-1}, \tau_n + \tau'_n; \zeta_1 + \zeta_2)$;
(7) $\Im(\tau_1, \ldots, \tau_n; \zeta)$ *is non-decreasing continuous in* ζ;
(8) $\lim_{\zeta\to\infty} \Im(\tau_1, \ldots, \tau_n; \zeta) = 1$ *and* $\lim_{\zeta\to 0} \Im(\tau_1, \ldots, \tau_n; \zeta) = 0$;
(9) $\Re(\tau_1, \ldots, \tau_n; \zeta) > 0$;
(10) $\Re(\tau_1, \ldots, \tau_n; \zeta) = 1$ *if and only if* τ_i *are linearly dependent for* $1 \leq i \leq n$;
(11) $\Re(\tau_1, \ldots, \tau_n; \zeta)$ *remains invariant for* $1 \leq i \leq n$;
(12) $\Re(\tau_1, \ldots, \tau_{n-1}, \alpha\tau_n; \zeta) = \Re\left(\tau_1, \ldots, \tau_{n-1}, \tau_n; \frac{\zeta}{|\alpha|}\right)$ *for* $\alpha \neq 0$, $\alpha \in F$;
(13) $\Re(\tau_1, \ldots, \tau_{n-1}, \tau_n; \zeta_1) \circledast \Re(\tau_1, \ldots, \tau_{n-1}, \tau'_n; \zeta_2) \geq \Re(\tau_1, \ldots, \tau_{n-1}, \tau_n + \tau'_n; \zeta_1 + \zeta_2)$;
(14) $\Re(\tau_1, \ldots, \tau_n; \zeta)$ *is non-decreasing continuous in* ζ;
(15) $\lim_{\zeta\to\infty} \Re(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n; \zeta) = 0$ *and* $\lim_{\zeta\to 0} \Re(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n; \zeta) = 1$;
(16) $\wp(\tau_1, \ldots, \tau_n; \zeta) > 0$;
(17) $\wp(\tau_1, \ldots, \tau_n; \zeta) = 1$ *if and only if* τ_i *are linearly dependent for* $1 \leq i \leq n$;
(18) $\wp(\tau_1, \ldots, \tau_n; \zeta)$ *remains invariant for* $1 \leq i \leq n$;
(19) $\wp(\tau_1, \ldots, \tau_{n-1}, \alpha\tau_n; \zeta) = \wp\left(\tau_1, \ldots, \tau_{n-1}, \tau_n; \frac{\zeta}{|\alpha|}\right)$ *for* $\alpha \neq 0$, $\alpha \in F$;
(20) $\wp(\tau_1, \ldots, \tau_{n-1}, \tau_n; \zeta_1) \circledast \wp(\tau_1, \ldots, \tau_{n-1}, \tau'_n; \zeta_2) \geq \wp(\tau_1, \ldots, \tau_{n-1}, \tau_n + \tau'_n; \zeta_1 + \zeta_2)$;
(21) $\wp(\tau_1, \ldots, \tau_n; \zeta)$ *is non-decreasing continuous in* ζ;
(22) $\lim_{\zeta\to\infty} \wp(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n; \zeta) = 0$ *and* $\lim_{\zeta\to 0} \wp(\tau_1, \tau_2, \ldots, \tau_{n-1}, \tau_n; \zeta) = 1$.

To keep things simple, we will refer to the neutrosophic n-norm as $N_n := N_n(\Im, \Re, \wp)$.

Definition 5. *Let $\{q_u\}$ be a sequence in an N-n-NLS. Let us choose $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$. Then $\{q_u\}$ is said to be convergent if there exists a $u_0 \in \mathbb{N}$, $q \in W$ such that*

$$\Im(q_u - q, \tau_1, \ldots, \tau_{n-1}; \zeta) > 1 - \rho,$$
$$\Re(q_u - q, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho, \qquad \wp(q_u - q, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho$$

for all $u \geq u_0$.

Here, we will write N_n-$\lim q_u = q$ or $q_u \overset{N_n}{\to} q$ and q is called N_n-limit of $\{q_u\}$.

Definition 6. *Let $K \subset \mathbb{N}$. Given the existence of the limit, the natural density of K, represented by $\delta(K)$, is defined as*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|,$$

where the vertical bars represent the cardinality of the contained set.

Definition 7. *Let $\{q_u\}$ be a sequence in an N-n-NLS. Choose any $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$. Then $\{q_u\}$ is identified as statistically convergent to $q \in W$ with regard to N_n if for every $\tau_1, \ldots, \tau_{n-1} \in W$ we have*

$$\delta\big(\{u \in \mathbb{N} : \Im\,(q_u - q, \tau_1, \ldots, \tau_{n-1}; \zeta) \leq 1-\rho \quad \text{or} \\ \Re\,(q_u - q, \tau_1, \tau_2, \ldots, \tau_{n-1}; \zeta) \geq \rho \quad \text{and} \quad \wp\,(q_u - q, \tau_1, \tau_2, \ldots, \tau_{n-1}; \zeta) \geq \rho\}\big) = 0.$$

Here, we will write $\mathcal{S}\,(N_n)$ -$\lim q_u = q$.

Deferred Cesàro mean D_p^q was introduced by R.P. Agnew [3] in 1932 as a compelling generalization of Cesàro mean of real (complex) valued sequence $t = \{t_u\}$ by

$$\left(D_p^q t\right)_n = \frac{1}{q(n)-p(n)} \sum_{u=p(n)+1}^{q(n)} t_u, \quad n = 1,2,3,\ldots,$$

where $q = q(n)$ and $p = p(n)$ are the sequences of non-negative integers satisfying $p(n) < q(n)$ and $q(n) \to \infty$ as $n \to \infty$.

A sequence $t = \{t_u\}$ is named to be D_p^q-convergent to t if $\lim\limits_{n\to\infty} \left(D_p^q t\right)_n = t_0$. The sequence $t = \{t_u\}$ is named to be strong D_p^q-convergent to q if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{u=p(n)+1}^{q(n)} |t_u - t_0| = 0.$$

3 Deferred statistical convergence in N-n-NLS

In this section, we define and study deferred statistical convergence of sequences with regard to N_n and we prove some interesting results.

Definition 8. *Let $(W, \Im, \Re, \wp, \otimes, \circledast)$ be an N-n-NLS. A sequence $\{q_{uv}\}$ in W is called to be strong deferred summable or $D_{p,q,r,s}$-summable to $q_0 \in W$ with respect to neutrosophic n-norm N_n if for every $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ there exists a $u_0 \in \mathbb{N}$ such that*

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \Im\,(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) > 1-\rho,$$

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \Re\,(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho$$

and

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \wp\,(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho$$

hold for all $\alpha, \beta \geq u_0$, where $\vartheta(\alpha) = q(\alpha) - p(\alpha)$ and $\varrho(\beta) = s(\beta) - r(\beta)$.

In this case, we write

$$D_{p,q,r,s}\left[(\Im, \Re, \wp)^n\right] \text{-}\lim q_{uv} = q_0 \quad \text{or} \quad q_{uv} \overset{D_{p,q,r,s}\left[(\Im,\Re,\wp)^n\right]}{\longrightarrow} q_0.$$

Definition 9. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. Then $\{q_{uv}\}$ is called to be deferred statistically convergent to $q_0 \in W$ with respect to N_n (in short $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent) if for every $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ we have*

$$\begin{aligned}\delta_{p,q,r,s}\big(\{&p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \le 1-\rho \\ &\text{or } \ \Re\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \ge \rho, \wp\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \ge \rho\}\big) = 0.\end{aligned}$$

This can also be rephrased as

$$\begin{aligned}\lim_{\alpha,\beta\to\infty} \frac{1}{\vartheta(\alpha)\varrho(\beta)} \big|\{&p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \\ &\Im\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \le 1-\rho \quad \text{or} \\ &\Re\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \ge \rho, \quad \wp\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \ge \rho\}\big| = 0.\end{aligned}$$

It is denoted as $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv} = q_0$ or $q_{uv} \overset{D_{p,q,r,s}[\mathcal{S}(N_n)]}{\longrightarrow} q_0$ and q_0 is identified as the $D_\alpha^\vartheta\left[\mathcal{S}\left(N_n\right)\right]$ limit of $\{q_{uv}\}$.

Remark 1. (1) *By taking $p(\alpha) = 0$, $q(\alpha) = \alpha$, $r(\beta) = 0$, $s(\beta) = \beta$ and $n = 2$, Definition 9 coincides with the description of statistical convergence of double sequences in N_2 [17].*

(2) *By taking $p(\alpha) = 0$, $q(\alpha) = \lambda_\alpha$, $r(\beta) = 0$, $s(\beta) = \lambda_\beta$ and $n = 2$, Definition 9 coincides with the idea of λ-statistical convergence of double sequences with respect to N_2 [1].*

(3) *By taking $p(\alpha) = k_{\alpha-1}$, $q(\alpha) = k_\alpha$, $r(\beta) = l_{\beta-1}$, $s(\beta) = l_\beta$ and $n = 2$, Definition 9 coincides with the concept of lacunary statistical convergence of double sequences with respect to N_2 [2].*

Example 1. *Let $(W, \mathbb{R}^n)$ be an n-normed space. For $v_1, v_2 \in [0,1]$, define t-norm, t-conorm by $\nu_1 \otimes \nu_2 = \nu_1\nu_2$ and $\nu_1 \circledast \nu_2 = \min\left(\nu_1+\nu_2, 1\right)$ and fuzzy sets $\Im, \Re, \wp$ on $\mathbb{R}^n \times (0,\infty)$ by*

$$\begin{aligned}\Im(\tau_1, \ldots, \tau_n; \zeta) &= \frac{\zeta}{\zeta + \|\tau_1, \ldots, \tau_n\|_n}, \\ \Re(\tau_1, \ldots, \tau_n; \zeta) &= \frac{\|\tau_1, \ldots, \tau_n\|_n}{\zeta + \|\tau_1, \ldots, \tau_n\|_n}, \\ \wp(\tau_1, \ldots, \tau_n; \zeta) &= \frac{\|\tau_1, \ldots, \tau_n\|_n}{\zeta}.\end{aligned}$$

Then W becomes an N-n-NLS.

We define a sequence $\{q_{uv}\} \in W$ as

$$q_{uv} = \begin{cases} (u^2v^2, 0, \ldots, 0) \in \mathbb{R}^n, & \begin{aligned}&[|\sqrt{q(\alpha)}|] - u_0 < u \le [|\sqrt{q(\alpha)}|], \\ &[|\sqrt{s(\beta)}|] - v_0 < v \le [|\sqrt{s(\beta)}|], \ \ \alpha, \beta = 1, 2, \ldots,\end{aligned} \\ (0, \ldots, 0) \in \mathbb{R}^n, & \text{otherwise,} \end{cases}$$

where $q(\alpha)$, $s(\beta)$ are monotonic increasing sequences with $0 < p(\alpha) \le [|\sqrt{q(\alpha)}|] - u_0$ and $0 < r(\beta) \le [|\sqrt{s(\beta)}|] - v_0$ and $u_0, v_0 \in \mathbb{N}$ are fixed.

Then for each $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$, we have

$$\begin{aligned}
K_{p,q,r,s}(\rho,\zeta) = &\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or} \\
&\Re(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\} \\
= &\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \frac{\zeta}{\zeta + \|q_{uv}, \tau_1, \ldots, \tau_n\|_n} \le 1-\rho \text{ or} \\
&\frac{\|\tau_1, \ldots, \tau_n\|_n}{\zeta + \|q_{uv}, \tau_1, \ldots, \tau_n\|_n} \ge \rho \text{ and } \frac{\|q_{uv}, \tau_1, \ldots, \tau_n\|_n}{\zeta} \ge \rho\Big\} \\
= &\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \\
&\|q_{uv}, \tau_1, \ldots, \tau_n\|_n > \frac{\zeta\rho}{1-\rho} > 0 \text{ and } \|q_{uv}, \tau_1, \ldots, \tau_n\|_n \ge \zeta\rho\Big\} \\
\subset &\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : q_{uv} = \big(u^2v^2, 0, \ldots, 0\big)\Big\} \\
= &\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : [|\sqrt{q(\alpha)}|] - u_0 < u \le [|\sqrt{q(\alpha)}|], \\
&[|\sqrt{s(\beta)}|] - v_0 < v \le [|\sqrt{s(\beta)}|], \ \alpha, \beta = 1, 2, \ldots\Big\}.
\end{aligned}$$

This gives $\delta_{p,q,r,s}\left(K_{p,q,r,s}(\rho,\zeta)\right) = \lim_{\alpha,\beta\to\infty} \frac{|K_{p,q,r,s}(\rho,\zeta)|}{\vartheta(\alpha)\varrho(\beta)} \le \lim_{\alpha,\beta\to\infty} \frac{u_0 v_0}{\vartheta(\alpha)\varrho(\beta)} = 0$. *Hence,* $D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim q_{uv} = 0$. *But it is clear that the sequence* $\{q_{uv}\}$ *is not convergent in* $(W, \Im, \Re, \wp, \otimes, \circledast)$ *with respect to N-n-norm* N_n.

From Definition 9, we can easily prove the following assertion.

Lemma 1. *Let* $\{q_{uv}\}$ *be a sequence in an N-n-NLS. Then for every* $\rho \in (0,1)$, $\zeta > 0$ *and* $\tau_1, \ldots, \tau_{n-1} \in W$ *the following properties hold:*

(1) $D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim q_{uv} = q_0$;

(2) deferred density of each of

$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho\},$$
$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Re(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\} \text{ and}$$
$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \wp(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}$$

is zero;

(3)

$$\begin{aligned}
\delta_{p,q,r,s}\big(\{&p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) > 1-\rho \\
&\text{or } \Re(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho, \ \wp(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho\}\big) = 1;
\end{aligned}$$

(4) deferred density of each of

$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) > 1-\rho\},$$
$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Re(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho\} \text{ and}$$
$$\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \wp(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) < \rho\}$$

is one;

(5)

$$\begin{aligned}
&D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim \Im(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) = 1, \\
&D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim \Re(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) = 0, \\
&D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim \wp(q_{uv} - q_0, \tau_1, \ldots, \tau_{n-1}; \zeta) = 0.
\end{aligned}$$

Theorem 1. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. If $\{w_k\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent, then the $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-limit of $\{q_{uv}\}$ is unique.*

Proof. Let $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q_1$ and $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q_2$ where $q_1 \neq q_2$. For a given $\rho \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \otimes (1-\varpi) > 1-\rho$ and $\varpi \circledast \varpi < \rho$. Then, using Lemma 1, for any $\zeta > 0$ and $\tau_1, \tau_2, \ldots, \tau_{n-1} \in W$ deferred density of each of the following

$$\begin{aligned}
P_{\mathfrak{J},1}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \mathfrak{J}\left(q_{uv}-q_1,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \le 1-\varpi\right\},\\
Q_{\mathfrak{J},2}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \mathfrak{J}\left(q_{uv}-q_2,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \le 1-\varpi\right\},\\
P_{\mathfrak{R},1}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \mathfrak{R}\left(q_{uv}-q_1,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \ge \varpi\right\},\\
Q_{\mathfrak{R},2}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \mathfrak{R}\left(q_{uv}-q_2,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \ge \varpi\right\},\\
P_{\wp,1}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \wp\left(q_{uv}-q_1,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \ge \varpi\right\},\\
Q_{\wp,2}(\varpi,\zeta) &= \left\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \wp\left(q_{uv}-q_2,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \ge \varpi\right\}
\end{aligned}$$

is zero. Let

$$\begin{aligned}
A_{(\mathfrak{J},\mathfrak{R},\wp)}(\rho,\zeta) = \left\{P_{\mathfrak{J},1}(\varpi,\zeta) \cup Q_{\mathfrak{J},2}(\varpi,\zeta)\right\} &\cap \left\{P_{\mathfrak{R},1}(\varpi,\zeta) \cup Q_{\mathfrak{R},2}(\varpi,\zeta)\right\}\\
&\cap \left\{P_{\wp,1}(\varpi,\zeta) \cup Q_{\wp,2}(\varpi,\zeta)\right\}.
\end{aligned}$$

Then we get $\delta_{p,q,r,s}\left(A_{(\mathfrak{J},\mathfrak{R},\wp)}(\rho,\zeta)\right) = 0$. Obviously, $\delta_{p,q,r,s}\left(\mathbb{N}^2 \setminus A_{(\mathfrak{J},\mathfrak{R},\wp)}(\rho,\zeta)\right) = 1$. So, let $(u,v) \in \mathbb{N}^2 \setminus A_{(\mathfrak{J},\mathfrak{R},\wp)}(\rho,\zeta)$. This leads to three cases:

(i) $(u,v) \in \mathbb{N}^2 \setminus \left(P_{\mathfrak{J},1}(\varpi,\zeta) \cup Q_{\mathfrak{J},2}(\varpi,\zeta)\right)$;

(ii) $(u,v) \in \mathbb{N}^2 \setminus \left(P_{\mathfrak{R},1}(\varpi,\zeta) \cup Q_{\mathfrak{R},2}(\varpi,\zeta)\right)$;

(iii) $(u,v) \in \mathbb{N}^2 \setminus \left(P_{\wp,1}(\varpi,\zeta) \cup Q_{\wp,2}(\varpi,\zeta)\right)$.

Consider the first case. If $(u,v) \in \mathbb{N}^2 \setminus (P_{\mathfrak{J},1}(\varpi,\zeta) \cup Q_{\mathfrak{J},2}(\varpi,\zeta))$, then

$$\begin{aligned}
\mathfrak{J}\left(\varrho_1-\varrho_2,\tau_1,\ldots,\tau_{n-1};\zeta\right) &\ge \mathfrak{J}\left(q_{uv}-q_1,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \otimes \mathfrak{J}\left(q_{uv}-q_2,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right)\\
&\ge (1-\varpi) \otimes (1-\varpi) > 1-\rho.
\end{aligned}$$

Since $\rho \in (0,1)$ is arbitrary, $\mathfrak{J}\left(\varrho_1-\varrho_2,\tau_1,\ldots,\tau_{n-1};\zeta\right) = 1$, which yields $\varrho_1 = \varrho_2$.

In the second case, if $(u,v) \in \mathbb{N}^2 \setminus (P_{\mathfrak{R},1}(\varpi,\zeta) \cup Q_{\mathfrak{R},2}(\varpi,\zeta))$, then

$$\begin{aligned}
\mathfrak{R}\left(\varrho_1-\varrho_2,\tau_1,\ldots,\tau_{n-1};\zeta\right) &\le \mathfrak{R}\left(q_{uv}-q_1,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right) \circledast \mathfrak{R}\left(q_{uv}-q_2,\tau_1,\ldots,\tau_{n-1};\tfrac{\zeta}{2}\right)\\
&< \varpi \circledast \varpi < \rho.
\end{aligned}$$

Since $\rho \in (0,1)$ is arbitrary, $\mathfrak{R}\left(\varrho_1-\varrho_2,\tau_1,\ldots,\tau_{n-1};\zeta\right) = 0$, which yields $\varrho_1 = \varrho_2$.

Using the similar technique, for the third case we can prove the same. Hence the $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-limit of $\{q_{uv}\}$ is unique. ◻

Theorem 2. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. If N_n-$\lim q_{uv} = q$, then*

$$D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q.$$

Proof. Suppose that N_n-$\lim q_{uv} = q$. Then for every $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ there exists $k_0 \in \mathbb{N}$ such that $\Im\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) > 1-\rho$, $\Re\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) < \rho$ and $\wp\left(q_{uv} - q, \tau_1, \tau_2, \ldots, \tau_{n-1}; \zeta\right) < \rho$ for all $u, v \geq k_0$. Then it is obvious that the set

$$A = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \leq 1-\rho$$
$$\text{or } \Re\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho \text{ and } \wp\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\Big\}$$

contains at most finite number of terms. Namely, $A \subseteq \{(1,1), (2,2), \ldots, (k_0-1, k_0-1)\}$. So, $\delta_{p,q,r,s}(A) = 0$. Hence $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q$. □

However, as the following examples show, the converse of Theorem 2 does not hold in general.

Example 2. *Let $(W, \mathbb{R}^n)$ be an n-normed space. For $\nu_1, \nu_2 \in [0,1]$, define t-norm, t-conorm by $\nu_1 \otimes \nu_2 = \nu_1\nu_2$ and $\nu_1 \circledast \nu_2 = \min(\nu_1+\nu_2, 1)$. We take N-n-NLS as defined in Example 1. Define a sequence $\{q_{uv}\} \in W$ by*

$$q_{uv} = \begin{cases} (1,0,\ldots,0) \in \mathbb{R}^n, & \text{if } u = i^2,\ v = j^2,\ i,j = 1,2,\ldots, \\ (0,0,\ldots,0) = \mathbf{0}, & \text{otherwise.} \end{cases}$$

Then, for each $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$, we get

$$K_{p,q,r,s}(\rho,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\left(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \leq 1-\rho$$
$$\text{or } \Re\left(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho \text{ and } \wp\left(q_{uv} - \mathbf{0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\Big\}$$
$$= \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \frac{\zeta}{\zeta + \|q_{uv}, \tau_1, \ldots, \tau_n\|_n} \leq 1-\rho$$
$$\text{or } \frac{\|q_{uv}, \tau_1, \ldots, \tau_n\|_n}{\zeta + \|q_{uv}, \tau_1, \ldots, \tau_n\|_n} \geq \rho \text{ and } \frac{\|q_{uv}, \tau_1, \ldots, \tau_n\|_n}{\zeta} \geq \rho\Big\}$$
$$= \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) :$$
$$\|q_{uv}, \tau_1, \ldots, \tau_n\|_n > \frac{\zeta\rho}{1-\rho} > 0 \text{ and } \|q_{uv}, \tau_1, \ldots, \tau_n\|_n \geq \zeta\rho\Big\}$$
$$\subset \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : q_{uv} = (1,0)\Big\}$$
$$= \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : u = i^2,\ v = j^2\Big\}.$$

This gives $\delta_{p,q,r,s}\left(K_{p,q,r,s}(\rho,\zeta)\right) \leq \lim_{\alpha,\beta\to\infty} \frac{\sqrt{\vartheta(\alpha)}\sqrt{\varrho(\beta)}}{\vartheta(\alpha)\varrho(\beta)} = 0$. So, $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = \mathbf{0}$. But, $\{q_{uv}\}$ is not convergent to $\mathbf{0}$ with regard to N_n.

Theorem 3. *Let $\{q_{uv}\}$ and $\{w_{uv}\}$ be two sequences in an N-n-NLS. Then the below statements hold true:*

(1) If $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q$ and $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim w_{uv} = w$, then

$$D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim(q_{uv} + w_{uv}) = q + w.$$

(2) If $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim q_{uv} = q$, then

$$D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]\text{-}\lim \kappa q_{uv} = \kappa q, \quad \kappa \neq 0.$$

Proof. Suppose that $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv} = q$ and $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim w_{uv} = w$. For a given $\rho \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \otimes (1-\varpi) > 1-\rho$ and $\varpi \circledast \varpi < \rho$. Then for every $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ the sets

$$A(\varpi,\zeta) = \Big\{ p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \leq 1-\varpi$$
$$\text{or } \Re\big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \geq \varpi \text{ and } \wp\big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \geq \varpi \Big\}$$

and

$$B(\varpi,\zeta) = \Big\{ p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(w_{uv}-w, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \leq 1-\varpi$$
$$\text{or } \Re\big(w_{uv}-w, \tau_1, \ldots, \tau_{n-1}; \frac{\zeta}{2}\big) \geq \varpi \text{ and } \wp\big(w_{uv}-w, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \geq \varpi \Big\}$$

have deferred density zero. Consider the set

$$\begin{aligned} C(\rho,\zeta) = \Big\{ & p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \\ & \Im\big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \leq 1-\rho \\ & \text{or } \Re\big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \geq \rho \\ & \text{and } \wp\big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\big) \geq \rho \Big\}. \end{aligned}$$

Then deferred density of $A^c(\varpi,\zeta)$ and $B^c(\varpi,\zeta)$ is 1. So, let $(u,v) \in A^c(\varpi,\zeta) \cap B^c(\varpi,\zeta)$. Then we have

$$\begin{aligned} & \Im\Big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) \\ & \quad \geq \Im\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) \otimes \Im\Big(w_{uv}-w, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) > (1-\varpi) \otimes (1-\varpi) > 1-\rho \end{aligned}$$

and

$$\begin{aligned} & \Re\Big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) \\ & \quad \leq \Re\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) \circledast \Re\Big(w_{uv}-w, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) < \varpi \circledast \varpi < \rho. \end{aligned}$$

Similarly, we get

$$\wp\Big((q_{uv}+w_{uv})-(q+w), \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) < \rho.$$

Therefore, $A^c(\varpi,\zeta) \cap B^c(\varpi,\zeta) \subset C^c(\rho,\zeta)$, i.e. $C(\rho,\zeta) \subset A(\varpi,\zeta) \cup B(\varpi,\zeta)$.

Hence $\delta_{p,q,r,s}(C(\rho,\zeta)) = 0$, i.e. $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim(q_{uv}+w_{uv}) = q+w$.

Let us prove the second statement. Suppose that $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv} = q$. Then for every $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ we get

$$\delta_{p,q,r,s}\Big(\Big\{ p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \leq 1-\rho$$
$$\text{or } \Re\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \geq \rho, \wp\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \geq \rho \Big\}\Big) = 0.$$

Since

$$\begin{aligned} & \Big\{ p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(\kappa q_{uv}-\kappa q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \leq 1-\rho \\ & \quad \text{or } \Re\big(\kappa q_{uv}-\kappa q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho \text{ and } \wp\big(\kappa q_{uv}-\kappa q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho \Big\} \\ & = \Big\{ p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \leq 1-\rho \\ & \quad \text{or } \Re\Big(\kappa q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \geq \rho \text{ and } \wp\Big(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{|\kappa|}\Big) \geq \rho \Big\}. \end{aligned}$$

Hence $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim \kappa q_{uv} = \kappa q$, $\kappa \neq 0$. $\square$

Theorem 4. *If $\{q_{uv}\}$ is a sequence in an N-n-NLS, then $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv} = q$ if and only if there exists a set*

$$K = \left\{u_1 < u_2 < \cdots < u_p < \cdots ; v_1 < v_2 < \cdots < v_s < \cdots\right\} \subset \mathbb{N}^2$$

such that $\delta_{p,q,r,s}(K) = 1$ and N_n-$\lim q_{u_p v_s} = q$.

Proof. First suppose that $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv} = q$. Then for any $j \in \mathbb{N}$, $\zeta > 0$ and for each $\tau_1, \ldots, \tau_{n-1} \in W$ the sets

$$A(j,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) > 1 - \tfrac{1}{j}$$
$$\text{and } \Re\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \tfrac{1}{j} \text{ and } \wp\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \tfrac{1}{j}\Big\},$$

and

$$B(j,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \leq 1 - \tfrac{1}{j}$$
$$\text{or } \Re\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \tfrac{1}{j} \text{ and } \wp\big(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \tfrac{1}{j}\Big\}$$

have the deferred density one and zero, respectively. It is obvious that $A(j+1,\zeta) \subset A(j,\zeta)$. We can express $A(j,\zeta)$ as $\left\{u_1 < \cdots < u_p < \cdots ; v_1 < \cdots < v_s < \cdots\right\}$. It is sufficient to prove the necessary part that for $\left(u_p, v_s\right) \in A(j,\zeta)$ we have N_n-$\lim q_{u_p v_s} = q$. If possible, let the subsequence $\left\{q_{u_p v_s}\right\}$ is not convergent to q with regard to N_n. Then for some $\rho \in (0,1)$ we get

$$\Im\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \leq 1 - \rho \text{ or } \Re\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho,$$
$$\text{and } \wp\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho$$

except for finite number of terms u_p, v_s. Consider the set

$$C(\rho,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) > 1 - \rho$$
$$\text{and } \Re\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \rho \text{ and } \wp\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \rho\Big\},$$

where $\rho > \frac{1}{j}$. Then $\delta_{p,q,r,s}(C(\rho,\zeta)) = 0$. As $\rho > \frac{1}{j}$, we obtain $A(j,\zeta) \subset C(\rho,\zeta)$. This gives $\delta_{p,q,r,s}(A(j,\zeta)) = 0$, which is a contradiction. Therefore N_n-$\lim q_{u_p v_s} = q$.

On the other hand, let us assume that there is a set

$$K = \left\{u_1 < \cdots < u_p < \cdots ; v_1 < \cdots < v_s < \cdots\right\} \subset \mathbb{N}^2$$

such that $\delta_{p,q,r,s}(K) = 1$ and N_n-$\lim q_{u_p v_s} = q$. Then for every $\rho \in (0,1)$, $\zeta > 0$, $\tau_1, \ldots, \tau_{n-1} \in W$ there exists $p_0 \in \mathbb{N}$ such that

$$\Im\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) > 1 - \rho \text{ and } \Re\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \rho,$$
$$\wp\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) < \rho$$

for all $p, s \geq p_0$. Thus,

$$\Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \leq 1 - \rho$$
$$\text{or } \Re\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho \text{ and } \wp\big(q_{u_p v_s} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\big) \geq \rho\Big\}$$
$$\subset \mathbb{N}^2 - \left\{\left(u_{p_0+1}, v_{p_0+1}\right), \left(u_{p_0+2}, v_{p_0+2}\right), \ldots\right\}.$$

Therefore,

$$\delta_{p,q,r,s}\big(\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Im\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \leq 1 - \rho$$
$$\text{or } \Re\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho, \wp\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\}\big) = 0,$$

i.e. $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv} = q$. The evidence is now complete. □

Corollary 1. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. Then $D_{p,q,r,s}[\mathcal{S}(N_n)]$-$\lim q_{uv} = q$ if and only if there exists a sequence $\{w_{uv}\} \in W$ such that N_n-$\lim w_{uv} = q$ and*

$$\delta_{p,q,r,s}\left(\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : q_{uv} = w_{uv}\}\right) = 1.$$

Theorem 5. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. Also, let $\left\{\frac{p(\alpha)}{\vartheta(\alpha)}\right\}$ and $\left\{\frac{r(\beta)}{\varrho(\beta)}\right\}$ be bounded sequences. If $\mathcal{S}(N_n)$-$\lim q_{uv} = q$, then $D_{p,q,r,s}[\mathcal{S}(N_n)]$-$\lim q_{uv} = q$.*

Proof. Suppose $\mathcal{S}(N_n)$-$\lim q_{uv} = q$. Then for every $\rho \in (0,1)$, $\zeta > 0$, $\tau_1, \ldots, \tau_{n-1} \in W$ we get

$$\lim_{\alpha,m\to\infty} \frac{1}{\alpha\beta}\Big|\{u \le \alpha, v \le \beta : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or }$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}\Big| = 0.$$

Since $q(\alpha) \to \infty$ and $s(\beta) \to \infty$ as $\alpha, \beta \to \infty$, we get

$$\lim_{n,m\to\infty} \frac{1}{q(\alpha)s(\beta)}\Big|\{u \le q(\alpha), v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or}$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}\Big| = 0.$$

Now, since the following inclusion

$$\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho$$
$$\text{or } \Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\Big\}$$
$$\subset \Big\{u \le q(\alpha), v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or}$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\Big\}$$

holds, we have

$$\frac{1}{\vartheta(\alpha)\varrho(m)}\Big|\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho$$
$$\text{or } \Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}\Big|$$
$$\le \frac{q(\alpha)s(\beta)}{\vartheta(\alpha)\varrho(\beta)}\frac{1}{q(\alpha)s(\beta)}\Big|\{u \le q(\alpha), v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or}$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}\Big|$$
$$= \left(1+\frac{p(\alpha)}{\vartheta(\alpha)}\right)\left(1+\frac{r(\beta)}{\varrho(\beta)}\right)\frac{1}{q(\alpha)s(\beta)}\Big|\{u \le q(\alpha), v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho$$
$$\text{or } (q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\}\Big|.$$

Given that $\left\{\frac{p(\alpha)}{\vartheta(\alpha)}\right\}$ and $\left\{\frac{r(\beta)}{\varrho(\beta)}\right\}$ are bounded in the inequality above, we should obtain the intended outcome. We can easily demonstrate this.

Since $q(\alpha) \le \alpha$, $s(\beta) \le \beta$ for all $\alpha, \beta \in \mathbb{N}$ the following inclusion

$$\Big\{p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or }$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\Big\}$$
$$\subseteq \Big\{1 \le u \le \alpha, 1 \le v \le \beta : \Im(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \le 1-\rho \text{ or }$$
$$\Re(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho \text{ and } \wp(q_{uv}-q, \tau_1, \ldots, \tau_{n-1}; \zeta) \ge \rho\Big\}$$

and the inequality

$$\begin{aligned}&\Big|\big\{p(\alpha)+1\le u\le q(\alpha), r(\beta)+1\le v\le s(\beta) : \mathfrak{I}\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\le 1-\rho \text{ or}\\ &\qquad \mathfrak{R}\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\ge\rho \text{ and } \wp\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\ge\rho\big\}\Big|\\ &\le\Big|\big\{1\le u\le\alpha, 1\le v\le\beta : \mathfrak{I}\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\le 1-\rho \text{ or}\\ &\qquad \mathfrak{R}\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\ge\rho \text{ and } \wp\left(q_{uv}-q,\tau_1,\ldots,\tau_{n-1};\zeta\right)\ge\rho\big\}\Big|\end{aligned}$$

hold. If we take limit as $\alpha,\beta\to\infty$, we get $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv}=q$. □

Throughout the proofs of following Theorem 6 and Theorem 7, for simplicity we denote

$$1-\mathfrak{I}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)=\overline{\mathfrak{I}}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right).$$

If $\mathfrak{I}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)>1-\rho$, then we restate it as $\overline{\mathfrak{I}}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho$.

Theorem 6. *Let $(W,\mathfrak{I},\mathfrak{R},\wp,\otimes,\circledast)$ be an N-n-NLS and (q_{uv}) be a sequence in W. Then $D_{p,q,r,s}\left[(\mathfrak{I},\mathfrak{R},\wp)^n\right]$ - $\lim q_{uv}=q_0$ implies $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv}=q_0$.*

Proof. Let $D_{p,q,r,s}\left[(\mathfrak{I},\mathfrak{R},\wp)^n\right]$ - $\lim q_{uv}=q_0$. Then for every $\rho\in(0,1)$, $\zeta>0$, $\tau_1,\ldots,\tau_{n-1}\in W$, there exists a $u_0\in\mathbb{N}$ such that

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\mathfrak{I}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)>1-\rho,$$

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\mathfrak{R}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho$$

and

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\wp\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho$$

hold for all $u,v\ge u_0$, or equivalently,

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\overline{\mathfrak{I}}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho, \tag{1}$$

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\mathfrak{R}\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho$$

and

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\wp\left(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta\right)<\rho$$

hold for all $\alpha, \beta \geq u_0$. Now, we get

$$\begin{aligned}
&\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \\
&= \frac{1}{\vartheta(\alpha)\varrho(\beta)} \left(\left(\sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)\geq\rho}}^{u=q(\alpha), v=s(\beta)} \sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)<\rho}}^{u=q(\alpha), v=s(\beta)} \right) \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \right) \qquad (2) \\
&\geq \frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)\geq\rho}}^{u=q(\alpha), v=s(\beta)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \\
&\geq \frac{\rho}{\vartheta(\alpha)\varrho(\beta)} \left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\right\}\right|.
\end{aligned}$$

Denote

$$A_{\alpha\beta} = \frac{\rho}{\vartheta(\alpha)\varrho(\beta)} \left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\right\}\right|.$$

Now, from (1) and (2) we obtain

$$\rho > \frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq A_{\alpha\beta}.$$

Since $\rho > 0$ is arbitrary, we get $\lim_{\alpha,\beta\to\infty} A_{\alpha\beta} = 0$. This implies $\delta_{p,q,r,s}\left(A_{\alpha\beta}\right) = 0$ as deferred density cannot be negative.

By similar process we get

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \Re\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq B_{\alpha\beta},$$

where

$$B_{\alpha\beta} = \frac{\rho}{\vartheta(\alpha)\varrho(\beta)} \left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \Re\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\right\}\right|.$$

Also, by similar process $\delta_{p,q,r,s}\left(B_{\alpha\beta}\right) = 0$. In addition

$$\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \wp\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq C_{\alpha\beta},$$

where

$$C_{\alpha\beta} = \frac{\rho}{\vartheta(\alpha)\varrho(\beta)} \left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \wp\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\right\}\right|.$$

Then, we have $\delta_{p,q,r,s}\left(C_{\alpha\beta}\right) = 0$. Hence, $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv} = q_0$. □

In general, the converse of Theorem 6 is not true. Take a look at the sample below for this.

Example 3. *Consider the N-n-NLS as in Example 1. Define a sequence*

$$q_{uv}=\begin{cases}\left(u^2v^2,0,\dots,0\right)\in\mathbb{R}^n, & \begin{array}{l}[|\sqrt{q(\alpha)}|]-u_0<u\leq[|\sqrt{q(\alpha)}|],\\ [|\sqrt{s(\beta)}|]-v_0<v\leq[|\sqrt{s(\beta)}|],\end{array}\quad \alpha,\beta=1,2,\dots,\\ (0,\dots,0)\in\mathbb{R}^n, & \text{otherwise,}\end{cases}$$

where $q(\alpha)$, $s(\beta)$ are monotonic increasing sequences with $0<p(\alpha)\leq[|\sqrt{q(\alpha)}|]-u_0$ and $0<r(\beta)\leq[|\sqrt{s(\beta)}|]-v_0$, and $u_0,v_0\in\mathbb{N}$ are fixed. We have shown in Example 1 that $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv}=0$. However, the sequence is not $D_{p,q,r,s}\left[(\Im,\Re,\wp)^n\right]$-convergent to zero as we have

$$\begin{aligned}&\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\Re\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)\\
&\quad=\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\frac{\|\tau_1,\dots,\tau_n\|_n}{\zeta+\|\tau_1,\dots,\tau_n\|_n}\\
&\quad=\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\frac{\left([|\sqrt{q(\alpha)}|]-u_0\right)^2}{\zeta+\left([|\sqrt{q(\alpha)}|]\right)^2}\frac{\left([|\sqrt{s(\beta)}|]-v_0\right)^2}{\zeta+\left([|\sqrt{s(\beta)}|]\right)^2}\\
&\quad\geq\frac{\left([|\sqrt{q(\alpha)}|]-u_0\right)^2}{\left([|\sqrt{q(\alpha)}|]\right)^2}\frac{\left([|\sqrt{s(\beta)}|]-v_0\right)^2}{\left([|\sqrt{s(\beta)}|]\right)^2}\to 1\quad\text{as }\alpha,\beta\to\infty.\end{aligned}$$

Theorem 7. *Let $(W,\Im,\Re,\wp,\otimes,\circledast)$ be an N-n-NLS and (q_{uv}) be a bounded sequence in W. Then $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv}=q_0$ implies $D_{p,q,r,s}\left[(\Im,\Re,\wp)^n\right]$-$\lim q_{uv}=q_0$.*

Proof. Suppose (q_{uv}) is a bounded sequence in W and $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-$\lim q_{uv}=q_0$. By the assumption on (q_{uv}), there exists a positive real number M such that

$$\Im\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)>1-M,\qquad\Re\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)<M$$

and

$$\wp\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)<M$$

hold for all u, v. Or equivalently,

$$\overline{\Im}\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)<M,\qquad\Re\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)<M$$

and

$$\wp\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)<M$$

hold for all u, v. Now, we have

$$\begin{aligned}&\frac{1}{\vartheta(\alpha)\varrho(\beta)}\sum_{u=p(\alpha)+1,v=r(\beta)+1}^{u=q(\alpha),v=s(\beta)}\Im\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)\\
&=\frac{1}{\vartheta(\alpha)\varrho(\beta)}\left(\left(\sum_{\substack{u=p(\alpha)+1,v=r(\beta)+1\text{ and}\\ \Im(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta)\geq\rho}}^{u=q(\alpha),v=s(\beta)}\sum_{\substack{u=p(\alpha)+1,v=r(\beta)+1\text{ and}\\ \Im(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta)<\rho}}^{u=q(\alpha),v=s(\beta)}\right)\Im\left(q_{uv}-q_0,\tau_1,\dots,\tau_{n-1};\zeta\right)\right).\end{aligned}\tag{3}$$

We rewrite (3) as follows

$$
\begin{aligned}
&\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \\
&= \frac{1}{\vartheta(\alpha)\varrho(\beta)}\left(\left(\sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)\geq\rho}}^{u=q(\alpha), v=s(\beta)} \sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)<\rho}}^{u=q(\alpha), v=s(\beta)}\right) \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)\right) \\
&< \frac{1}{\vartheta(\alpha)\varrho(\beta)}\left(M \sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)\geq\rho}}^{u=q(\alpha), v=s(\beta)} 1+\rho \sum_{\substack{u=p(\alpha)+1, v=r(\beta)+1 \text{ and} \\ \overline{\Im}(q_{uv}-q_0,\tau_1,\ldots,\tau_{n-1};\zeta)<\rho}}^{u=q(\alpha), v=s(\beta)} 1\right) \\
&< \frac{M}{\vartheta(\alpha)\varrho(\beta)}\left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\right\}\right| \\
&\quad +\rho\left|\left\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)<\rho\right\}\right| .
\end{aligned}
$$

Now, we take the limit by considering $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ - $\lim q_{uv}=q$. Then we get

$$
\frac{1}{\vartheta(n)\varrho(m)} \sum_{u=p(n)+1, v=r(m)+1}^{u=q(n), v=s(m)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)<M \cdot 0+\rho \cdot 1,
$$

so

$$
\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \overline{\Im}\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)<\rho,
$$

or equivalently

$$
\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \Im\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)>1-\rho.
$$

By similar process as above we have

$$
\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \Re\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)<\rho
$$

and

$$
\frac{1}{\vartheta(\alpha)\varrho(\beta)} \sum_{u=p(\alpha)+1, v=r(\beta)+1}^{u=q(\alpha), v=s(\beta)} \wp\left(q_{uv}-q_0, \tau_1, \ldots, \tau_{n-1}; \zeta\right)<\rho.
$$

Therefore, $D_{p,q,r,s}\left[(\Im, \Re, \wp)^n\right]$ - $\lim q_{uv}=q_0$. □

4 Deferred Statistical Completeness in N-n-NLS

In this section, we discuss deferred statistical Cauchy sequences and deferred statistical completeness with regard to N_n.

Definition 10. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. Then $\{q_{uv}\}$ is said to be deferred statistical Cauchy sequence with regard to N_n (in short $D_{p,q,r,s}[\mathcal{S}(N_n)]$-Cauchy) if for every $\rho \in (0,1)$, $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$, there exists $k_0 = k_0(\rho) \in \mathbb{N}$ and $l_0 = l_0(\rho) \in \mathbb{N}$ such that*

$$\delta_{p,q,r,s}\big(\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \mathfrak{I}\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \leq 1-\rho \text{ or } \mathfrak{R}\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho,\ \wp\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \geq \rho\}\big) = 0.$$

Theorem 8. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. If $\{q_{uv}\}$ is $D_{p,q,r,s}[\mathcal{S}(N_n)]$-convergent, it is $D_{p,q,r,s}[\mathcal{S}(N_n)]$-Cauchy sequence.*

Proof. Suppose that $D_{p,q,r,s}[\mathcal{S}(N_n)]\text{-}\lim q_{uv} = q$. For $\rho \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \otimes (1-\varpi) > 1-\rho$ and $\varpi \circledast \varpi < \rho$. Then for every $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ deferred density of the set

$$A(\varpi,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \mathfrak{I}\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \leq 1-\varpi \text{ or } \mathfrak{R}\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq \varpi \text{ and } \wp\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq \varpi\Big\}$$

is zero. Then $\delta_{p,q,r,s}\left(A^c(\varpi,\zeta)\right) = 1$. So, there exist elements $k_0, l_0 \in A^c(\varpi,\zeta)$. Therefore, we have

$$\mathfrak{I}\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) > 1-\varpi,\ \mathfrak{R}\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) < \varpi,\ \wp\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) < \varpi.$$

Let

$$B(\rho,\zeta) = \Big\{p(\alpha)+1 \leq u \leq q(\alpha), r(\beta)+1 \leq v \leq s(\beta) : \mathfrak{I}\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \leq 1-\rho \text{ or } \mathfrak{R}\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq \rho \text{ and } \wp\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq \rho\Big\}.$$

We assert that $B(\varpi,\zeta) \subset A(\varpi,\zeta)$. If this inclusion does not hold, then there must exist some $(i,j) \in B(\varpi,\zeta) \setminus A(\varpi,\zeta)$, which immediately leads to

$$\mathfrak{I}\Big(q_{ij} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\Big) \leq 1-\rho, \qquad \mathfrak{I}\left(q_{ij} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq 1-\varpi.$$

In particular, $\mathfrak{I}\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \geq 1-\varpi$. Therefore, we get

$$\begin{aligned} 1-\rho &\geq \mathfrak{I}\Big(q_{ij} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\Big) \\ &\geq \mathfrak{I}\Big(q_{ij} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\Big) \otimes \mathfrak{I}\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \\ &> (1-\varpi) \otimes (1-\varpi) > 1-\rho, \end{aligned}$$

which is absurd.

Again

$$\begin{aligned}\rho &\le \Re\Big(q_{ij} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\Big) \\ &\le \Re\left(q_{ij} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \oplus \Re\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \\ &< \varpi \circledast \varpi < \rho,\end{aligned}$$

which is absurd.

Similarly we achieve the impossibility $\rho \le \wp\left(q_{ij} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) < \rho$. Hence, $B(\rho, \zeta) \subset A(\varpi, \zeta)$. This gives

$$\delta_{p,q,r,s}(B(\varpi, \zeta)) \le \delta_{p,q,r,s}(A(\varpi, \zeta)),$$

i.e. $\delta_{p,q,r,s}(B(\rho, \zeta)) = 0$, which shows that $\{q_{uv}\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequence. □

Theorem 9. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. If $\{q_{uv}\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequence, it is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent.*

Proof. Let $\{q_{uv}\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequence, but not $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent. For a given $\rho \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \otimes (1-\varpi) > 1-\rho$ and $\varpi \circledast \varpi < \rho$. Then for every $\zeta > 0$ and $\tau_1, \ldots, \tau_{n-1} \in W$ there exist $k_0, l_0 \in \mathbb{N}$ such that $\delta_{p,q,r,s}(B(\rho, \zeta)) = 0$, where

$$\begin{aligned}B(\rho, \zeta) = \Big\{&p(\alpha)+1 \le u \le q(\alpha), r(\beta)+1 \le v \le s(\beta) : \Im\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) \le 1-\rho \\ &\text{or } \Re\left(w_k - w_{k_0}, v; \zeta\right) \ge \rho \text{ and } \wp\left(w_k - w_{k_0}, v; \zeta\right) \ge \rho\Big\}.\end{aligned}$$

Since $\{q_{uv}\}$ is not $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent to $q \in W$, we have

$$\begin{aligned}\Im\Big(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\Big) &\ge \Im\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \otimes \Im\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \\ &> (1-\varpi) \otimes (1-\varpi) > 1-\rho, \\ \Re\Big(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\Big) &\le \Re\left(q_{uv} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \circledast \Re\left(q_{k_0 l_0} - q, \tau_1, \ldots, \tau_{n-1}; \tfrac{\zeta}{2}\right) \\ &< \varpi \circledast \varpi < \rho,\end{aligned}$$

and $\wp\left(q_{uv} - q_{k_0 l_0}, \tau_1, \ldots, \tau_{n-1}; \zeta\right) < \rho$. Therefore, $\delta_{p,q,r,s}\left(B^c(\rho, \zeta)\right) = 0$, i.e. $\delta_{p,q,r,s}(B(\rho, \zeta)) = 1$, a contradiction. Hence, the sequence is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent to q. □

Definition 11. *An N-n-NLS is said to be deferred statistically complete with regard to N_n (in short, $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-complete) if every $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequence is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent.*

Remark 2. *In the light of Theorem 9, we conclude that every N-n-NLS is N_n-complete (in short, $D_{p,q,r,s}\left[\mathcal{S}(N_n)\right]$-complete).*

On the basis of Theorems 4, 8 and 9, we state an equivalent result.

Theorem 10. *Let $\{q_{uv}\}$ be a sequence in an N-n-NLS. Then the below properties are equivalent:*

(1) $\{q_{uv}\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-convergent;

(2) $\{q_{uv}\}$ is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequence;

(3) there exists a set $K = \left\{u_1 < \cdots < u_p < \cdots ; v_1 < \cdots < v_s < \cdots\right\} \subset \mathbb{N}^2$ such that $\delta_{p,q,r,s}(K) = 1$ and the subsequence $\left\{q_{u_p v_s}\right\}$ is N_n-Cauchy sequence.

5 Conclusion

In this research paper, we have explored the convergence properties of $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$ and the concept of $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-Cauchy sequences. Our findings demonstrate that every N-n-NLS is $D_{p,q,r,s}\left[\mathcal{S}\left(N_n\right)\right]$-complete. Building on this research, future work could focus on generalizing these notions within the context of ideals and extending them to sequences of sets of order α with regard to N_n. Additionally, the principles discussed in this study hold potential for addressing convergence-related problems across various fields in science and engineering. Future research aims to gain different features of the stated notion and produce its equivalent in various sequence spaces.

References

[1] Aslam S., Kumar V., Sharma A. *On λ-statistical and V_λ-statistical summability in neutrosophic 2-normed spaces*. Neutrosophic Sets Syst. 2024, **63**, 188–202. doi:10.5281/zenodo.10531797

[2] Aslam S., Sharma A. Kumar V. *On S_θ-summability in neutrosophic 2-normed spaces*. Neutrosophic Sets Syst. 2024, **63**, 153–168. doi:10.5281/zenodo.10531785

[3] Agnew R.P. *On deferred Cesàro mean*. Ann. of Math. (2) 1932, **33** (3), 413–421. doi:10.2307/1968524

[4] Atanassov K. *Intuitionistic fuzzy sets*. Fuzzy Sets and Systems 1986, **20** (1), 87–96. doi:10.1016/S0165-0114(86)80034-3

[5] Dağadur I., Sezgek Ş. *Deferred Cesàro mean and deferred statistical convergence of double sequences*. J. Inequal. Spec. Funct. 2016, **7** (4), 118–136.

[6] Fast H. *Sur la convergence statistique*. Colloq. Math. 1951, **2** (3-4), 241–244.

[7] George A., Veeramani P. *On some results in fuzzy metric spaces*. Fuzzy Sets and Systems 1994, **64** (3), 395–399. doi:10.1016/0165-0114(94)90162-7

[8] Gürdal M., Huban M.B. *On I-convergence of double sequences in the topology induced by random* 2*-norms*. Mat. Vesnik 2014, **66** (1), 73–83.

[9] Gürdal M., Şahiner A., Açık I. *Approximation theory in* 2*-Banach spaces*. Nonlinear Anal. 2009, **71** (5-6), 1654–1661. doi:10.1016/j.na.2009.01.030

[10] Hossain N., Esi A. *Deferred statistical convergence in neutrosophic 2-normed spaces*. J. Class. Anal. 2025, **25** (1), 43–61. doi:10.7153/jca-2024-25-03

[11] Kaleva O., Seikkala S. *On fuzzy metric spaces*. Fuzzy Sets and Systems 1984, **12** (3), 215–229. doi:10.1016/0165-0114(84)90069-1

[12] Katsaras A.K. *Fuzzy topological vector spaces II*. Fuzzy Sets and Systems 1984, **12** (2), 143–154. doi:10.1016/0165-0114(84)90034-4

[13] Kirişci M., Şimşek N. *Neutrosophic normed spaces and statistical convergence*. J. Anal. 2020, **28** (4), 1059–1073. doi:10.1007/s41478-020-00234-0

[14] Kişi Ö., Choudhury C. *Deferred statistical convergence of double sequences in neutrosophic normed spaces*. J. Anal. 2024, **32** (2), 1057–1078. doi:10.1007/s41478-023-00650-y

[15] Kumar V., Sharma A., Murtaza S. *On neutrosophic n-normed linear spaces*. Neutrosophic Sets Syst. 2023, **61**, 275–288.

[16] Mursaleen M., Edely H.H.E. *Statistical convergence of double sequences*. J. Math. Anal. Appl. 2003, **288** (1), 223–231. doi:10.1016/j.jmaa.2003.08.004

[17] Murtaza S., Sharma A., Kumar V. *Neutrosophic 2-normed spaces and generalized summability*. Neutrosophic Sets Syst. 2023, **55** (1), 415–426.

[18] Nabiev A.A., Savaş E., Gürdal M. *Statistically localized sequences in metric spaces*. 2019, **9** (2), 739–746. doi: 10.11948/2156-907X.20180157

[19] Park J.H. *Intuitionistic fuzzy metric spaces*. Chaos, Solitons & Fractals 2004, **22** (5), 1039–1046. doi: 10.1016/j.chaos.2004.02.051

[20] Şahiner A., Gürdal M., Yigit T. *Ideal convergence characterization of the completion of linear n-normed spaces*, Comput. Math. Appl. 2011, **61** (3), 683–689. doi:10.1016/j.camwa.2010.12.015

[21] Savaş E., Gürdal M. *I-Statistical convergence in probabilistic normed spaces*. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 2015, **77** (4), 195–204.

[22] Smarandache F. Neutrosophy: Neutrosophic Probability, Set, and Logic: analytic synthesis & synthetic analysis. Rehoboth, NM: American Research Press, Michigan, 1998.

[23] Smarandache F. *Neutrosophic set, a generalisation of the intuitionistic fuzzy sets*. Int. J. Pure Appl. Math. 2005, **24** (3), 287–297.

[24] Zadeh L.A. *Fuzzy sets*. Inform. Control 1965, **8** (3), 338–353. doi:10.1016/S0019-9958(65)90241-X

Received 27.08.2024
Revised 19.10.2024

Кіші О., Гюрдал М. *Подальші результати щодо відкладеної статистичної збіжності в нейтрософічних n-нормованих просторах* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 735–753.

У цій статті проведено поглиблений аналіз сильної відкладеної підсумовності та відкладеної статистичної збіжності, а також отримано основні результати в контексті нейтрософічних n-нормованих просторів. Крім того, детально досліджено поняття відкладених статистичних послідовностей Коші та показано, що кожний нейтрософічний n-нормований простір є відкладено статистично повним у межах цього підходу.

Ключові слова і фрази: нейтрософічний n-нормований простір, відкладена статистична збіжність, відкладена статистична послідовність Коші, відкладена статистична повнота.