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Some identities on degenerate trigonometric functions

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In this paper, we study several degenerate trigonometric functions, which are degenerate versions of the ordinary trigonometric functions, and derive some identities among such functions by using elementary methods. Especially, we obtain multiple-angle formulas for the degenerate cotangent and degenerate sine functions.

Key words and phrases: degenerate tangent function, degenerate cotangent function, degenerate sine function, degenerate cosine function.

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1 Introduction

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined (see [2–6]) by

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!},$$

where

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots\big(x-(n-1)\lambda\big), \quad n \in \mathbb{N}.$$

Note that $\lim\limits_{\lambda\to 0} e_\lambda^x(t) = e^{xt}$ and $e_\lambda(t) = e_\lambda^1(t)$. In [4], the degenerate hyperbolic functions are introduced by

$$\cosh_\lambda\big(x:a\big) = \frac{e_\lambda^x(a)+e_\lambda^{-x}(a)}{2}, \quad \sinh_\lambda\big(x:a\big) = \frac{e_\lambda^x(a)-e_\lambda^{-x}(a)}{2},$$
$$\tanh_\lambda\big(x:a\big) = \frac{\sinh_\lambda(x:a)}{\cosh_\lambda(x:a)}, \quad \coth_\lambda(x:a) = \frac{\cosh_\lambda(x:a)}{\sinh_\lambda(x:a)}, \quad x\neq 0,\ a\neq 0.$$

It is well known [1] that

$$\cos a = \frac{e^{ia}+e^{-ia}}{2}, \qquad \sin a = \frac{e^{ia}-e^{-ia}}{2i}, \tag{1}$$

where $i=\sqrt{-1}$.

In light of (1), we consider the degenerate cosine and degenerate sine functions, which are respectively given by

$$\cos_\lambda\big(x:a\big) = \frac{e_\lambda^{xi}(a)+e_\lambda^{-xi}(a)}{2}, \qquad \sin_\lambda\big(x:a\big) = \frac{e_\lambda^{xi}(a)-e_\lambda^{-xi}(a)}{2i}. \tag{2}$$

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Note that

$$\lim_{\lambda\to 0}\cos_\lambda\left(x:a\right)=\cos ax,\qquad \lim_{\lambda\to 0}\sin_\lambda\left(x:a\right)=\sin ax.$$

In addition, we define the degenerate tangent and degenerate cotangent functions by

$$\tan_\lambda\left(x:a\right)=\frac{\sin_\lambda(x:a)}{\cos_\lambda(x:a)},\qquad \cot_\lambda\left(x:a\right)=\frac{\cos_\lambda(x:a)}{\sin_\lambda(x:a)},$$

where $x\neq 0$ and $a\neq 0$.

Recently, degenerate versions of many special numbers and polynomials and some transcendental functions have been investigated. The aim of this paper is to study several degenerate trigonometric functions, namely the degenerate sine, degenerate cosine, degenerate tangent and degenerate cotangent functions. We derive some identities on those degenerate trigonometric functions by using elementary methods. In particular, we show some multiple-angle fomulas for the degenerate cotangent and degenerate sine functions (see formulas (10), (14), (26) below).

2 Some identities on degenerate trigonometric functions

In Theorems 1 and 2 below, we show that exactly the same analogues of double-angle and addition formulas for sine and cosine functions also hold for the degenerate sine and cosine functions.

From (2), we note that

$$\sin_\lambda^2\left(x:a\right)+\cos_\lambda^2\left(x:a\right)=\left(\frac{e_\lambda^{xi}(a)+e_\lambda^{-xi}(a)}{2}\right)^2+\left(\frac{e_\lambda^{xi}(a)-e_\lambda^{-xi}(a)}{2i}\right)^2=1. \tag{3}$$

We observe that

$$\begin{aligned}\cos_\lambda\left(2x:a\right)&=\frac{e_\lambda^{2xi}(a)+e_\lambda^{-2xi}(a)}{2}=1+\frac{e_\lambda^{2xi}(a)+e_\lambda^{-2xi}(a)-2}{2}\\&=1-2\left(\frac{e_\lambda^{xi}(a)-e_\lambda^{-xi}(a)}{2i}\right)^2=1-2\sin_\lambda^2\left(x:a\right).\end{aligned} \tag{4}$$

By (3) and (4), we get

$$\cos_\lambda\left(2x:a\right)=1-2\sin_\lambda^2(x:a)=2\cos_\lambda^2(x:a)-1.$$

Moreover, by (2), we get

$$\begin{aligned}\sin_\lambda\left(2x:a\right)&=\frac{e_\lambda^{2xi}(a)-e_\lambda^{-2xi}(a)}{2i}=2\frac{e_\lambda^{xi}(a)-e_\lambda^{-ix}(a)}{2i}\frac{e_\lambda^{xi}(a)+e_\lambda^{-xi}(a)}{2}\\&=2\sin_\lambda\left(x:a\right)\cos_\lambda\left(x:a\right).\end{aligned}$$

So, we get the following theorem.

Theorem 1. *For $a\in\mathbb{R}$, we have*

$$\cos_\lambda\left(2x:a\right)=1-2\sin_\lambda^2\left(x:a\right)=2\cos_\lambda^2\left(x:a\right)-1,$$

and

$$\sin_\lambda\left(2x:a\right)=2\sin_\lambda\left(x:a\right)\cos_\lambda\left(x:a\right).$$

We observe that

$$\begin{aligned}\sin_\lambda\left(x+y:a\right) &= \frac{1}{2i}\Big(e_\lambda^{(x+y)i}-e_\lambda^{-(x+y)i}(a)\Big)\\ &= \frac{e_\lambda^{xi}(a)-e_\lambda^{-xi}(a)}{2i}\frac{e_\lambda^{yi}(a)+e_\lambda^{-yi}(a)}{2}+\frac{e_\lambda^{xi}(a)+e_\lambda^{-xi}(a)}{2}\frac{e_\lambda^{yi}(a)-e_\lambda^{-yi}(a)}{2i}\\ &= \sin_\lambda\left(x:a\right)\cos_\lambda\left(y:a\right)+\cos_\lambda\left(x:a\right)\sin_\lambda\left(y:a\right)\end{aligned}$$

and

$$\begin{aligned}\cos_\lambda\left(x+y:a\right) &= \frac{e_\lambda^{(x+y)i}(a)+e_\lambda^{-(x+y)i}(a)}{2}\\ &= \frac{e_\lambda^{xi}(a)+e_\lambda^{-xi}(a)}{2}\frac{e_\lambda^{yi}(a)+e_\lambda^{-yi}(a)}{2}-\frac{e_\lambda^{xi}(a)-e_\lambda^{-xi}}{2i}\frac{e_\lambda^{yi}(a)-e_\lambda^{-yi}(a)}{2i}\\ &= \cos_\lambda\left(x:a\right)\cos_\lambda\left(y:a\right)-\sin_\lambda\left(x:a\right)\sin_\lambda\left(y:a\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2. *For $a\in\mathbb{R}$, we have*

$$\begin{aligned}\sin_\lambda\left(x\pm y:a\right) &= \sin_\lambda\left(x:a\right)\cos_\lambda\left(y:a\right)\pm\cos_\lambda\left(x:a\right)\sin_\lambda\left(y:a\right),\\ \cos_\lambda\left(x\pm y:a\right) &= \cos_\lambda\left(x:a\right)\cos_\lambda\left(y:a\right)\mp\sin_\lambda\left(x:a\right)\sin_\lambda\left(y:a\right).\end{aligned}$$

We observe that

$$\frac{d}{dx}\cos_\lambda\left(x:a\right)=-\log e_\lambda(a)\sin_\lambda(x:a), \tag{5}$$

$$\frac{d}{dx}\sin_\lambda\left(x:a\right)=\log e_\lambda(a)\cos_\lambda(x:a). \tag{6}$$

Next, in Theorems 3–5, we derive multiple-angle formulas for the degenerate cotangent and degenerate sine functions.

From (2) and noting that

$$\prod_{j=0}^{2m-1}(x+e^{\frac{2\pi ij}{2m}})=x^{2m}-1,$$

we get

$$\begin{aligned}\prod_{j=0}^{2m-1}\cos_\lambda\left(t+\frac{j\pi}{2m\log e_\lambda(a)}:a\right) &= \prod_{j=0}^{2m-1}\left(\frac{e_\lambda^{(t+\frac{j\pi}{2m\log e_\lambda(a)})i}(a)+e_\lambda^{-(t+\frac{j\pi}{2m\log e_\lambda(a)})i}(a)}{2}\right)\\ &= \frac{1}{4^m}\prod_{j=0}^{2m-1}\Big(e_\lambda^{it}(a)e_\lambda^{-\frac{j\pi}{2m\log e_\lambda(a)}i}(a)\Big(e_\lambda^{\frac{2j\pi}{2m\log e_\lambda(a)}i}(a)+e_\lambda^{-2it}(a)\Big)\Big)\\ &= \frac{1}{4^m}e^{-\frac{\pi}{2m}i\frac{2m(2m-1)}{2}}e_\lambda^{2mti}(a)\prod_{j=0}^{2m-1}\left(e_\lambda^{-2it}(a)+e^{\frac{2\pi ij}{2m}}\right)\\ &= \frac{1}{4^m}e_\lambda^{2mti}(a)e^{-m\pi i}e^{\frac{\pi}{2}i}\Big(e_\lambda^{-4mti}(a)-1\Big)\\ &= \frac{1}{4^m}e^{-m\pi i}e^{\frac{\pi}{2}i}\Big(e_\lambda^{-2mti}(a)-e_\lambda^{2mti}(a)\Big)\\ &= -\frac{1}{4^m}e^{-m\pi i}e^{\frac{\pi}{2}i}2i\left(\frac{e_\lambda^{2mti}(a)-e_\lambda^{-2mti}(a)}{2i}\right)\\ &= -\frac{i}{2^{2m-1}}e^{-m\pi i}e^{\frac{\pi}{2}i}\sin_\lambda(2mt:a).\end{aligned} \tag{7}$$

Note that

$$\left| -\frac{i}{2^{2m-1}} e^{-m\pi i} e^{\frac{\pi}{2} i} \sin_\lambda \left(2mt : a\right) \right| = \left| \frac{1}{2^{2m-1}} \sin_\lambda \left(2mt : a\right) \right|.$$

Thus, by (7), we get

$$\log \left| \frac{1}{2^{2m-1}} \sin_\lambda \left(2mt : a\right) \right| = \sum_{j=0}^{2m-1} \log \left| \cos_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right) \right|. \tag{8}$$

By using (5), we take the derivatives with respect to t on both sides of (8) and get the following identity

$$\frac{2m \log(e_\lambda(a)) \cos_\lambda(2mt : a)}{\sin_\lambda(2mt : a)} = -\sum_{j=0}^{2m-1} \log\left(e_\lambda(a)\right) \frac{\sin_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a\right)}{\cos_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a\right)}. \tag{9}$$

Therefore, by (9), we obtain the following theorem.

Theorem 3. *For $m \in \mathbb{N}$, we have*

$$-2m \cot_\lambda \left(2mt : a\right) = \sum_{j=0}^{2m-1} \tan_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right). \tag{10}$$

We remark here that

$$\tan_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right) = \tan \left(t \log e_\lambda(a) + \frac{j\pi}{2m} \right).$$

Taking $\lambda \to 0$ in (10), we obtain (see [1])

$$-2m \cot\left(2mat\right) = \sum_{j=0}^{2m-1} \tan\left(at + \frac{j\pi}{2m} \right).$$

Working with $\prod_{j=0}^{2m-1} \sin_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right)$ and proceeding analogously to (7), we obtain

$$\prod_{j=0}^{2m-1} \sin_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right) = -\frac{i}{2^{2m-1}} (-1)^m e^{-m\pi i} e^{\frac{\pi}{2} i} \sin_\lambda(2mt : a). \tag{11}$$

Thus, from (11), we have

$$\log \left| \frac{1}{2^{2m-1}} \sin_\lambda \left(2mt : a\right) \right| = \sum_{j=0}^{2m-1} \log \left| \sin_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a \right) \right|. \tag{12}$$

Using (6), we take the derivatives with respect to t on both sides of (12) and get the identity

$$\frac{2m \log(e_\lambda(a)) \cos_\lambda(2mt : a)}{\sin_\lambda(2mt : a)} = \sum_{j=0}^{2m-1} \log\left(e_\lambda(a)\right) \frac{\cos_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a\right)}{\sin_\lambda \left(t + \frac{j\pi}{2m \log e_\lambda(a)} : a\right)}. \tag{13}$$

Therefore, by (13), we obtain the following theorem.

Theorem 4. *For $m \in \mathbb{N}$, we have*

$$2m \cot_\lambda \big(2mt : a\big) = \sum_{j=0}^{2m-1} \cot_\lambda \Big(t + \frac{j\pi}{2m \log e_\lambda(a)} : a\Big). \tag{14}$$

Taking $\lambda \to 0$ in (14), we obtain (see [1])

$$2m \cot \big(2mat\big) = \sum_{j=0}^{2m-1} \cot \Big(at + \frac{j\pi}{2m}\Big).$$

Now, we observe that

$$\cos_\lambda \big((k+1)x : a\big) + \cos_\lambda \big((k-1)x : a\big) = 2\cos_\lambda \big(kx : a\big) \cos_\lambda \big(x : a\big). \tag{15}$$

Thus, by (15), we get

$$\begin{aligned} \cos_\lambda \big((k+1)x : a\big) &= 2\cos_\lambda \big(kx : a\big) \cos_\lambda (x : a) - \cos_\lambda \big((k-1)x : a\big) \\ &= 2^2 \cos_\lambda^2 \big(x : a\big) \cos_\lambda \big((k-1)x : a\big) \\ &\quad - 2\cos_\lambda \big(x : a\big) \cos_\lambda \big((k-2)x : a\big) - \cos_\lambda \big((k-1)x : a\big) \\ &\cdots \\ &= 2^{k+1} \cos_\lambda^{k+1} \big(x : a\big) + \cdots . \end{aligned} \tag{16}$$

From (16), we note that there is a polynomial $T_n(x)$ with degree n such that

$$\cos_\lambda \big(nx : a\big) = T_n\big(\cos_\lambda \big(x : a\big)\big), \quad n \in \mathbb{N}. \tag{17}$$

By using Theorem 2, we get

$$\sin_\lambda \big((2k+1)x : a\big) - \sin_\lambda \big((2k-1)x : a\big) = 2\cos_\lambda \big(2kx : a\big) \sin_\lambda \big(x : a\big). \tag{18}$$

By (17) and Theorem 1, we get

$$\cos_\lambda \big(2kx : a\big) = T_k\big(\cos_\lambda \big(2x : a\big)\big) = T_k\big(1 - 2\sin_\lambda^2 \big(x : a\big)\big). \tag{19}$$

From (17), (18) and (19), we show that

$$\sin_\lambda \big((2m+1)x : a\big) = \sin_\lambda \big(x : a\big)\Bigg(1 + 2\sum_{k=1}^{m} T_k\big(1 - 2\sin_\lambda^2 \big(x : a\big)\big)\Bigg),$$

and hence there exist polynomials $K_m(x)$ with degree m such that

$$\sin_\lambda \big((2m+1)x : a\big) = \sin_\lambda(x : a) K_m\big(\sin_\lambda^2(x : a)\big). \tag{20}$$

Now, from (20) we note that

$$\begin{aligned} 0 &= \sin_\lambda\left((2m+1)\frac{k\pi}{(2m+1)\log(e_\lambda(a))} : a\right) \\ &= \sin_\lambda\left(\frac{k\pi}{(2m+1)\log(e_\lambda(a))} : a\right) K_m\left(\sin_\lambda^2\left(\frac{k\pi}{(2m+1)\log e_\lambda(a)} : a\right)\right) \\ &= \sin\left(\frac{k\pi}{2m+1}\right) K_m\left(\sin^2\left(\frac{k\pi}{2m+1}\right)\right), \end{aligned} \tag{21}$$

which implies that $K_m\left(\sin^2\left(\frac{k\pi}{2m+1}\right)\right) = 0$ for $k = 1, 2, 3, \dots, m$.

By (21), the polynomials $K_m(x)$ can be written as

$$K_m(x) = C \prod_{k=1}^{m} \left(1 - \frac{x}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right). \tag{22}$$

Note from (20) that

$$\begin{aligned} C = K_m(0) &= \lim_{x\to 0} K_m\left(\sin_\lambda^2(x : a)\right) \\ &= \lim_{x\to 0} \frac{\sin_\lambda\left((2m+1)x : a\right)}{\sin_\lambda(x : a)} = \frac{(2m+1)\log(e_\lambda(a))}{\log e_\lambda(a)} = 2m+1. \end{aligned} \tag{23}$$

By (22) and (23), we get

$$K_m(x) = (2m+1) \prod_{k=1}^{m} \left(1 - \frac{x}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right). \tag{24}$$

From (20) and (24), we have

$$\begin{aligned} \sin_\lambda\left((2m+1)x : a\right) &= \sin_\lambda\left(x : a\right) K_m\left(\sin_\lambda^2(x : a)\right) \\ &= (2m+1)\sin_\lambda\left(x : a\right) \prod_{k=1}^{m} \left(1 - \frac{\sin_\lambda^2(x : a)}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right). \end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 5. *For $m \in \mathbb{N}$, we have*

$$\sin_\lambda\left((2m+1)x : a\right) = (2m+1)\sin_\lambda\left(x : a\right) \prod_{k=1}^{m} \left(1 - \frac{\sin_\lambda^2(x : a)}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right). \tag{26}$$

Taking $\lambda \to 0$ in (26), we obtain

$$\sin\left((2m+1)ax\right) = (2m+1)\sin ax \prod_{k=1}^{m} \left(1 - \frac{\sin^2 ax}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right).$$

3 Conclusion

The degenerate trigonometric and degenerate hyperbolic functions are obtained by replacing the ordinary exponentials by the degenerate exponentials in the definitions of trigonometic and hyperbolic functions. In this paper, we considered the degenerate sine, degenerate cosine, degenerate tangent and degenerate cotangent functions and derived several identities among them. Especially, we were able to show the degenerate versions of some multiple-angle formulas (see (10), (14), (26)).

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У цій статті ми вивчаємо кілька вироджених тригонометричних функцій, які є виродженими версіями звичайних тригонометричних функцій, та виводимо деякі тотожності для таких функцій за допомогою елементарних методів. Зокрема, ми отримуємо формули кратних кутів для виродженого котангенса та виродженого синуса.

Ключові слова і фрази: вироджений тангенс, вироджений котангенс, вироджений синус, вироджений косинус.