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Multivariate growth and cogrowth

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We deal with a multivariate growth series $\Gamma_L(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^d$, associated with a regular language L over an alphabet of cardinality $d \geq 2$. Our focus is on languages coming from subgroups of the free group F_m of finite rank m and from the subshifts of finite type. We suggest a tool for computing the rate of growth $\varphi_L(\mathbf{r})$ of L in the direction $\mathbf{r} \in \mathbb{R}^d$. Using the concave growth condition introduced by the second author in [Comment. Math. Helv. 2002, **77** (3), 563–608] and the results of Convex Analysis we represent $\psi_L(\mathbf{r}) = \log(\varphi_L(\mathbf{r}))$ as a support function of a convex set that is the closure of the Relog image of the domain of absolute convergence of $\Gamma_L(\mathbf{z})$. This allows us to compute $\psi_L(\mathbf{r})$ in some cases, including a Fibonacci language or a language of freely reduced words representing elements of a free group F_2. Also we show that the methods of the Large Deviation Theory can be used as an alternative approach.

Key words and phrases: growth, cogrowth, regular language, multivariate growth exponent, free group, Fibonacci subshift, subshift of finite type, large deviations principle.

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1 Introduction

The goal of this article is to introduce a growth of coefficients in a given direction $\mathbf{r}$ of multivariate power series $\Gamma(\mathbf{z})$ denoted $\psi(\mathbf{r})$ and called indicatrice of growth, introduce a condition (called here (CG)) that gives a possibility to express $\psi(\mathbf{r})$ as infimum of a linear function over the convex subset and to show how this can be used in some combinatorial problems associated with groups and formal languages.

Let us begin by recalling some standard facts, along with key details related to (co)growth. The study of asymptotic properties of a sequence $\{\gamma_n\}_{n\geq 0}$ of real (or complex) numbers is related to study of analytic properties of the function $\Gamma(z)$ presented by a power series $\sum_{n=0}^{\infty} \gamma_n z^n$. This includes inspection of singularities on the border of the domain of (absolute) convergence of the series and local behavior of $\Gamma(z)$ in their neighborhood. If $\Gamma(z)$ is a rational function, i.e. $\Gamma(z) = \frac{G(z)}{H(z)}$, $G(z), H(z) \in \mathbb{C}[z]$, then all the needed information about the coefficients γ_n can be gained from the polynomials $G(z)$ and $H(z)$. The same is true, when $\Gamma(z)$ is an algebraic function.

In algebra, and especially in modern group theory, there are many notions and concepts that associate to the algebraic object a sequence $\{\gamma_n\}_{n\geq 0}$. This include growth, cogrowth, sub-

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group growth etc. Recall that given a finitely generated group G with a system of generators S, one can consider a function $\gamma_n = \#\{g \in G : |g| = n\}$, where $n \in \mathbb{N}$ and $|g|$ is the length of the element g with respect to S. If the pair (G, S) has a regular geodesic normal form (in other terminology a rational cross section [9]), then the power series $\Gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ represents a rational function and the asymptotic of γ_n is either polynomial or exponential. There are many groups (for instance groups of intermediate growth constructed in [14]) for which $\Gamma(z)$ is irrational for any system of generators and the study of asymptotic properties of $\{\gamma_n\}_{n=0}^{\infty}$ becomes much more complicated.

Now let F_m be a free group of rank m. Every group generated by m elements can be presented as a quotient F_m/N, for a suitable normal subgroup $N \lhd F_m$. Let $A = \{a_1, \ldots, a_m\}$ be a basis of F_m. Elements of F_m are presented by freely reduced words over the alphabet $\Sigma = \{a_1, \ldots, a_m, a_1^{-1}, \ldots, a_m^{-1}\}$ and there is $2m(2m-1)^{n-1}$ such words of length $n \geq 1$. The function

$$\Gamma_{F_m}(z) = 1 + \sum_{n=1}^{\infty} 2m(2m-1)^{n-1} z^n = \frac{1+z}{1-(2m-1)z}$$

is a spherical growth function of F_m with respect to the basis A. Now let $H < F_m$ be a subgroup and H_n be the set of elements in H of length n with respect to generators $\{a_1, \ldots, a_m\}$ of F_m. The sequence $\{|H_n|\}_{n=0}^{\infty}$ of cardinalities of these sets is a cogrowth sequence,

$$\Gamma_H(z) = \sum_{n=0}^{\infty} |H_n| z^n$$

is a cogrowth series, and

$$\alpha_H = \limsup_{n \to \infty} |H_n|^{\frac{1}{n}}$$

is a cogrowth of H (or an exponent of relative growth of subgroup H). The range for α_H is inside interval $[1, 2m-1]$ and the range of α_H, when H is nontrivial and normal subgroup, is inside $\left(\sqrt{2m-1}, 2m-1\right]$ (see [10–13]). The spectral radius χ of a simple random walk on $G = F_m/N$, when $N \lhd F_m$ is a normal subgroup, is related with α_N as

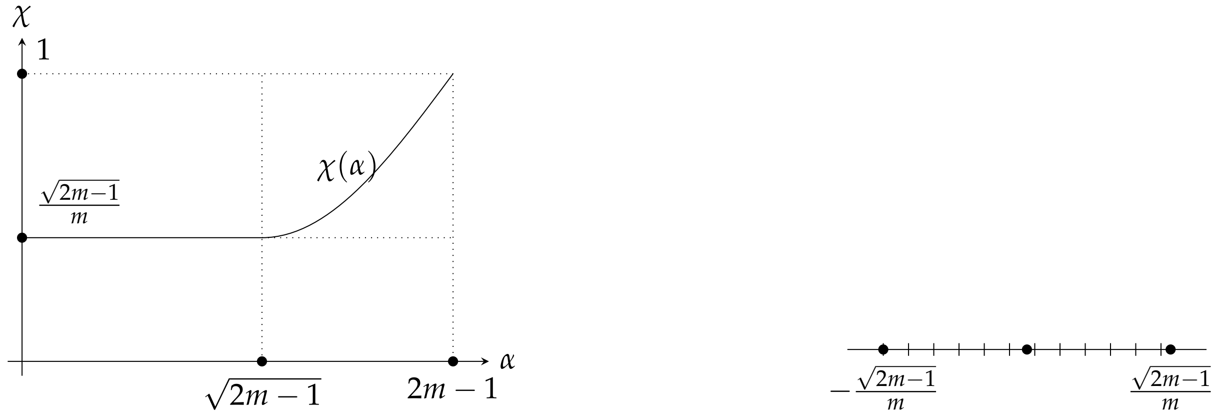
$$\chi = \frac{\sqrt{2m-1}}{2m} \left(\frac{\sqrt{2m-1}}{\alpha_N} + \frac{\alpha_N}{\sqrt{2m-1}} \right)$$

and the group G is amenable if and only if $\alpha_N = 2m-1$ (that is α_N takes its maximum possible value).

In the case, when $H < F_m$ is not a normal subgroup, one can consider a Schreier graph $\Lambda = \Lambda(F_m, H, \Sigma)$. Then the dependence on α_H of the spectral radius χ of a simple random walk on Λ is given by

$$\chi = \begin{cases} \dfrac{\sqrt{2m-1}}{2m} \left(\dfrac{\sqrt{2m-1}}{\alpha_H} + \dfrac{\alpha_H}{\sqrt{2m-1}} \right), & \text{if } \alpha_H \geq \sqrt{2m-1}, \\ \dfrac{\sqrt{2m-1}}{m}, & \text{if } \alpha_H \leq \sqrt{2m-1} \end{cases}$$

(see Figure 1 (a)).



(a) **(b)**

Figure 1. The graph of $\chi = \chi(\alpha)$ and the interval $\left[-\frac{\sqrt{2m-1}}{m}, \frac{\sqrt{2m-1}}{m}\right]$

Hence, again the graph Λ is amenable (on the notion of amenable graph see [3]) if and only if $\alpha_H = 2m-1$. On the other hand, in the case when Λ is infinite, it is a Ramanujan graph if and only if $\alpha_H \leq \sqrt{2m-1}$. Recall that the value $\frac{\sqrt{2m-1}}{m}$ is a spectral radius of a simple random walk on F_m computed by H. Kesten [17] and if the spectrum of the Laplacian operator on graph Λ minus two points set $\{-1, 1\}$ is a subset of the interval given by Figure 1 (b), then the graph is called Ramanujan. The terminology "Ramanujan graph" was introduced in the case of finite graphs by A. Lubotzky, R. Phillips and P. Sarnak in 1988 (see [19]). The definition adopted to the infinite case is given in [15].

Observe that the analogue of the above formula for χ in the context of Riemannian geometry was obtained by D. Sullivan in [24], where the role of parameter α_H plays a critical exponent of the group of isometries of the hyperbolic space.

The goal of this article is to investigate a finer growth characteristics: multivariate growth and multivariate cogrowth.

Let $\Sigma = \{a_1, \ldots, a_d\}$, $d \geq 2$, be an alphabet, Σ^* be a set of all finite words (or strings) over Σ. The set Σ^* with concatenation as a binary operation, can be interpreted as a monoid (with empty word serving as the identity element). In fact, Σ^* is a free monoid. Its growth sequence is $d^n, n = 0, 1, \ldots$. Any subset $L \subset \Sigma^*$ is a (formal) language. With $w \in \Sigma^*$ we can associate the length $|w|$ and the vector $\wp(w) = (|w|_{a_1}, \ldots, |w|_{a_d}) \in \mathbb{N}^d$, where $|w|_{a_i}$ is a number of occurrences of the symbol a_i in w.

Let $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and

$$\Gamma_L(\mathbf{z}) = \sum_{w \in L} \mathbf{z}^{\wp(w)}, \tag{1}$$

where $\mathbf{z}^{\wp(w)} = z_1^{|w|_{a_1}} \cdots z_d^{|w|_{a_d}}$. The series (1) is a multivariate series associated with L and can be rewritten as

$$\Gamma_L(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^d} \gamma_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}, \tag{2}$$

where $\gamma_{\mathbf{i}}$ is the number of words w in L with $\wp(w) = \mathbf{i}$.

The vector $\tilde{\wp}(w) = \dfrac{1}{|w|} \wp(w)$ is a vector of frequencies. It belongs to the simplex M_d of probability vectors, defined by

$$M_d = \left\{ \mathbf{r} \in \mathbb{R}^d_{\geq 0} : \mathbf{r} = (r_1, \dots, r_d),\ r_i \geq 0,\ \|\mathbf{r}\|_1 = \sum_{i=1}^{d} r_i = 1 \right\}, \tag{3}$$

here $\| \cdot \|_1$ is the l_1 norm.

The multivariate growth *indicatrice*, that we are going to study, is the number $\psi(\mathbf{r})$, depending on a vector $\mathbf{r} \in M_d$, which characterizes the growth of coefficients $\gamma_{\mathbf{i}}$, when $\|\mathbf{i}\|_1 \to \infty$ and index $\mathbf{i}$ "keeps" the direction of the vector $\mathbf{r}$. When $\mathbf{r} \in \mathbb{Q}^d$ is a rational vector, then one can define $\psi(\mathbf{r})$ by

$$\psi(\mathbf{r}) = \limsup_{n \to \infty} \frac{1}{n} \log |\gamma_{n\mathbf{r}}|,$$

where $n \in \mathbb{N}$ and $\gamma_{n\mathbf{r}} = 1$ if $n\mathbf{r} \notin \mathbb{N}^d$. This approach is used in S. Melczer's book [20], specifically on page 11, which draws on the work of the mathematicians cited therein. The case of an irrational direction vector is also briefly discussed in the book, but a rigorous definition of the corresponding function is not provided. The Definition 2 given below in the Section 4 defines $\psi(\mathbf{r})$ for arbitrary $\mathbf{r} \in M_d$. It follows the idea used in [22].

A crucial assumption concerning the involved languages that we use is the "concavity" assumption (CG) given in the Definition 3, which allows to apply the powerful methods from Convex Analysis (as well as the results from [22]). In fact, the definition can be adopted to arbitrary multivariate power series (2) with real coefficients. The main goal is to present the indicatrice of growth $\psi(\mathbf{r})$ (assuming the condition (CG)) in the form

$$\psi(\mathbf{r}) = \inf_{\theta \in \Omega'} \langle \mathbf{r}, \theta \rangle, \tag{4}$$

where $\Omega' = -\Omega \subset \mathbb{R}^d_{\geq 0}$ and Ω is a closed convex set representing the Relog image of the domain of absolute convergence of (2), where

$$\mathrm{Relog}(\mathbf{z}) = (\log |z_1|, \dots, \log |z_d|)$$

(see Theorem 2).

We apply (4) for two languages: the language L_{F_m} of freely reduced words associated with a free group F_m of rank $m \geq 2$ and the Fibonacci language L_{Fib} (see Sections 5 and 6). These languages belong to class of regular languages, that is languages accepted by finite automaton-aceptor (more on this see in Section 2).

Regular languages play important role in many areas of mathematics, including dynamical systems and algebra. Regular normal form of elements in the group is a bijective presentation of elements of the group by elements of a regular language over the alphabet of generators and inverses. By definition, a regular geodesic normal form is a presentation in which the length of an element with respect to the generating set coincides with the length of the corresponding word. Virtually abelian groups and Gromov hyperbolic groups have a regular geodesic normal form for any system of generators.

Regular languages are good in particular because their growth series (in one or multivariate case) are rational functions. This fact, even in a stronger form, was known already to N. Chomsky and M.P. Schützenberger [5]. Proposition 1 of Section 2 gives a rational expression for a multivariate growth series associated with regular language. The condition (CG) mentioned earlier holds under the assumption of ergodicity of the automaton presenting the language (condition (E) in Section 2). It is satisfied in the examples given in this article (and in many other cases) and we summaries the results of computations from Sections 5 and 6 as the following assertion.

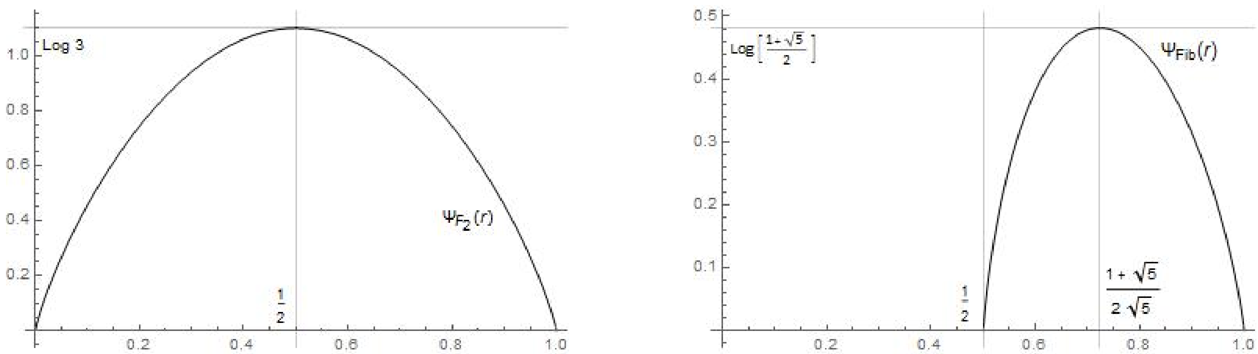
Theorem 1. *The indicatrices* $\psi_{F_2}(\mathbf{r})$ *and* $\psi_{Fib}(\mathbf{r})$ *are given by*

$$\text{(i)} \qquad \psi_{F_2}(\mathbf{r}) = \mathbf{H}(\mathbf{r}) + p\log\left(2q - p + 2\sqrt{p^2 - pq + q^2}\right) + q\log\left(2p - q + 2\sqrt{p^2 - pq + q^2}\right),$$

$$\text{(ii)} \qquad \psi_{Fib}(\mathbf{r}) = \begin{cases} p\log\left(\frac{p}{p-q}\right) + q\log\left(\frac{p-q}{q}\right), & \text{if } p \geq \frac{1}{2}, \\ -\infty, & \text{if } p < \frac{1}{2}, \end{cases}$$

where $\mathbf{r} = (p, q) \in M_2$ *and* $\mathbf{H}(\mathbf{r}) = -p\log p - q\log q$ *is the Shannon's entropy.*

The graphics of these functions are presented by Figure 2.



(a) **(b)**

Figure 2. Graphs of $\psi_{F_2}(\mathbf{r})$ and $\psi_{Fib}(\mathbf{r})$

In fact, $\psi_{F_2}(\mathbf{r})$ is computed for the 2-variable series $\Delta_{F_2}(\mathbf{z})$, $\mathbf{r} \in M_2$, defined in Section 3, which is a kind of symmetrization of the 4-variate series $\Gamma_{F_2}(\mathbf{z})$.

Regular languages quite often appear in Dynamical systems, for instance as languages associated with the subshifts of finite type (SFT), i.e. subshifts determined by a finite set $\mathbf{F}$ of forbidden words (patterns). For example, a subshift of the full shift $\{0,1\}^{\mathbb{Z}}$ determined by a single forbidden pattern 11 is a Fibonacci subshift and the corresponding language is a Fibonacci language. If $\Sigma = \{a_1, \ldots, a_m, a_1^{-1}, \ldots, a_m^{-1}\}$, $\mathbf{F} = \{a_i a_i^{-1}, a_i^{-1} a_i : i = 1, \ldots, m\}$ we get a subshift corresponds to the language L_{F_m} of freely reduced words representing elements of the free group F_m over Σ.

There is an alternative way to present $\psi_\Delta(\mathbf{r})$ in the case of languaages associated with SFT. This approach uses methods of Large Deviation Theory (LDT) [7]. One of the basic results of the LDT is Sanov's Theorem (see [7, Theorem 3.1.6]) stating that the Large Deviation Principle holds for finite Markov chains and giving the expressions for the rate function $I(\mathbf{r})$. In Section 7, we relate formula (4) to Sanov's expression for $I(\mathbf{r})$.

The paper is organized as follows. In Section 2, we recall some of the basic definitions from the theory of finite automata and formal languages that will be needed later. Then we introduce the condition (E) for the regular languages. Section 3 is devoted to the computations of the modified multivariate growth series of the language L_{F_m} of reduced elements of a free group F_m using two approaches: a classical one (due to N. Chomsky and M.P. Schützenberger) and the alternative, based on the use of symmetries hyded in the given combinatorial problem. In Section 4, we discuss the condition (CG) and then prove Theorem 2. In Sections 5 and 6, we present computations of indicatrice $\psi(\mathbf{r})$ for F_2 and for the Fibonacci language, respectively. Section 7 is devoted to application of Large Deviations Theory. We get a relationship between $\psi(\mathbf{r})$ and the rate function $I(\mathbf{r})$. Then using result from asymptotic combinatorics in several variables presented in [20], in Section 8, we get a finer asymptotic associated with F_2. Finally, Section 9 contains concluding remarks and some open questions.

2 Regular languages, their growth series and the condition (E)

We first recall the definitions of finite automaton (acceptor) and language accepted by it. A finite automaton $\mathscr{A}$ is given by a quintuple $(Q, \Sigma, \kappa, q_0, \mathscr{F})$, where Q is a finite set whose elements are called states, Σ is a finite alphabet, $\kappa : Q \times \Sigma \to Q$ is a transition function, the state $q_0 \in Q$ is a special state called the initial state and the set $\mathscr{F} \subset Q$ is nonempty set whose elements are called final states.

It is convenient to visualize $\mathscr{A}$ as a labeled directed graph (called also a diagram) $\Theta_{\mathscr{A}}$ with the vertex set Q, edge set

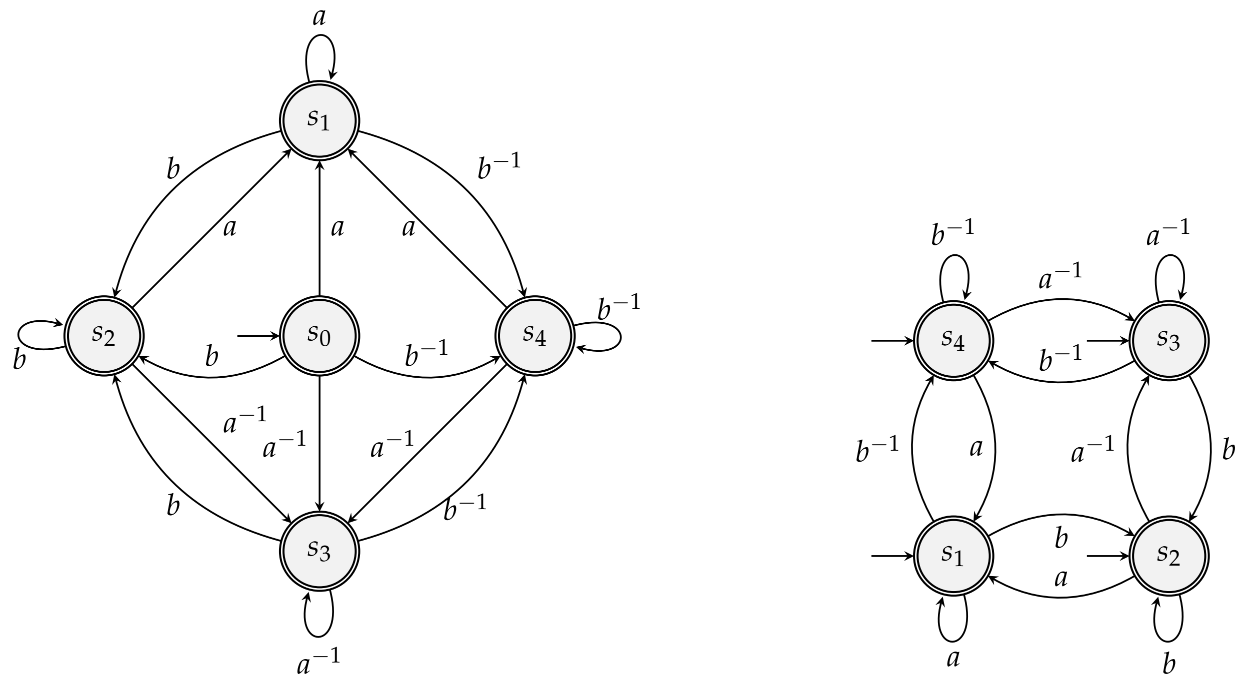
$$E = \{(q,s) : q, s \in Q, \kappa(q, a_i) = s \text{ for some } a_i \in \Sigma\},$$

and each such edge (q,s) with $\kappa(q, a_i) = s$ is supplied by the label a_i. Multiple edges and loops are allowed. The graph $\Theta_{\mathscr{A}}$ is called the diagram of $\mathscr{A}$. The example of these diagrams are presented by Figures 3 (a) and 7.

Observe that so defined $\mathscr{A}$ is deterministic and complete automaton, i.e. given any $q \in Q$ and any $a \in \Sigma$ we know what would be the next state $\kappa(q, a)$. A word $w \in \Sigma^*$ is accepted by $\mathscr{A}$ if starting with the initial state q_0 and traveling in diagram $\Theta_{\mathscr{A}}$ along the path p_w determined by w we end up at final state. Let $\mathscr{L}(\mathscr{A})$ be the set of words accepted by $\mathscr{A}$. A language $L \subset \Sigma^*$ is regular if there is a finite automaton $\mathscr{A}$ such that $L = \mathscr{L}(\mathscr{A})$.

One can generalize the above definition by replacing a singleton $\{q_0\}$ by a nonempty subset $\mathscr{I} \subset Q$ whose elements are called initial states and defining $\mathscr{L}(\mathscr{A})$ as a set of words w for which there is an initial state $i \in \mathscr{I}$ such that the path $p_{i,w}$ that begins at i and follow the word w ends up at $\mathscr{F}$. Surprisingly, this does not lead to a larger class of languages (as it is always possible to replace $\mathscr{A}$ by automaton $\mathscr{A}'$ with a single initial state such that $\mathscr{L}(\mathscr{A}) = \mathscr{L}(\mathscr{A}')$). The automaton with a single initial state are unambiguous in the sense that for each $w \in \mathscr{L}(\mathscr{A})$ there is a unique path p_w that recognizes w. Nevertheless, there are situations (for instance in the case of the language of freely reduced words over the alphabet of generators of a free group) when it is better to use ambiguous automata (see Figure 3 (b) and Section 5).

One can also consider non-deterministic and incomplete automata, which, despite their broader scope, still define the same class of languages – the regular languages. Non-deterministic automata, for instance, are particularly relevant in the study of languages associated with sofic subshifts [18].



(a) **(b)**

Figure 3. Diagrams of automata $\mathscr{A}$ and $\mathscr{A}'$ associated with the language L_{F_2} of reduced words representing elements of the free group F_2, where the initial states are denoted by horizontal unlabeled arrow and the final states are represented by "double" circles

Recall that given $w \in \Sigma^*$ the vector $\wp(w) = (|w|_{a_1}, \ldots, |w|_{a_d})$, where $|w|_{a_i}$ denotes the number of occurrences of $a_i \in \Sigma$ in w. Let also $|w|$ denotes the length of w. With $L \subset \Sigma^*$ one can associate formal series

$$\Gamma_L(z) = \sum_{w \in L} z^{|w|}, \quad z \in \mathbb{C}, \tag{5}$$

and the multivariate series

$$\Gamma_L(\mathbf{z}) = \Gamma_L(z_1, \ldots, z_d) = \sum_{w \in L} \mathbf{z}^{\wp(w)}, \tag{6}$$

where $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $\mathbf{z}^{\wp(w)} = z_1^{|w|_{a_1}} \ldots z_d^{|w|_{a_d}}$. Also one can consider the formal sum

$$\sum_{w \in L} w z^{|w|} \in \mathbb{Z}[\Sigma^*][\![z]\!] \tag{7}$$

viewed as a formal power series with coefficients in the ring $\mathbb{Z}[w^*]$ (the semi-group ring of the free semi-group Σ^*). The consideration of these type of series go back at least to 50's of 20th century and is related first of all with the names of N. Chomsky and M.P. Schützenberger [5].

In the case when L is a regular language the series (5), (6) and (7) are rational, i.e. ratio of two polynomials. This is a well known fact but for convenience of the reader we include its proof (given in the next proposition).

Let $A = (a_{ij})$ be the adjacency matrix of $\Theta_{\mathscr{A}}$, i.e. a $|Q| \times |Q|$ matrix whose rows and columns correspond to the states and a_{ij} is equal to the number of edges in $\Theta_{\mathscr{A}}$ joining i with j, where $i, j \in Q$.

We use the ordering on Q in such a way that the first state is q_0. Let $u = (1, 0, \dots, 0)$ and $v = \left(v_1, \dots, v_{|Q|}\right)^t$ be a row and column vectors of dimension $|Q|$, where $v_q = 1$, if $q \in \mathscr{F}$ and $v_q = 0$, if $q \notin \mathscr{F}$.

Then the standard technique of counting paths (see, e.g., [23, Theorem 4.7.2]) in finite graph (or in finite Markov chain) leads to the relations

$$\Gamma_L(z) = u \left[\sum_{n \geq 0} (zA)^n \right] v = u \left[I - zA \right]^{-1} v$$

and the rationality of $L(z)$ is obvious. A similar argument works for multivariate case, only the matrix zA should be replaced by

$$A(\mathbf{z}) = \left(a_{st}(\mathbf{z})\right)_{s,t=1}^{|Q|},$$

where $\mathbf{z} = (z_1, \dots, z_d)$ and

$$a_{st}(\mathbf{z}) = \begin{cases} \sum\limits_i z_i, & \text{where summation is taken over such } i \text{ that } \kappa(s, a_i) = t, \\ 0, & \text{if there is no edge from } s \text{ to } t. \end{cases}$$

With this notations we have the following assertion.

Proposition 1. *The multivariate growth series of a regular language $L = \mathscr{L}(\mathscr{A})$ satisfies*

$$\Gamma_L(\mathbf{z}) = u \left[I - A(\mathbf{z}) \right]^{-1} v.$$

For example, in the case of automaton presented by Figure 7, we have

$$A(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ z_1 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_L(\mathbf{z}) = \frac{1 + z_2}{1 - z_1 - z_1 z_2}.$$

For us the following condition will be crucial.

Definition 1. *We say that a language $L \subset \Sigma^*$ satisfy the condition (E) if there exists an integer $N \geq 0$ such that, for every $U, V \in L$, there exists $w \in \Sigma^*$, $|w| \leq N$, with $UwV \in L$.*

The examples of regular languages satisfying the condition (E) come from ergodic (or primitive, connected, etc.) automata $\mathscr{A}$, i.e. automata in which each state can be connected to the any other state by a path in the diagram of automaton.

This is because if $\mathscr{A}$ is ergodic automaton, $U, V \in \mathscr{L}(\mathscr{A})$ and s, t are the end states of the paths p_U, p_V, then connecting state s with the initial state q_0 by some path q whose length does not exceed the diameter D of the graph $\Theta_{\mathscr{A}}$ (i.e. maximum of combinatorial distances between states in $\mathscr{A}$) we get a word $UwV \in \mathscr{L}(\mathscr{A})$ where w is a word read along the path q, $|w| \leq D$ (see Figure 4).

Unfortunately, not every regular language satisfying (E) can be accepted by ergodic unambiguous automaton. The example is the language L_{F_2} over the alphabet $\Sigma = \{a, a^{-1}, b, b^{-1}\}$ of freely reduced words (i.e. $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$ are forbidden) of the free group F_2 of rank 2 (see [4]). Figure 3 illustrates two automata that accept the language L_{F_2}. The first automaton presented in the Figure 3 (a) is unambiguous but not ergodic. In the second automaton 3 (b), all states are initial, it is ergodic automaton but not unambiguous.

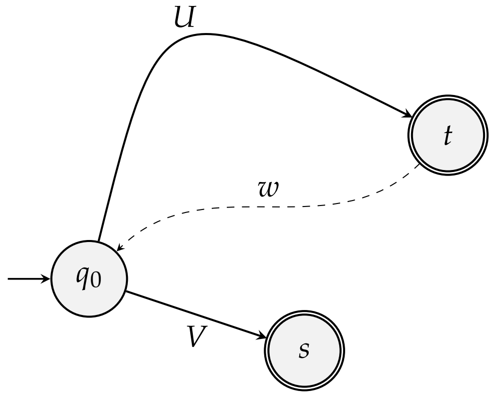


Figure 4. Part of Moore diagram of an ergodic automaton $\mathscr{A}$

Later, we will introduce a condition (CG) for multivariate power series and it will follow that $\Gamma_L(\mathbf{z}), \mathbf{z} \in \mathbb{C}^d$ satisfies the condition (CG) if the associated language L satisfies the condition (E).

3 The case of a free group

Let $F_m = \langle a_1, \ldots, a_m \rangle$ be a free group of rank $m \geq 2$ and L_{F_m} be the language of freely reduced words over the alphabet $\Sigma = \{a_1, \ldots, a_m, a_1^{-1}, \ldots, a_m^{-1}\}$. This is a regular language (the automaton accepting it in the case $m = 2$ is given by Figure 3 (a)) and hence a corresponding multivariate growth series $\Gamma_{F_m}(\mathbf{z})$ is rational.

Instead of $\Gamma_{F_m}(\mathbf{z})$, we will focus on a relative series $\Delta_{F_m}(\mathbf{z})$ defined as

$$\Delta_{F_m}(\mathbf{z}) = 1 + \sum_{g \in F_m,\ g \neq e} \mathbf{z}^{\wp(g)},$$

where $\wp(g) = (|g|_{a_1^\epsilon}, \ldots, |g|_{a_m^\epsilon})$ and $|g|_{a_i^\epsilon}$ is a number of occurrences of symbols a_i and a_i^{-1} in the freely reduced word presenting element g, i.e. $|g|_{a_i^\epsilon} = |g|_{a_i} + |g|_{a_i^{-1}}$. The study of this series is related with the study of symmetric random walk on a free group and its quotients (as outlined by H. Kesten in [17]).

Instead of using the Proposition 1, we will compute $\Delta_{F_m}(\mathbf{z})$ exploring hidden symmetries in the given combinatorial problem.

Proposition 2. *The function* $\Delta_{F_m}(\mathbf{z})$ *satisfies*

$$\Delta_{F_m}(\mathbf{z}) = \frac{1}{1 - 2\sum_{i=1}^{m} \frac{z_i}{1 + z_i}}. \tag{8}$$

Proof. Recall that the elements of F_m are identified with freely reduced words over the alphabet $\Sigma = \{a_1, \ldots, a_m, a_1^{-1}, \ldots, a_m^{-1}\}$. We have

$$\Delta_{F_m}(\mathbf{z}) = 1 + \sum_{e \neq g \in F_m} \mathbf{z}^{\wp(g)}, \quad \mathbf{z} = (z_1, \ldots, z_m),$$

where e is the identity in F_m, $\wp(g) = (|g|_{a_1^\epsilon}, \ldots, |g|_{a_m^\epsilon})$ and $\mathbf{z}^{\wp(g)} = z_1^{|g|_{a_1^\epsilon}} \ldots z_m^{|g|_{a_m^\epsilon}}$.

For each $i = 1, \dots, m$, $\epsilon = \pm 1$, let $F^{\epsilon}_{m,i} = \{g \in F_m \backslash \{e\} : \text{ the first letter of } g \text{ is } a_i^{\epsilon}\}$. Define

$$\Delta_i^{\epsilon}(\mathbf{z}) = \Delta^{\epsilon}_{F_{m,i}}(\mathbf{z}) = \sum_{e \neq g \in F^{\epsilon}_{m,i}} \mathbf{z}^{\wp(g)},$$

so that we get

$$\Delta_{F_m}(\mathbf{z}) = 1 + \sum_{i,\epsilon} \Delta_i^{\epsilon}(\mathbf{z}).$$

Observe that

$$\Delta_i(\mathbf{z}) = z_i + z_i \Delta_i(\mathbf{z}) + z_i \sum_{j \neq i, \epsilon} \Delta_j^{\epsilon}(\mathbf{z}).$$

This implies $\Delta_i(\mathbf{z}) = z_i\big(\Delta_{F_m}(\mathbf{z}) - \Delta_i^{-1}(\mathbf{z})\big)$. Similarly we can write $\Delta_i^{-1}(\mathbf{z}) = z_i\big(\Delta_{F_m}(\mathbf{z}) - \Delta_i(\mathbf{z})\big)$. Adding $\Delta_i(\mathbf{z})$ and $\Delta_i^{-1}(\mathbf{z})$ we get

$$\Delta_i(\mathbf{z}) + \Delta_i^{-1}(\mathbf{z}) = \left(\frac{2z_i}{1+z_i}\right) \Delta_{F_m}(\mathbf{z}).$$

Hence

$$\Delta_{F_m}(\mathbf{z}) = 1 + \sum_i \left(\Delta_i(\mathbf{z}) + \Delta_i^{-1}(\mathbf{z})\right) = 1 + 2\sum_{i=1}^{m} \left(\frac{z_i}{1+z_i}\right) \Delta_{F_m}(\mathbf{z})$$

and we come to (8). □

4 The multivariate growth exponent and the condition (CG)

Let

$$\Gamma(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = \sum_{i_1, \dots, i_d} f_{i_1, \dots, i_d} z_1^{i_1} \cdots z_d^{i_d} \tag{9}$$

be a multivariate power series. We denote by $\mathscr{D}$ the interior of the domain of absolute convergence of Γ, which we assume to be non empty. Observe that $\mathbf{0}$ belongs to $\mathscr{D}$.

We are going to define for each $\mathbf{r} \in M_d$ (recall that M_d is a simplex of probabilistic vectors (3)) a growth exponent $\varphi(\mathbf{r})$ for coefficients $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^d}$ in the direction of $\mathbf{r}$. When $\mathbf{r}$ is a vector with rational entries, then one can define $\varphi(\mathbf{r})$ as

$$\varphi(\mathbf{r}) = \limsup_{n \to \infty} |f_{n\mathbf{r}}|^{\frac{1}{n}}, \tag{10}$$

where only coefficients $n\mathbf{r} = (nr_1, \dots, nr_d) \in \mathbb{N}^d$ with integer entries are taken into account. This approach is considered in the book [20] and literature quoted there.

To get the analogue of $\varphi(\mathbf{r})$, which will be denoted by $\tilde{\varphi}(\mathbf{r})$ for arbitrary $\mathbf{r} \in M_d$ (in many cases it gives the same function when restricted to rational $\mathbf{r}$), we will define the function $\psi(\mathbf{r})$ and define $\tilde{\varphi}(\mathbf{r})$ as the exponent $e^{\psi(\mathbf{r})}$. For $\tilde{\varphi}(\mathbf{r})$ we will use the name *multivariate growth exponent*.

Definition 2. *Let $C \subset \mathbb{R}^d$ be an open (linear) cone. Define*

$$\tau_C = \limsup_{R \to \infty} \frac{1}{R} \log \left(\sum_{\substack{\mathbf{i} \in C \\ R \leq \|\mathbf{i}\|_1 \leq R+1}} |f_{\mathbf{i}}| \right)$$

and for $\mathbf{r} \in \mathbb{R}^d, \mathbf{r} \neq \mathbf{0}$,

$$\psi(\mathbf{r}) = \|\mathbf{r}\|_1 \inf_{\mathbf{r} \in C} \tau_C,$$

where inf *is taken over open cones containing* $\mathbf{r}$. *Also set* $\psi(\mathbf{0}) = \mathbf{0}$.

Following [22] we call $\psi(\mathbf{r})$ *indicatrice* of growth. The assumption that $\mathbf{0}$ is an interior point of the domain of absolute convergence of Γ ensures that ψ can not take the value $+\infty$. But it takes the value $-\infty$ for vectors $\mathbf{r}$ with at least one negative coordinate and could take the value $-\infty$ even for $\mathbf{r}$ with positive entries (as the example of the series associated with the Fibonacci language shows (see part *(ii)* of Theorem 1)).

Recall that the Relog map is a map Relog : $\mathbb{C}_*^d \to \mathbb{R}^d$, where $\mathbb{C}_* = \mathbb{C}\setminus\{0\}$, given by

$$\mathrm{Relog}(\mathbf{z}) = (\log|z_1|, \ldots, \log|z_d|).$$

In what follows, we will use the notation $\mathcal{E} = \mathbb{R}^d$ and $\mathcal{E}^*$ for its dual space, i.e. the space of continuous linear functionals $\theta : \mathcal{E} \to \mathbb{R}$ with a natural identification of $\mathcal{E}^*$ with $\mathcal{E}$ via the standard inner product

$$\langle \mathbf{x}, \theta \rangle = \theta(\mathbf{x}) = \sum_{i=1}^{d} x_i \theta_i, \qquad \mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{E}, \;\; \theta = (\theta_1, \ldots, \theta_d) \in \mathcal{E}^*.$$

The height function $h_{\mathbf{r}}(\mathbf{z})$ for $\mathbf{z} \in \mathcal{E}$ is the function

$$h_{\mathbf{r}}(\mathbf{z}) = -\sum_{i=1}^{d} r_i \log|z_i| = -\langle \mathbf{r}, \mathrm{Relog}(\mathbf{z}) \rangle.$$

If $\mathbf{r} \in M_d \cap \mathbb{Q}^d$ is a rational vector, then for $\varphi(\mathbf{r})$ given by (10) there is an upper bound

$$\varphi(\mathbf{r}) = \limsup_{n\to\infty} |f_{n\mathbf{r}}|^{\frac{1}{n}} \le |z'_1|^{-r_1} \ldots |z'_d|^{-r_d} = e^{-h_{\mathbf{r}}(\mathbf{z}')}, \tag{11}$$

which holds for every point $\mathbf{z}'$ in the closure $\overline{\mathscr{D}}$ of $\mathscr{D}$, as can be seen, for example, in [20, formula (5.15)]. In the case of $\tilde{\varphi}(\mathbf{r})$, we will provide sufficient conditions for replacement of the inequality (11) by the equality for properly chosen $\mathbf{z}'$, that holds for all directions (not only rational). In many cases where the condition (CG), defined a few lines below, holds, we have the coincidence $\tilde{\varphi}(\mathbf{r}) = \varphi(\mathbf{r})$.

In what follows, it will be convenient to associate with the power series $\Gamma(\mathbf{z})$ a non-negative Radon measure $\nu = \nu_\Gamma$, defined on Borel subsets S of $\mathcal{E}$, by

$$\nu(S) = \sum_{\mathbf{i}\in S} |f_{\mathbf{i}}|. \tag{12}$$

Following [22] we define the condition (CG).

Definition 3 (condition (CG))**.** *We say that the series* $\Gamma(\mathbf{z})$ *has a concave growth of coefficients if there are* $a, b, c > 0$ *such that for all* $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ *and the measure* ν*, associated with this series, satisfies*

$$\nu\left(B_{\mathbf{x}+\mathbf{y}}(a)\right) \ge c\nu\left(B_{\mathbf{x}}(b)\right)\nu\left(B_{\mathbf{y}}(b)\right),$$

where $B_{\mathbf{x}}(a)$ *is the ball of radius* a *with the center at* $\mathbf{x} \in \mathcal{E}$ *for the norm* $\|\cdot\|_1$.

A large class of power series $\Gamma(\mathbf{z})$, satisfying the condition (CG), are multivariate generating series of regular languages satisfying the condition (E), as was mentioned in Section 2.

Given a Radon measure ν satisfying condition (CG) for a convex open cone $C \subset \mathcal{E}$ we define

$$\tau_{\nu,C} = \limsup_{R\to\infty} \frac{1}{R} \log \nu\Big(\big(B_{\mathbf{0}}(R+1) \smallsetminus B_{\mathbf{0}}(R)\big) \cap C\Big)$$

and

$$\psi_\nu(\mathbf{x}) = \|\mathbf{x}\|_1 \inf_{\mathbf{x}\in C} \tau_{\nu,C}, \quad \mathbf{x} \in \mathcal{E},$$

with $\psi_\nu(\mathbf{0}) = \mathbf{0}$. The above infimum is taken over open convex cones containing $\mathbf{x}$. Then assuming that $\tau_\nu < \infty$, where

$$\tau_\nu = \sup_{\mathbf{x}\in\mathcal{E}, \|\mathbf{x}\|_1=1} \psi_\nu(\mathbf{x}), \tag{13}$$

we know from [22, Lemma 3.1.7] that the function $\psi_\nu : \mathcal{E} \to \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous. Obviously it is positively homogeneous, i.e. for any $t > 0$ we have $\psi_\nu(t\mathbf{x}) = t\psi_\nu(\mathbf{x})$. Finally, if the condition (CG) holds, then $\psi_\nu(\mathbf{x})$ is concave, i.e. $\psi_\nu(\mathbf{x}+\mathbf{y}) \geq \psi_\nu(\mathbf{x}) + \psi_\nu(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{E}$. As was already mentioned, because the measure $\nu = \nu_\Gamma$ is supported only on the subset $\mathbb{N}^d \subset \mathbb{R}^d_{\geq 0}$, we have $\psi_\nu(\mathbf{x}) = -\infty$ for $\mathbf{x} \notin \mathbb{R}^d_{\geq 0}$.

Let $\Omega = \operatorname{Relog}(\mathscr{D} \cap \mathbb{C}^d_*) \subset \mathcal{E}$, where $\mathscr{D} \subset \mathbb{C}^d$ is the interior of the domain of absolute convergence of series $\Gamma(\mathbf{z})$ given by (9). It is well known (see [20, Proposition 3.4]) that Ω is convex. Recall that the support function $h_E(x)$ of a closed convex subset $E \subset \mathcal{E}$ is the supremum of the values of linear form $\langle \mathbf{x}, \theta\rangle$ when $\theta \in E$. Let $\psi = \psi_\Gamma$ be the function given by Definition 2.

Theorem 2. *Let $\Gamma(\mathbf{z})$ be a power series given by* (9) *with non-negative coefficients $f_\mathbf{i}$, $\mathbf{i} \in \mathbb{N}^d$, such that $f_\mathbf{i}$ satisfy the concavity condition (CG) and $\tau_\nu < \infty$, where $\nu = \nu_\Gamma$ is given by* (12) *and τ_ν is given by* (13). *Let $\mathscr{D} \subset \mathbb{C}^d$ be the interior of the set of points of absolute convergence of $\Gamma(\mathbf{z})$, $\Omega = \operatorname{Relog}(\mathscr{D} \cap \mathbb{C}^d_*)$ and $\overline{\Omega}$ be the closure of Ω. Then $-\psi_\Gamma(\mathbf{x})$ is the support function of the closure $\overline{\Omega}$ and for $\mathbf{x} \in \mathbb{R}^d_{\geq 0}$ we have*

$$\psi_\Gamma(\mathbf{x}) = \inf_{\theta\in-\overline{\Omega}} \langle \mathbf{x}, \theta\rangle. \tag{14}$$

Proof. Because of the condition (CG), the function $-\psi_\Gamma(\mathbf{x})$ is a lower semi-continuous, convex and positively homogeneous. Hence, it is a support function of a one and only one closed convex subset $S \subset \mathcal{E}^*$ given by

$$S = \big\{\theta \in \mathcal{E}^* : -\psi_\Gamma(\mathbf{x}) \geq \langle \mathbf{x}, \theta\rangle \ \ \forall \mathbf{x} \in \mathcal{E}\big\} = \big\{\theta \in \mathcal{E}^* : \psi_\Gamma(\mathbf{x}) \leq -\langle \mathbf{x}, \theta\rangle \ \ \forall \mathbf{x} \in \mathcal{E}\big\}. \tag{15}$$

The domain of absolute convergence of $\Gamma(\mathbf{z})$ is determined by the condition that the integral given in the following equation

$$\Gamma(|z_1|, \ldots, |z_d|) = \sum_{\mathbf{i}} f_\mathbf{i} e^{\sum_{k=1}^d i_k \log|z_k|} = \int_{\mathcal{E}} e^{\langle \mathbf{x}, \theta\rangle} d\nu_\Gamma(\mathbf{x}) \tag{16}$$

is convergent, where $\theta = \operatorname{Relog}(\mathbf{z})$, $\mathbf{z} = (z_1, \ldots, z_d)$.

Consider the set

$$\Delta^\circ_\Gamma = \big\{\theta \in \mathcal{E}^* : -\langle \mathbf{x}, \theta\rangle > \psi_\Gamma(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathcal{E}\backslash\{\mathbf{0}\}\big\}.$$

Using [22, Lemma 3.1.3], we conclude that if $\theta \in \Delta_\Gamma^\circ$, then the integral (16) converges, while if there is $\mathbf{x} \in \mathcal{E}\backslash\{\mathbf{0}\}$ with $-\langle \mathbf{x}, \theta\rangle < \psi_\Gamma(\mathbf{x})$, then it diverges. The closure $\Delta_\Gamma = \overline{\Delta_\Gamma^\circ}$ is the set

$$\Delta_\Gamma = \big\{\theta \in \mathcal{E}^* : -\langle \mathbf{x}, \theta\rangle \geq \psi_\Gamma(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathcal{E}\big\},$$

which is the set S given by (15), for whom $-\psi_\Gamma(\mathbf{x})$ is the support function. Thus

$$-\psi_\Gamma(\mathbf{x}) = \sup_{\theta\in S}\langle \mathbf{x}, \theta\rangle, \qquad \psi_\Gamma(\mathbf{x}) = -\sup_{\theta\in S}\langle \mathbf{x}, \theta\rangle = \inf_{\theta\in -S}\langle \mathbf{x}, \theta\rangle.$$

Also, we conclude that $S = \mathrm{Relog}(\overline{\mathscr{D}}) = \overline{\Omega}$. This leads us to (14). □

As a consequence we have the following assertion.

Corollary 1.

$$\psi(\mathbf{x}) = \inf_{\mathbf{z}\in\mathscr{D}\backslash\{0\}} \left(-\sum x_i \log|z_i|\right).$$

Example. *Let $d = 2$, $\Sigma = \{a_1, a_2\}$ and $L = \Sigma^*$ be the language of a free monoid. Then*

$$\Gamma_L(\mathbf{z}) = \frac{1}{1 - z_1 - z_2} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i} z_1^i z_2^{n-i}.$$

It is obvious that the condition (CG) holds and the direct computations based on the use of Stirling's formula or Theorem 2 show that for $\mathbf{r} \in M_2$ the function $\psi_L(\mathbf{r})$ is Shannon's entropy $\mathbf{H}(\mathbf{r})$, i.e.

$$\psi_L(\mathbf{r}) = \mathbf{H}(\mathbf{r}) = -r_1 \log r_1 - r_2 \log r_2.$$

5 Multivariate growth exponent in the case of a free group

In the Section 3, we got the expression for the multivariate growth series $\Delta_{F_m}(\mathbf{z})$ associated with the language L_{F_m} of freely reduced words in the alphabet $\{a_1, \ldots, a_m, a_1^{-1} \ldots, a_m^{-1}\}$.

Consider the case $m = 2$. Then

$$\Delta_{F_2}(\mathbf{z}) = \frac{(1+z_1)(1+z_2)}{1 - z_1 - z_2 - 3z_1z_2}. \tag{17}$$

We use the notations $\mathscr{D}, \Omega$, Relog from Section 4. To visualize the real slice of the domain of the absolute convergence of the power series (17), we observe that the domain $\mathscr{D}$ is described as

$$\mathscr{D} = \big\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| + 3|z_1||z_2| < 1\big\}.$$

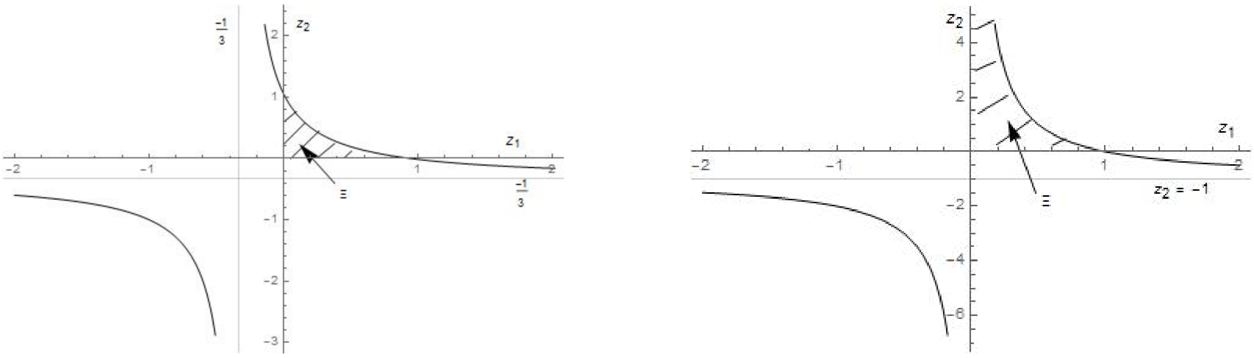
The real slice of the curve $H(z_1, z_2) = f_1(\mathbf{z}) = 1 - z_1 - z_2 - 3z_1z_2 = 0$ is presented by the hyperbola from Figure 5 (a).

And the part of the real slice of $\mathscr{D}$ from the positive quarter of the plane is presented by the tented area Ξ. The set $-\Omega$ is obtained from Ξ by application of the map $z_1 = e^{-s}, z_2 = e^{-t}$. To get the clear picture of the shape of the set Ω (and hence $-\Omega$) we use the notion from algebraic geometry called *amoeba*.

Recall that given a Laurent polynomial $f(\mathbf{z}), \mathbf{z} \in \mathbb{C}^d$, the amoeba of f is the set

$$amoeba(f) = \big\{\,\mathrm{Relog}(\mathbf{z}) : \mathbf{z} \in \mathbb{C}_*^d, f(z) = 0\big\} \subset \mathbb{R}^d,$$

where $\mathbb{C}_* = \mathbb{C}\backslash\{0\}$. The amoeba's complement is $amoeba(f)^c = \mathbb{R}^d \backslash amoeba(f)$.



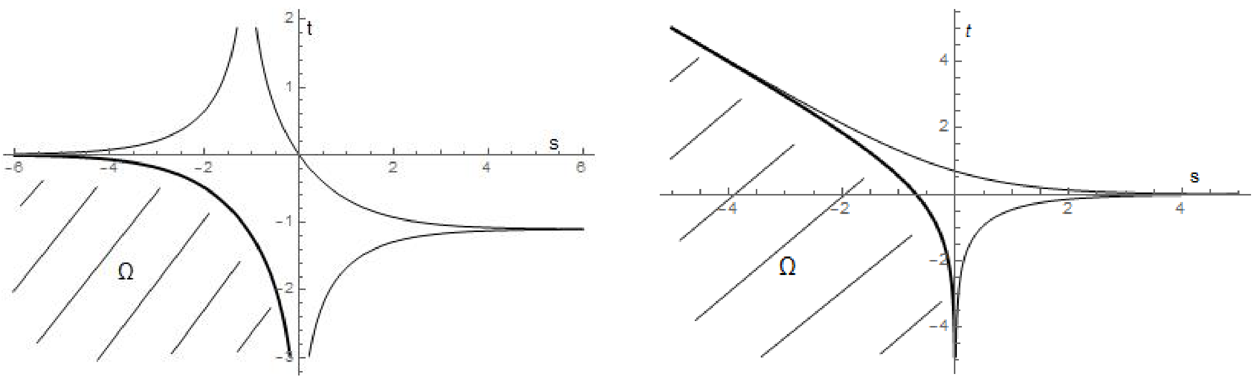
(a) (b)

Figure 5. The real part of the Domain $\mathscr{D}$ for the free group F_2 and Fibonacci cases respectively

The following result follows from [8, Chapter 6, Proposition 1.5 and Corollary 1.6].

Proposition 3. *If $f(\mathbf{z})$ is a Laurent polynomial, then all connected components of the complement $amoeba(f)^c$ are convex subsets of $\mathbb{R}^d$. These real convex subsets are in bijection with the Laurent series expansions of the rational function $\frac{1}{f(\mathbf{z})}$. When $\frac{1}{f(\mathbf{z})}$ has a power series expansion, then it corresponds to the component of $amoeba(f)^c$ containing all points $(-N, \ldots, -N)$ for N positive and sufficiently large.*

The techniques of drawing amoeba are well developed. The amoebas of polynomials $f_1(\mathbf{z}) = 1 - z_1 - z_2 - 3z_1z_2$ and $f_2(\mathbf{z}) = 1 - z_1 - z_1z_2$ that correspond to the cases of language L_{F_2} and Fibonacci language, respectively, are presented in the Figure 6.



(a) (b)

Figure 6. The $amoeba(f_1)$, $amoeba(f_2)$ along with sets Ω associated to the language L_{F_2} and the Fibonacci language, where $f_1(\mathbf{z}) = 1 - z_1 - z_2 - 3z_1z_2$ and $f_2(\mathbf{z}) = 1 - z_1 - z_1z_2$

To compute $\psi(\mathbf{r})$ we apply a standard method of Lagrange multipliers.

In coordinates $(s, t) \in \mathbb{R}^2$, $z_1 = e^{-s}, z_2 = e^{-t}$, the boundary of $-\Omega$ is a curve l given by the equation

$$e^{s+t} - e^s - e^t - 3 = 0. \tag{18}$$

Let $\mathbf{r} = (p, q) \in M_2$. We have to minimize $ps + qt$, when $(s, t) \in l$. The associated Lagrange function is

$$\Phi(s, t, \lambda) = ps + qt - \lambda(e^{s+t} - e^s - e^t - 3).$$

Equating partial derivatives to zero we obtain

$$\begin{aligned}
\frac{\partial \Phi}{\partial s} &= p - \lambda\left(e^{s+t} - e^{s}\right) = 0 \implies p = \lambda\left(e^{s+t} - e^{s}\right), \\
\frac{\partial \Phi}{\partial t} &= q - \lambda\left(e^{s+t} - e^{t}\right) = 0 \implies q = \lambda\left(e^{s+t} - e^{t}\right), \\
\frac{\partial \Phi}{\partial \lambda} &= e^{s+t} - e^{s} - e^{t} - 3 = 0.
\end{aligned}$$

This gives

$$\rho = \frac{p}{q} = \frac{1 - e^{-t}}{1 - e^{-s}} \implies e^{-t} = 1 - (1 - e^{-s})\rho.$$

Substituting the value of e^{-t} in (18) after standard transformations we get

$$-3\rho e^{-2s} + (2\rho - 4)\, e^{-s} + \rho = 0,$$

that is a quadratic equation with repect to unknown $x = e^{-s}$. Solving it gives

$$x_{1,2} = \frac{2\rho \pm \sqrt{1 - \rho + \rho^2}}{6\rho}$$

(the positive sign is chosen because we know from [16] that the function is real analytic). Resubstituting $x = e^{-s}$ and $\rho = \dfrac{p}{q}$, we get

$$e^{-s} = \frac{p - 2q + 2\sqrt{p^2 - pq + q^2}}{3p}, \qquad e^{s} = \frac{2q - p + 2\sqrt{p^2 - pq + q^2}}{p},$$

and by symmetry,

$$e^{t} = \frac{2p - q + 2\sqrt{p^2 - pq + q^2}}{q}.$$

Hence,

$$s = \log\left(\frac{2q - p + 2\sqrt{p^2 - pq + q^2}}{p}\right), \qquad t = \log\left(\frac{2p - q + 2\sqrt{p^2 - pq + q^2}}{q}\right),$$

and

$$\psi_{F_2}(\mathbf{r}) = p\log\left(\frac{2q - p + 2\sqrt{p^2 - pq + q^2}}{p}\right) + q\log\left(\frac{2p - q + 2\sqrt{p^2 - pq + q^2}}{q}\right)$$

or

$$\psi_{F_2}(\mathbf{r}) = \mathbf{H}(\mathbf{r}) + p\log\left(2q - p + 2\sqrt{p^2 - pq + q^2}\right) + q\log\left(2p - q + 2\sqrt{p^2 - pq + q^2}\right),$$

where $\mathbf{H}(\mathbf{r}) = -p\log p - q\log q$. See Figure 2 (a) for the graph of $\psi_{F_2}(\mathbf{r}) = \psi_{F_2}(p, 1-p)$.

6 Multivariate growth in the case of Fibonacci language

Fibonacci language L_{Fib} is a language over a binary alphabet $\{0,1\}$ consisting of words that do not contain 11 as a subword, i.e. the word 11 is forbidden. It is a language associated with Fibonacci subshift, one of the simplest but important examples of a subshift of finite type [18]. The reason why Fibonacci name is associated to it is coming from the fact that the number of words of length n in L_{Fib} is equal to the $(n+2)th$ Fibonacci number

$$\frac{1}{\sqrt{5}}\left(\lambda_1^{n+2}-\lambda_2^{n+2}\right),$$

where $\lambda_1=\frac{1+\sqrt{5}}{2}$ and $\lambda_2=\frac{1-\sqrt{5}}{2}$ are eigenvalues of matrix $A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, which is the matrix of transitions of the Fibonacci subshift. At the same time A is the adjacency matrix of the automaton $\mathscr{A}_{Fib}$, given by the Figure 7, which accepts the language L_{Fib}, if instead of $\{0,1\}$ one

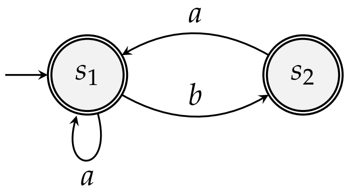


Figure 7. Moore diagram of Fibonacci automaton $\mathscr{A}_{Fib}$

uses the alphabet $\{a,b\}$. The growth represents the function

$$\Gamma_{Fib}(\mathbf{z})=\frac{1+z_2}{1-z_1-z_1z_2}$$

showed before in the Definition 1.

From Figure 7 it is clear that the condition (CG) holds as the automaton $\mathscr{A}_{Fib}$ is ergodic (conencted).

Proposition 4. *The indicatrice ψ_{Fib} viewed as a function of p with $0<p<1$, where $\mathbf{r}=(p,1-p)\in M_2$ is the direction vector, is given by*

$$\psi_{Fib}(\mathbf{r})=\begin{cases} p\log\left(\frac{p}{2p-1}\right)+(1-p)\log\left(\frac{2p-1}{1-p}\right), & \text{if } p\geq\frac{1}{2},\\ -\infty, & \text{if } p<\frac{1}{2}.\end{cases}$$

The graph of $\psi_{Fib}(\mathbf{r})$ on $[\frac{1}{2},\infty)$ is shown in the Figure 2 (b).

Proof. As before, we switch to the variables x,y instead of z_1,z_2. The amoeba of the function $f_2(x,y)=1-x-xy$ and the set $-\Omega$ are shown in the Figures 6 (b) and 8 (b), respectively.

First we will show that $\psi_{Fib}(p)=-\infty$, when $0<p<\frac{1}{2}$. The power series expansion of $(1-x-xy)^{-1}$ is

$$\frac{1}{1-x-xy}=\sum_{n=1}^{\infty}\sum_{i=0}^{n}\binom{n}{i}x^{n-i}(xy)^i=\sum_{n=1}^{\infty}x^n\sum_{i=0}^{n}\binom{n}{i}y^i.$$

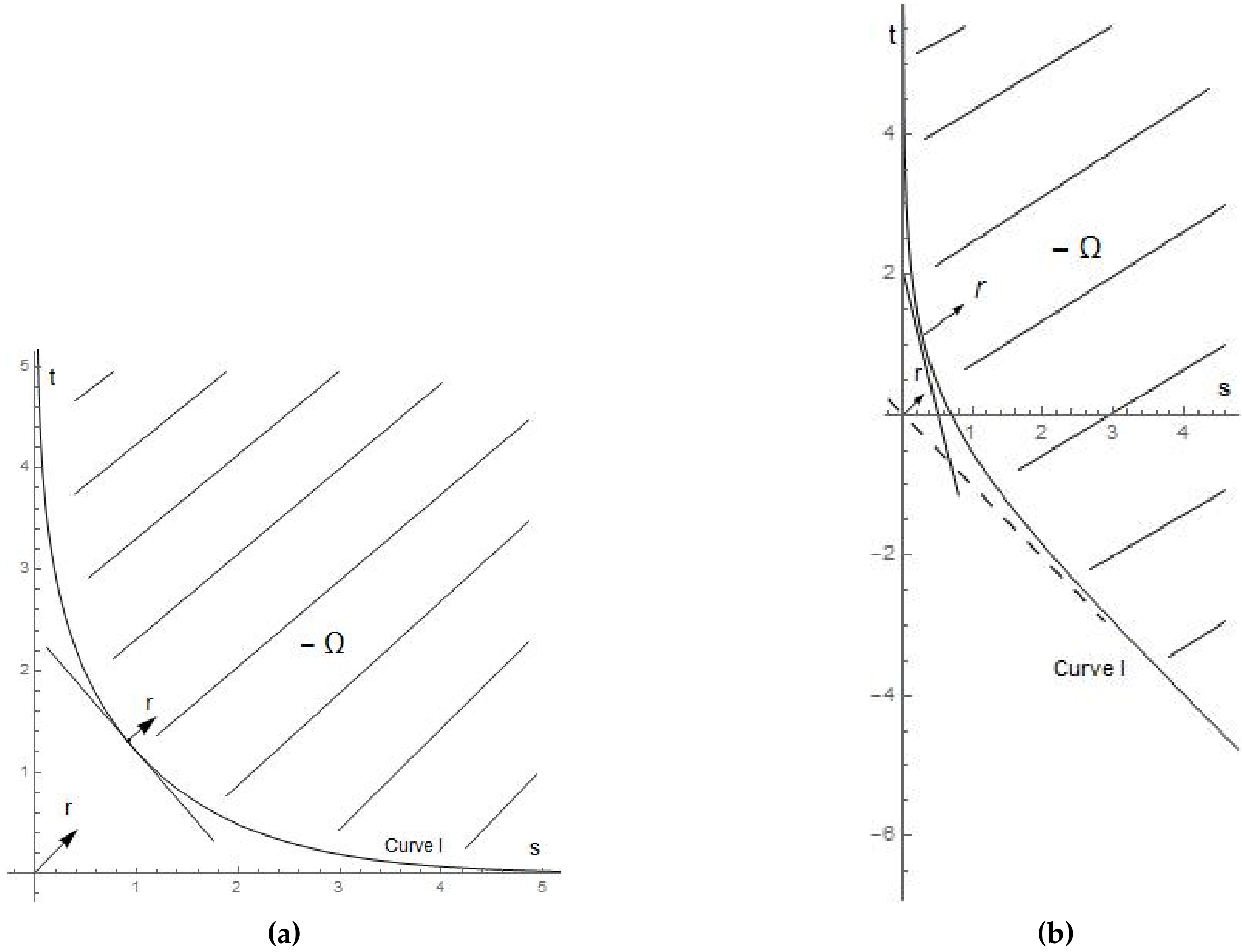


Figure 8. The sets $-\Omega$, associated with languages L_{F_2} and L_{Fib}, respectively

Hence,

$$\begin{aligned}\frac{1+y}{1-x-xy} &= \sum_{n=1}^{\infty} x^n \sum_{i=0}^{n} \binom{n}{i} y^i + \sum_{n=1}^{\infty} x^n \sum_{i=0}^{n} \binom{n}{i} y^{i+1} \\ &= \sum_{n=1}^{\infty} x^n \left\{ 1 + y + \sum_{i=0}^{n} \left[\binom{n}{i} + \binom{n}{i-1} \right] y^i \right\}\end{aligned}$$

and we see that coefficients corresponding to the indices (n,i) with $i > n+1$ are zero. Hence, any direction $\mathbf{r} = (p,q)$ with $\frac{p}{q} < 1$ gives value $-\infty$ to ψ_{Fib}. We also can appeal to the geometric explanation of the latter fact using the Figures 6 (b) and 8 (b). The domain $-\Omega$ is shown in Figure 8 (b) and it is bounded by the curve l given by the equation

$$1 - e^{-s} - e^{-s-t} = 0. \tag{19}$$

If $p < \frac{1}{2}$, then $\frac{1-p}{p} > \frac{1}{2}$ and the line f_c given by $ps + (1-p)t = c$ intersect $-\Omega$ for an arbitrary $c \in \mathbb{R}$. Hence,

$$\psi_{Fib}(p) = \inf_{(s,t)\in-\Omega} (ps + (1-p)t) = -\infty.$$

If $p > \frac{1}{2}$, then $\frac{1-p}{p} < \frac{1}{2}$ and there is a unique value of c such that the line f_c is tangent to the curve l. To find coordinates (s_0, t_0) of the tangency point that gives minimum value of the linear form $ps + (1-p)t$ when $(s,t) \in -\Omega$ we again apply Lagrange method. Denote $q = 1-p$ and rewrite equation (19) as $e^{s+t} - e^t - 1 = 0$.

The Lagrange function is $\Phi(s,t,\lambda) = ps + qt - \lambda(e^{s+t} - e^t - 1)$ and the corresponding system of equations has the form

$$\frac{\partial \Phi}{\partial s} = p - \lambda e^{s+t} = 0,$$
$$\frac{\partial \Phi}{\partial t} = q - \lambda e^{s+t} + \lambda e^t = 0,$$
$$\frac{\partial \Phi}{\partial \lambda} = e^{s+t} - e^t - 1 = 0.$$

Resolving it, we find

$$e^t = \frac{2p-1}{1-p}, \qquad e^s = \frac{p}{2p-1},$$

which determines the point (s_0, t_0). Making substitution in the linear form we get needed expression. □

7 Multivariate growth and Large Deviation Theory

An alternative approach to computing $\psi(\mathbf{r})$ relies on the application of results and methods from Large Deviation Theory (LDT). Here we discuss a special case related to languages associated with subshifts of finite type. Let us first recall the basic facts about subshifts of finite type (SFT). For further details, we refer the interested reader to [18].

Let $\Sigma = \{a_1, \ldots, a_d\}$ be a finite alphabet and $\Sigma^{\mathbb{Z}}$ be a space of bilateral infinite sequences over Σ indexed by integers. $\Sigma^{\mathbb{Z}}$ is supplied by a product topology and is homeomorphic to a Cantor set when $d \geq 2$. A shift map $U : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is a homeomorphism given by

$$(Uw)_n = w_{n+1}, \quad w = (w)_{n=-\infty}^{\infty} \in \Sigma^{\mathbb{Z}}.$$

A closed U-invariant subset $X \subset \Sigma^{\mathbb{Z}}$ is a subshift. Let $L_X \subset \Sigma^*$ be a language of subshift consisting of (finite) words that appear as a subwords of $w \in X$. A subshift X is said to be a SFT if there is a finite subset $\mathbf{F} \subset \Sigma^*$ (set of forbidden words) such that $X = X_{\mathbf{F}}$ consist of sequences $w \in \Sigma^{\mathbb{Z}}$ that do not contain forbidden subwords. It is obvious that $X_{\mathbf{F}}$ is closed and U-invariant. For instance, in the case $\Sigma = \{0,1\}$ and $\mathbf{F} = \{11\}$ we get a Fibonacci subshift. Alternative way to define SFT is based on the use of a finite directed graph $\Xi = (V, E)$ or equivalently, via its adjacency matrix $A = (a_{ij})$ of size $|V| \times |V|$, whose rows and columns correspond to the vertices and a_{ij} is a number of edges between vertices a_i and a_j. For instance, for the Fibonacci subshift the matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the graph is given by the Figure 9 and is similar to the diagram of the automaton from Figure 7.

It is well known that the language L_X associated with a subshift of finite type is regular. Hence, its multivariate growth series represents a rational function and the technique of computation of multivariate growth rate described in the previous sections is applicable. Now, we will see how the results of LDT can be used for the same goal.

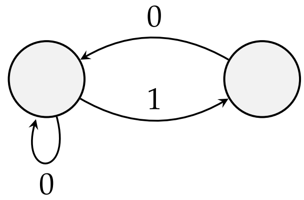


Figure 9. The graph Ξ of the Fibonacci subshift

Let us recall some other notions related to SFT. The SFT (U, X) is *irreducible* if the graph Ξ_X is irreducible (i.e. strongly connected, this means that for any vertices in graph there is a path in the graph connecting them). In this case, the associated matrix A also is called irreducible. The Perron-Frobenius theorem (see, for instance, [18, Theorem 4.5.11]) states that an irreducible aperiodic matrix A with non-negative entries has a simple real eigenvalue $\rho = \rho(A)$ (called the Perron-Frobenius eigenvalue) such that any other eigenvalue λ satisfies $|\lambda| < \rho$. Also there are two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ satisfying

$$A\mathbf{v} = \rho\mathbf{v}, \mathbf{u}^t A = \rho \mathbf{u}^t,$$

where $\mathbf{u}^t$ is a transpose of the column vector $\mathbf{u}$. These vectors are unique up to scalar factor. In what follows, a given matrix we will fix a pair of such u and v with the condition $u\dot{v} = 1$, where $u\dot{v}$ denotes the inner product. Subshifts usually are studied up to conjugacy and it is well known (see [18]) that each SFT is conjugate to the one with adjacency matrix A consisting only of 0 and 1. From now we will assume that this is the case and call it 0-1-condition.

Next, we recall the notion of a Markov measure on $\Sigma^{\mathbb{Z}}$. Given a stochastic $d \times d$ matrix $P = (p_{ij})$, where $p_{ij} \geq 0$ and $\sum_{j=1}^{d} p_{ij} = 1$, $i = 1, \ldots, d$, and a stationary probability vector $p = (p_1, \ldots, p_d)$, $pP = p$, one can define a Borel probability measure $\mu = \mu_P$ on $\Sigma^{\mathbb{Z}}$ by

$$\mu\left([\omega_0, \ldots, \omega_n]\right) = p_{\omega_0} \prod_{i=0}^{n-1} p_{\omega_i, \omega_{i+1}},$$

where $[\omega_0, \ldots, \omega_n]$ is a cylinder subset of $\Sigma^{\mathbb{Z}}$ consisting of all $\omega \in \Sigma^{\mathbb{Z}}$ with the prescribed entries $\omega_0, \ldots, \omega_n$ at coordinates $0, 1, \ldots, n$ (such sets generate the sigma-algebra of Borel subsets and hence, values of μ on them determine μ completely). The Perron-Frobenius eigenvalue of P is 1, vector p exists and is unique if P is irreducible. The measure μ_P is shift-invariant and the system $\left(\Sigma^{\mathbb{Z}}, U, \mu\right)$ (called Markov shift) is ergodic, i.e. every U-invariant subset of $\Sigma^{\mathbb{Z}}$ has a measure 0 or 1.

One of the main results of the theory of SFT is the statement (assuming irreducibility of subshift X) about the existence and uniqueness of the probability measure $\eta = \eta_X$ of maximal entropy. Not getting into details we just mention that it is a U-invariant probability measure that has a maximal Kolmogorov-Sinai entropy among all shift invariant probability measures supported on the subshift. The measure η is called *Parry* measure. It is a Markov type measure determined by the stochastic matrix $P = (p_{ij})$, with

$$p_{ij} = \frac{1}{\rho} a_{ij} \frac{v_j}{v_i},$$

where $A = (a_{ij})$ is the $\{0, 1\}$-matrix determining subshift, $\rho = \rho(A)$ and $A\mathbf{v} = \rho\mathbf{v}$. For such measure we have

$$\eta\left([i, x_1, \ldots, x_{n-1}, j]\right) = \frac{u_i v_j}{\rho^n}.$$

See [21] for further details. Starting from this moment we assume that the measure of maximal entropy is associated with SFT and keep the notation η for it.

Now, we recall some basic notions of LDT, namely, the notion of the rate function I and the Large Deviation Principal (LDP). A rate function is a lower semi-continuous function $I : W \to [0, +\infty]$ defined on a topological space W (for us $W = \mathbb{R}^d$) such that for each $a \in [0, \infty)$ the level set

$$Y_I(a) = \{\mathbf{x} \in W : I(\mathbf{x}) \leq a\}$$

is a closed subset of W. A *good rate function* is a rate function for which all level sets $Y_I(a)$ are compact. A sequence $\{\mu_n\}_{n=1}^{\infty}$ of Borel measures on W satisfies LDP if for every Borel subset $B \subset W$ we have

$$-\inf_{\mathbf{x} \in B^\circ} I(\mathbf{x}) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mu_n(B) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mu_n(B) \leq -\inf_{\mathbf{x} \in \overline{B}} I(\mathbf{x}),$$

where B°, $\overline{B}$ are the interior and the closure of B, respectively.

Given a finite Markov chain on a set $\Sigma = \{a_1, \ldots, a_d\}$ determined by a stochastic matrix $P = (p_{ij})_{i,j=1}^d$ and a function $f : \Sigma \to \mathbb{R}^d$, $d \geq 1$, one can consider for each $x = (x_i)_{i=1}^{\infty} \in \Sigma^{\mathbb{N}}$ the empirical means

$$Z_n(x) = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

and the corresponding distributions μ_n (discrete measures in $\mathbb{R}^d$) determined by the Markov shift given by the matrix P and stationary vector p, $pP = p$.

Associate with every $\mathbf{y} \in \mathbb{R}^d$ matrix $\Pi(\mathbf{y})$ whose elements are

$$\pi_{ij}(\mathbf{y}) = p_{ij} e^{\langle \mathbf{y}, f(j) \rangle}, \quad i, j \in \Sigma.$$

$\Pi(\mathbf{y})$ is a matrix with non-negative entries and it is irreducible if and only if P is irreducible. Let $\rho(\Pi(\mathbf{y}))$ be the Perron-Frobenious eigenvalue of $\Pi(\mathbf{y})$. In [7, Theorem 3.1.2] it is stated that the means Z_n (or corresponding distributions μ_n) satisfy the LDP with the convex and good rate function

$$I(\mathbf{z}) = \sup_{\mathbf{y} \in \mathbb{R}^d} \left\{ \langle \mathbf{y}, \mathbf{z} \rangle - \log \rho\left(\Pi(\mathbf{y})\right) \right\}, \quad \mathbf{z} \in \mathbb{R}^d.$$

There is a version of this result due to Sanov which is more suitable for our goals. Let $f : \Sigma \to \mathbb{R}^d$, $|\Sigma| = d$, be a function such that

$$f(a_i) = \left(0, \ldots, 1, \ldots, 0\right)^T$$

where 1 is at ith position, $i = 1, \ldots, d$.

In this case, the means $Z_n(x)$ give the distribution of the vector of frequencies describing how many times the Markov chain given by the corresponding matrix P visits different states. In [7, Theorem 3.1.6] it is stated that

$$I(\mathbf{r}) = \begin{cases} \sup\limits_{\mathbf{u} \gg \mathbf{0}} \sum\limits_{j=1}^{d} r_j \log\left(\dfrac{u_j}{(\mathbf{u}P)_j}\right), & \mathbf{r} \in M_d, \\ \infty, & \mathbf{r} \notin M_d, \end{cases} \tag{20}$$

where the supremum is taken over the strictly positive vectors $\mathbf{u}$, i.e. $u_i > 0$ for all i.

Now we have all needed to describe the connection between ψ and I. We assume that the language $L \subset \Sigma^*$ is a language determined (accepted) by the automaton $\mathscr{A}$ with the property that for each state $q \in Q$ all incoming edges are labeled by the same symbol (like in the examples given by the Figures 3 or 7. The sets of initial and final states could be any nonempty sets. Also we assume the 0-1-condition and that $\mathscr{A}$ is strictly connected (and hence, the condition (CG) is satisfied). Let A be the adjacency matrix of $\mathscr{A}$ and X_A be the corresponding subshift. Let L_1 be a language associated with X_A and $L_1(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^d} l_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$ be its multivariate growth series.

In described situation we identify alphabet Σ with Q attaching to each state $q \in Q$ symbol a_q which is a label of the entering edge. After such identification, it become obvious that $L \subset L_1$. Because of ergodicity (i.e. irreducibility of A) as easily can be shown the indicatrice of growth $\psi(\mathbf{r})$ is same for languages L and L_1.

Associate with $A = (a_{ij})$ the stochastic matrix $P = (p_{ij})$, where $p_{ij} = \frac{a_{ij} v_j}{\rho v_i}$, $\rho = \rho(A)$ and $A\mathbf{v} = \rho \mathbf{v}$, $\mathbf{v} \gg 0$. Let p be a stationary probability vector, i.e. $pP = p$, and $\nu = \nu_p$ be a corresponding Markov measure, i.e. Perry measure. From

$$\mu\left([i, w_1, \ldots, w_{n-1}, j]\right) = \frac{p_i v_j}{\rho^n}, \quad i, j, w_1, \ldots, w_{n-1} \in \Sigma, \tag{21}$$

(assuming normalization $\langle p, \mathbf{v} \rangle = 1$) we know that the measure μ is almost equidistributed on the cylinder sets C_w determined by the words $w \in L_1$ of the fixed length. Let $\mathbf{r} \in M_d$ be a rational vector with positive entries and $\mathcal{C} \subset M_d$ a small neighborhood of $\mathbf{r}$. Let

$$B_n = \left\{ w \in \Delta_{\mathscr{A}} : Z_n(w) \in \mathcal{C} \right\}.$$

From LDP, we know that $\frac{1}{n} \log \mu(B_n)$ is close to $-I(\mathbf{r})$ when n is large. On the other hand, from (21) we get that $\frac{1}{n} \log \mu(B_n)$ is close to $\frac{1}{n} \log \left(\rho^{-n} \cdot l_{n\mathbf{r}}\right)$, where

$$L_1(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^d} l_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

is a multivariate growth series of L_1. Hence, in the limit when $n \to \infty$ and the neiboorhood $\mathcal{C}$ shrinks to the vector $\mathbf{r}$ we get the equality

$$I(\mathbf{r}) = \log \rho - \limsup_{n \to \infty} \frac{1}{n} \log l_{n\mathbf{r}} = \log \rho - \psi(\mathbf{r}). \tag{22}$$

It is easy to see that in considered situation the functions $\varphi(\mathbf{r})$ and $\tilde{\varphi}(\mathbf{r})$ coincide.

Using the result on analyticity of $\psi(\mathbf{r})$ from [16] we conclude that (22) holds for all $\mathbf{r} \in M_d$, not just rationals. We summarize this as the following assertion.

Proposition 5. *The indicatrice of growth $\psi(\mathbf{r})$ of a language determined by ergodic automaton $\mathscr{A}$ of type described above satisfies*

$$I(\mathbf{r}) = \log \rho - \psi(\mathbf{r}),$$

where $I(\mathbf{r})$ is the rate function associated with the Markov chain determined by the stochastic matrix P corresponding to A and

$$I(\mathbf{r}) = \sup_{\mathbf{u} \gg \mathbf{0}} \sum_{j=1}^{d} r_j \log \left[\frac{u_j}{(\mathbf{u}P)_j} \right], \quad \mathbf{r} \in M_d. \tag{23}$$

Finally, we make one more observation. Theorem 2 represents $\psi(\mathbf{r})$ as a supremum of the inner product take over the set $-\bar{\Omega}$. In non-degenerate cases the supremum is enough to be taken only over the boundary of $-\bar{\Omega}$. We are going to show that in our situation only a part of the boundary should be taken into account.

Recall that entries of A belong to the set $\{0,1\}$. Let as before $\mathbf{v} = (v_1, \cdots, v_d)$ be a right eigenvector of A corresponding to $\rho = \rho(A)$, $A\mathbf{v} = \rho\mathbf{v}$. Assume that $\mathbf{v}$ is a probability vector. Let

$$M_d^* = \{\mathbf{r} = (r_1, \ldots, r_d) \in M_d : r_i > 0 \ \ \forall\, i\}.$$

Define the map $T : M_d^* \to \mathbb{R}^d$ given by $T(\mathbf{q}) = \mathbf{s} = (s_1, \ldots, s_d)$, where

$$s_j = \frac{q_j}{v_j \sum_{i=1}^d \frac{q_i a_{ij}}{v_i}}, \quad j = 1, \ldots, d. \tag{24}$$

Let $A(\mathbf{z})$ be the matrix from Proposition 1, i.e. matrix obtained from A by replacing each 1 in the jth column of A by z_j and let $\mathbf{t}$ be a vector obtained from $\mathbf{q} \in M_d^*$ by

$$\mathbf{t} = \left(\frac{q_1}{v_1}, \ldots, \frac{q_d}{v_d}\right).$$

Lemma 1. *For each $\mathbf{q} \in M_d^*$ vector $\mathbf{t}$ satisfies $\mathbf{t}A(\mathbf{s}) = \mathbf{t}$, where $\mathbf{s} = (s_1, \ldots, s_d)$ is given by* (24).

Proof. For $1 \le j \le d$, we have

$$(\mathbf{t}A(\mathbf{s}))_j = s_j \sum_{i=1}^d \frac{q_i a_{ij}}{v_i} = \frac{q_j}{v_j \sum_{i=1}^d \frac{q_i a_{ij}}{v_i}} \cdot \sum_{i=1}^d \frac{q_i a_{ij}}{v_i} = \frac{q_j}{v_j} = t_j.$$

In the above relations we used the fact that $a_{ij} \in \{0,1\}$. □

Recall that the vector $\mathbf{s}$ depends on the vector $\mathbf{q}$, so we can write $\mathbf{s} = \mathbf{s}(\mathbf{q})$.

Corollary 2. *The following condition holds*

$$\det(I - A(\mathbf{s}(\mathbf{q}))) = 0 \qquad \forall\, \mathbf{q} \in M_d^*.$$

Recall that the multivariate growth series of a language L satisfies Proposition 1, i.e.

$$\Gamma_L(\mathbf{z}) = \frac{G(\mathbf{z})}{\det(I - A(\mathbf{z}))},$$

where $G(\mathbf{z})$ is some polynomial and singularities of $\Gamma_L(\mathbf{z})$ are determined by the roots of denominator. Hence, $T(M_d^*)$ is a part of the set of the real singularities of $\Gamma_L(\mathbf{z})$ and hence is a part of the boundary $\partial\mathscr{D}$.

We already know from Proposition 5 that

$$I(\mathbf{r}) = \log\rho - \psi(\mathbf{r}).$$

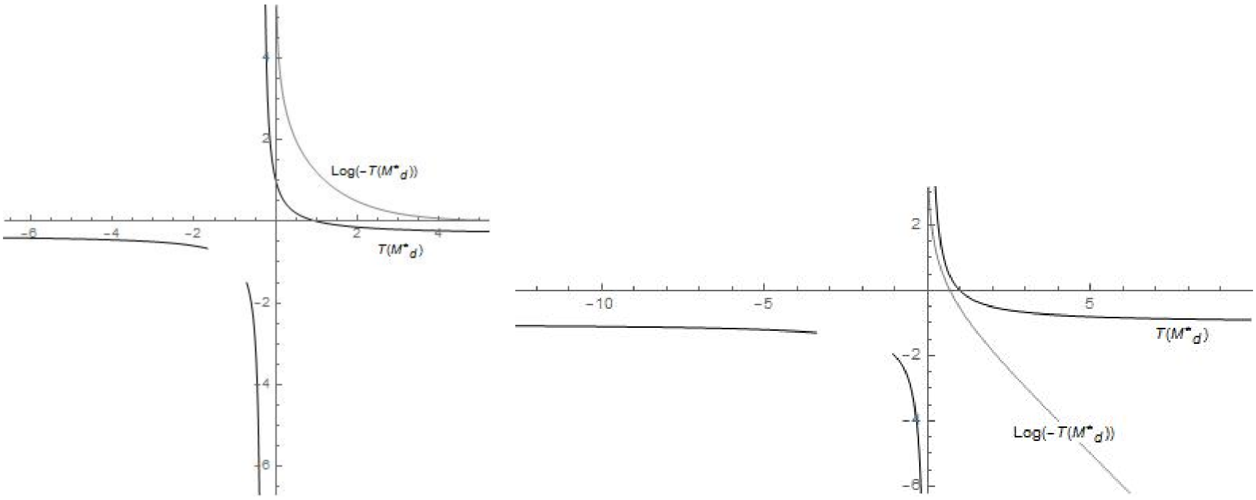
Recall that $P = (p_{ij})$ and $p_{ij} = \frac{v_j a_{ij}}{\rho v_i}$. Rewriting (23) as

$$\begin{aligned} I(\mathbf{r}) &= \sup_{\mathbf{q}\in M_d^*} \left\{ \sum_{j=1}^d r_j \log\rho - \sum_{j=1}^d r_j \log\left(\frac{v_j \sum_{i=1}^d \frac{q_i a_{ij}}{v_i}}{q_j}\right)\right\} \\ &= \log\rho - \inf_{\mathbf{q}\in M_d^*} \sum_{j=1}^d r_j(-1)\log[T(\mathbf{q})]_j = \log\rho - \inf_{\mathbf{s}\in T(M_d)}\left(-\sum_{j=1}^d r_j \log s_j\right), \end{aligned}$$

we conclude

$$\psi(\mathbf{r}) = \inf_{\mathbf{s}\in T(M_d)}\left(-\sum_{j=1}^d r_j \log s_j\right)$$

Figures 10 shows the set $T(M_d)$ in the two examples.



(a) **(b)**

Figure 10. The graphics of the sets $T(M_d^*)$ and $\log(-T(M_d^*))$ presented in (a) and (b) are associated with languages L_{F_2} and L_{Fib}, respectively

8 Finer asymptotic

A powerful results of the theory of ACSV (asymptotic combinatorics in several variables) presented in [20] allow not only to compute in many cases $\varphi(\mathbf{r})$, for rational functions $\Gamma(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})}$ and rational vectors $\mathbf{r}$ but also to describe much finer asymptotics of the diagonal coefficients $f_{n\mathbf{r}}$ (see [20, Theorems 5.1, 5.2, 5.3]). Also there are statements in the book on the smoothness of $\varphi(\mathbf{r})$ as a function of rational $\mathbf{r}$.

The article [16] uses a different approach to the finer asymptotics of $\psi(\mathbf{r})$ and hence $\tilde{\varphi}(\mathbf{r})$.

Now we recall few definitions and results presented in [20] and apply them to our examples.

In [20, Theorem 5.1] it is stated the following. Let $\mathbf{r} \in \mathbb{Q}^d$ and let $G(\mathbf{z}), H(\mathbf{z}) \in \mathbb{Q}[\mathbf{z}]$ be coprime polynomials such that $\Gamma(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})}$ admits a power series expansion $\Gamma(\mathbf{z}) = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$. Suppose that the system of equations

$$H(\mathbf{z}) = r_2 z_1 H_{z_1}(\mathbf{z}) - r_1 z_2 H_{z_2}(\mathbf{z}) = r_d z_1 H_{z_1}(\mathbf{z}) - r_1 z_d H_{z_d}(\mathbf{z}) \tag{25}$$

admits a finite number of solutions, exactly one of which $\mathbf{z}' \in \mathbb{C}_*^d$ is minimal, i.e. no other singularity $\mathbf{z}$ of $\Gamma(\mathbf{z})$ satisfies $|z_j| < |z_j'|$ for all $1 \le j \le d$.

Suppose that $H_{z_d}(\mathbf{z}) \neq 0$, $G(\mathbf{z}') \neq 0$ and $\mathbf{r}$ is a rational direction. Then

$$f_{n\mathbf{r}} = \mathbf{z}'^{-n\mathbf{r}} n^{\frac{1-d}{2}} \frac{(2\pi r_d)^{\frac{1-d}{2}}}{\sqrt{\det(\mathcal{H})}} \cdot \frac{-G(\mathbf{z}')}{z_d' H_{z_d}(\mathbf{z}')} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{as } n \to \infty, \tag{26}$$

when $n\mathbf{r} \in \mathbb{N}^d$, where $\mathcal{H}$ is a $(d-1)\times(d-1)$ matrix defined by [20, equation (5.25)] and it is supposed that $\det(\mathcal{H}) \neq 0$.

It is also claimed and justified in [20] that as $\mathbf{r}$ varies in any sufficiently neighborhood in $\mathbb{R}_{>0}^d$, the solution $\mathbf{z}' = \mathbf{z}'(\mathbf{r})$ varies smoothly with $\mathbf{r}$. The factor $(\mathbf{z}')^{-n\mathbf{r}}$ in (26) gives us

$$\limsup_{n\to\infty} |f_{n\mathbf{r}}|^{\frac{1}{n}} = |(\mathbf{z}')^{-\mathbf{r}}| = e^{-\sum r_i \log|z_i'|} = e^{-h_{\mathbf{r}}(\mathbf{z}')}.$$

The system of equations (25) is equivalent to the system of equations coming from the Lagrange multipliers method because if

$$\Phi(\mathbf{z},\lambda)=\sum_{i=1}^{d} r_i \log z_i-\lambda H(\mathbf{z}),$$

then the critical points are solutions of the system

$$\begin{aligned}&\frac{\partial\Phi}{\partial z_i}=\frac{r_i}{z_i}-\lambda H_{z_i}=0,\quad i=1,\dots,d,\\ &H(\mathbf{z})=0,\end{aligned}$$

which is equivalent to (25).

When the condition (CG) and conditions of [20, Theorem 5.1] are satisfied, then the functions $\varphi(\mathbf{r})$ and $\tilde{\varphi}(\mathbf{r})$ coincide. The smooth dependence of $\mathbf{z}'=\mathbf{z}'(\mathbf{r})$ (and hence of $\varphi(\mathbf{r})=\mathbf{z}'^{-\mathbf{r}}$) on $\mathbf{r}\in\mathbb{Q}^d_{>0}$ can be strengthen to the claim about the analytic dependence on $\mathbf{r}$ for all $\mathbf{r}\in\mathbb{R}^d_{>0}$, where $\psi(\mathbf{r})\geq 0$ as shown in [16, Corollary 9.1].

An useful tool in discussed topics is the logarithmic gradient map

$$\nabla_{\log} f=(z_1 f_{z_1},\dots,z_d f_{z_d})\,.$$

In [20, Proposition 3.13] it is stated that for any minimal singular point $\mathbf{z}'$ of $\Gamma(\mathbf{z})=\frac{G(\mathbf{z})}{H(\mathbf{z})}$, where G,H are coprime, there exist $\mathbf{r}\in\mathbb{R}^d_{\geq 0}$ and $\tau\in\mathbb{C}$ such that

$$\left(\nabla_{\log} H^{(\mathbf{s})}\right)(\mathbf{z}')=\tau\cdot\mathbf{r},$$

where $H^{(\mathbf{s})}$ is a square free part of H. In this situation, $\mathbf{z}'$ is either a minimizer or a maximizer of the height function $h_{\mathbf{r}}(\mathbf{z})$ on $\overline{\mathscr{D}}$.

Let us apply [20, Theorem 5.1] to the case of the free group of rank 2. We assume that $\mathbf{r}=(p,1-p)$ is rational, $\mathbf{z}=(x,y)$. Recall that the multivariate growth series of free group F_2 is

$$\Gamma_{F_2}(x,y)=\frac{(1+x)(1+y)}{1-x-y-3xy}=\frac{G(x,y)}{H(x,y)}$$

and that the singularities of $\Gamma_{F_2}(x,y)$ are the points

$$\mathbf{z}'=(x,y)=\left(\frac{3p-2+2\sqrt{3p^2-3p+1}}{3p},\frac{1-3p+2\sqrt{3p^2-3p+1}}{3(1-p)}\right),$$

whose coordinates are real numbers with positive coordinates. Hence,

$$(\mathbf{z}')^{-n\cdot\mathbf{r}}=\left(\frac{2-3p+2\sqrt{3p^2-3p+1}}{p},\frac{3p-1+2\sqrt{3p^2-3p+1}}{(1-p)}\right)^{n\cdot\mathbf{r}}=e^{n\cdot\psi_{F_2}(\mathbf{r})}.$$

We quickly check that the assumptions of [20, Theorem 5.1] are satisfied. In other words, we need to check that the partial derivative $\frac{\partial H}{\partial y}$ does not vanish at $\mathbf{z}'$ and that the matrix $\mathscr{H}$ from [20, equation (5.25)] is non singular (with $\mathbf{w}=\mathbf{z}'$). Indeed, a direct computation gives

$$\frac{\partial H}{\partial y}(x,y)=-1-3x,$$

which is non zero at $\mathbf{z}'$. Now, the dimension d being two, still with the notation of [20], the matrix $\mathscr{H}$ is the scalar

$$\begin{aligned}\mathscr{H} &= V_1 + V_1^2 + U_{1,1} - 2V_1U_{1,2} + V_1^2U_{2,2}\\ &= \frac{x(1+3y)}{y(1+3x)} + \left(\frac{x(1+3y)}{y(1+3x)}\right)^2 + 0 - 2\frac{x(1+3y)}{y(1+3x)}\frac{3xy}{y(1+3x)} + 0\\ &= \frac{xy + 3x^2y + 3xy^2 + x^2}{y^2(1+3x)^2} > 0.\end{aligned}$$

Therefore, following [20, equation (5.1)], for rational $\mathbf{r}$ we get

$$f_{n\cdot\mathbf{r}} = ce^{n\cdot\psi_{F_2}(\mathbf{r})}n^{-\frac{1}{2}}\left(1 + O\left(\frac{1}{n}\right)\right), \tag{27}$$

where $c = c(p)$ does not depend on p. In fact the results of [16] allows us to claim that the relation (27) holds also when $\mathbf{r}$ is irrational, only the left hand side should be replaced by the sum of the coefficients $\gamma_{\mathbf{i}}$ in the uniformly bounded neighborhood of the point $n\mathbf{r}$.

9 Concluding remarks and open questions

Finding of $\psi_{F_m}(\mathbf{r})$, where $\mathbf{r} = (p, q, 1-p-q)$, for free group F_3 of rank 3 leads to the solving of the following polynomial equation of degree 4 in variable $z = e^s$

$$\begin{aligned}&3p^2z^4 + 4p(7p-2)z^3\\ &\quad + 2\big(33p^2 - 32pq - 8p - 32q^2 + 32q - 8\big)z^2 + 12p(5p-6)z - 45p^2 = 0,\end{aligned} \tag{28}$$

and it can be solved in radicals. Substituting $p = q = \frac{1}{3}$ in (28) we obtain that $s = \log 5$ and hence we get a value $\psi_{F_3}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \log 5$. So the multivariate growth in this case coincides with the ordinary growth and $\log 5$ is a maximal value of $\psi(\mathbf{r})$. The higher ranks $m = 4, 5, \dots$ lead to polynomial equations of degree > 5 and most probably obtaining of the precise analytic expressions for $\psi_{F_m}(\mathbf{r})$ is impossible. But at least we know that $\psi_{F_m}(\mathbf{r})$ is a real analytic concave function [16] with a maximum value $\log(2m-1)$ achieved at unique point $\mathbf{r} = \left(\frac{1}{m}, \dots, \frac{1}{m}\right)$.

Now, let us go back to cogrowth. It can be shown that the condition (CG) always holds for a subgroup $H < F_m$ and so the formula (4) is applicable. If $H < F_m$ is a finitely generated subgroup, then it is represented by a regular language [6] and hence, its cogrowth and multivariate cogrowth series are rational. The conjecture, claiming that $\Gamma_H(z)$, $z \in \mathbb{C}$, is rational if and only if H is finitely generated, was stated in [6]. It is known that this conjecture is true in the case of normal subgroups. A similar conjecture can be stated for multivariate cogrowth series $\Gamma_H(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^m$. Also, we state the following assumption.

Conjecture. *Let $N \lhd F_m$. Then F_m/N is amenable if and only if $\psi_N(\mathbf{r}) = \psi_{F_m}(\mathbf{r})$.*

L. Bartholdi [2] showed that, at least in one direction, the statement of this conjecture is true. Namely, if quotient F_m/N is amenable then $\psi_N(\mathbf{r}) = \psi_{F_m}(\mathbf{r})$.

By cogrowth criteria of amenability we know that in the case, when F_m/N is amenable, the relation

$$\log(2m-1) = \max_{\mathbf{r}} \psi_{F_m}(\mathbf{r}) = \max_{\mathbf{r}} \psi_N(\mathbf{r}) \tag{29}$$

holds. It is unclear if $\psi_N(\mathbf{r})$ may have values less than the values of $\psi_{F_m}(\mathbf{r})$ in the case when the equation (29) holds. Even the case when $N = [F_2, F_2]$ is a commutator subgroup of F_2 deserves a separate consideration.

And finally, there is a formula (see [1])

$$\chi(p) = 2\min_t \left[\sum_{i=1}^m \sqrt{t^2 + p_i^2} - (m-1)t \right]$$

for the spectral radius $\chi(p)$ of a symmetric random walk on a free group F_m given by a positive vector $p = (p_1, \ldots, p_m)$, $2\sum p_i = 1$, where $p(a_i) = p(a_i^{-1}) = p_i$.

Computation of $\chi(p)$ in the case of rank 2 leads to the following equation of degree 4 in variable $x = t^2$

$$3x^4 + 4(p_1^2 + p_2^2)x^3 + 6p_1^2 p_2^2 x^2 - p_1^4 p_2^4 = 0, \tag{30}$$

and hence $\chi(p)$ can be expressed in radicals. Taking $p_1 = p_2 = \frac{1}{4}$, (30) leads to the equation $(1 + 16x)^3(-1 + 48x) = 0$, which gives a value $\chi(p) = \frac{\sqrt{3}}{2}$.

The latter number is known since 1959 due to H. Kesten [17], who in particular proved that for a simple random walk on F_m the spectral radius is $\chi = \frac{\sqrt{2m-1}}{m}$. Higher rank leads to solving polynomial equations of degree > 5 and expressing $\chi(p)$ in radicals seems to be impossible for F_m, $m \geq 3$, and arbitrary p.

Let $H < F_m$ and $\chi_{F_m/H}(p)$ be a spectral radius of a random walk on a Schreier graph $\Lambda = \Lambda(F_m, H, \Sigma)$ given by probabilities $p_i, 1 \leq i \leq m$. We end up with the following question.

Problem. *Is there a formula expressing $\chi_{F_m/H}$ via $\alpha_H(p)$, where $\alpha_H(p) = \varphi_H(p)$ is a multivariate growth of $\Delta_H(2p)$ in the direction prescribed by the vector $2p \in M_m$? Does such a formula exist, when H is normal subgroup in F_m and hence $\Lambda = \Lambda(F_m, H, \Sigma)$ is a Cayley graph?*

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Григорчук Р., Квінт Ж.-Ф., Шаікх А. *Багатовимірний ріст та коріст* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 82–109.

Розглядається багатовимірний ряд росту $\Gamma_L(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^d$, що асоційований з регулярною мовою L над алфавітом потужності $d \geq 2$. Основну увагу приділено мовам, що виникають із підгруп вільної групи F_m скінченного рангу m, а також із підзсувів скінченного типу. Запропоновано інструмент для обчислення швидкості зростання $\varphi_L(\mathbf{r})$ мови L в напрямку $\mathbf{r} \in \mathbb{R}^d$. Використовуючи умову опуклого зростання, введену другим автором у [Comment. Math. Helv. 2002, **77** (3), 563–608], та результати опуклого аналізу, функцію $\psi_L(\mathbf{r}) = \log(\varphi_L(\mathbf{r}))$ подано як опорну функцію опуклої множини, що є замиканням образу функції Relog області абсолютної збіжності $\Gamma_L(\mathbf{z})$. Це дозволяє обчислювати $\psi_L(\mathbf{r})$ у деяких випадках, зокрема для мови Фібоначчі або мови вільно скорочених слів, що представляють елементи вільної групи F_2. Також показано, що методи теорії великих відхилень можуть слугувати альтернативним підходом.

Ключові слова і фрази: ріст, коріст, регулярна мова, багатовимірна експонента зростання, вільна група, підзсув Фібоначчі, підзсув скінченного типу, принцип великих відхилень.