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Inverse initial problem for a time-fractional diffusion-wave equation

Boyko P.[1], Lopushansky A.[2], Lopushanska H.[3], Pukach P.[1,✉]

We find sufficient conditions for a unique classical solvability of the inverse problem of restoration of two functions in initial conditions of the Cauchy problem for a time-fractional diffusion-wave equation with the Caputo-Djrbashian-Nersesian derivative and the right-hand side with values in Schwartz-type spaces of smooth functions rapidly decreasing to zero at infinity.

We use two time-integral overdetermination conditions

$$\frac{1}{T}\int_0^T u(x,t)\eta_1(t)dt = \Phi_1(x), \quad \frac{1}{T}\int_0^T u(x,t)\eta_2(t)dt = \Phi_2(x), \quad x \in \mathbb{R}^n,$$

where u is the solution of the Cauchy problem for such equation, Φ_1, Φ_2 are given functions from the Schwartz-type space, η_1, η_2 are given functions from $C^2[0,T]$.

We use the method of the Green's vector-function. The initial data sought are expressed through the solution of a certain linear Fredholm integral equation of the second kind in the space of continuous functions with values in Schwartz-type spaces.

Key words and phrases: Schwartz-type functional space, fractional derivative, Cauchy problem, inverse problem, Green's vector-function.

[1] Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine
[2] University of Rzeszow, 1 Pigonia str., 35-310, Rzeszow, Poland
[3] Ivan Franko Lviv National University, 1 Universytetska str., 79000, Lviv, Ukraine
✉ Corresponding author
E-mail: boyko.petro@gmail.com (Boyko P.), alopushanskyj@gmail.com (Lopushansky A.), lhp@ukr.net (Lopushanska H.), petro.y.pukach@lpnu.ua (Pukach P.)

Introduction

Equations with fractional derivatives and inverse problems for them are arising in various applications with memory, in nano-technology, signal and image processing, control theory, biology, geology, geophysics, medicine, economy (see, for example, [2,7,9,10,13,23,26,27,29] and references therein). It was shown in some works (see [2], for example) that equations with fractional derivatives are more effective in describing of frequency-dependent behavior of viscoelastic polymers.

Solvability of the Cauchy and boundary-value problems for diffusion and diffusion-wave equations with fractional derivatives has been the subject of research in [1,3–5,17,24,25,28,31] and other works. In anomalous diffusion different aspects attract attention. The most works on inverse problems for such equations and differential equations with partial derivatives of entire orders are devoted to problems with unknown right-hand sides of equations, for example, [11,16,19,28,32–34].

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In this paper, we study the inverse problem of identifying initial data of the solution for a time-fractional diffusion-wave equation with a source from Schwartz-type spaces of smooth functions decreasing to zero at infinity.

In [21, 30, 35, 36], the inverse boundary-value problems of identifying initial values were obtained by a final time data. Unique solvability of the inverse problem of restoring one unknown function (for example, the source in [19], the equation's minor coefficient in [12], the solution's initial data in [18,20]) were obtained by using a time integral overdetermination condition, and in [11, 16] and some other papers, integral conditions over spatial variables were used as additional ones. Two integral conditions were used to find two unknown functions in [14, 15] and [22], in [14, 15] with unknowns of different arguments.

We use two time integral overdetermination conditions to find two unknown functions from Schwartz-type spaces in initial conditions. Such problems have not yet been studied.

1 Definitions and auxiliary results

We use the following: $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$, $j \in \{1, \dots, n\}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$, $D^\alpha v(x,t) = \frac{\partial^{|\alpha|} v(x,t)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $f*g$ is the convolution of f and g,

$$f_\lambda(t) = \begin{cases} \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, \\ f'_{1+\lambda}(t), & \lambda \le 0, \end{cases}$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$.

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined by the formula $v^{(\beta)}(t) = f_{-\beta}(t) * v(t)$, the Caputo (Caputo-Djrbashian-Nersesian or regularized) fractional derivative is defined by $D^\beta v(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\beta-1} \frac{d^m}{d\tau^m} v(\tau) d\tau$ for $m-1 < \beta < m$, $m \in \mathbb{N}$, and therefore,

$$D^\beta v(t) = v^{(\beta)}(t) - \sum_{j=0}^{m-1} f_{j+1-\beta}(t) v^{(j)}(0).$$

Let $Q = \mathbb{R}^n \times (0, T]$, $C_{2,\beta}(Q) = \{v \in C(Q) : \Delta v, D_t^\beta v \in C(Q)\}$, $C_{2,\beta}(\bar{Q}) = C_{2,\beta}(Q) \cap C(\bar{Q})$, $S(\mathbb{R}^n)$ be the space of infinitely differentiable functions v in $\mathbb{R}^n$ such that $x^\gamma D^\alpha v$ are bounded in $\mathbb{R}^n$ for all multi-indexes α, γ (the Schwartz space of smooth rapidly decreasing functions), $S_\gamma(\mathbb{R}^n)$, $\gamma > 0$, be the space of type $S(\mathbb{R}^n)$ (see [8, p. 201]):

$$S_\gamma(\mathbb{R}^n) = \left\{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \le C_\alpha e^{-a|x|^{\frac{1}{\gamma}}}, \;\; x \in \mathbb{R}^n, \;\; \forall\, \alpha\right\}$$

with some positive constants $C_\alpha = C_\alpha(v)$ and $a = a(v)$. The sequence $v_m(x)$ converges to zero as $m \to +\infty$ in the space $S_\gamma(\mathbb{R}^n)$ if the sequence $D^\alpha v_m(x)$ converges to zero uniformly on an arbitrary compact $|x| \le C < +\infty$ for each multi-index α and the estimates

$$|D^\alpha v_m(x)| \le C_\alpha e^{-a|x|^{\frac{1}{\gamma}}}, \;\; x \in \mathbb{R}^n, \;\; \forall\, \alpha \text{ and } m \in \mathbb{N},$$

hold with some $a > 0$. It is known (see [8, p. 211]) that $S_\gamma(\mathbb{R}^n) = \cup_{a>0} S_{\gamma,(a)}(\mathbb{R}^n)$ where

$$\begin{aligned} S_{\gamma,(a)}(\mathbb{R}^n) &= \{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \le C_{\alpha,\epsilon}(v) e^{-(a-\epsilon)|x|^{\frac{1}{\gamma}}}, \;\; x \in \mathbb{R}^n, \;\; \forall\, \alpha, \;\; \forall\, \epsilon > 0\} \\ &= \{v \in C^\infty(\mathbb{R}^n) : \|v\|_{k,(a)} = \sup_{|\alpha| \le k,\, x \in \mathbb{R}^n} e^{a(1-\frac{1}{k})|x|^{\frac{1}{\gamma}}} |D^\alpha v(x)| < +\infty \;\; \forall k \in \mathbb{N}, \; k \ne 1\}. \end{aligned}$$

The sequence $v_m(x)$ converges to zero as $m \to +\infty$ in the space $S_{\gamma,(a)}(\mathbb{R}^n)$ if the sequence $D^\alpha v_m(x)$ converges to zero uniformly on an arbitrary compact $|x| \le C < +\infty$ for each multi-index α and the norms $\|v_m\|_{k,(a)}$ are limited for all $m, k \in \mathbb{N},\ k \neq 1$.

We say that $v \in S_{\gamma,(a)}(\bar{Q})$ if $v(\cdot,t) \in S_{\gamma,(a)}(\mathbb{R}^n)$ for all $t \in [0,T]$, and the norm is defined by $\|v\|_{S_{\gamma,(a)}(\bar{Q})} = \max\limits_{t\in[0,T]} \|v(\cdot,t)\|_{S_{\gamma,(a)}(\mathbb{R}^n)}$.

For $\beta \in (1,2)$ we study the inverse problem

$$D_t^\beta u - \Delta u = F_0(x,t), \quad (x,t) \in Q, \tag{1}$$

$$u(x,0) = F_1(x), \quad \frac{\partial}{\partial t} u(x,0) = F_2(x), \quad x \in \mathbb{R}^n, \tag{2}$$

$$\frac{1}{T}\int_0^T u(x,t)\eta_1(t)\,dt = \Phi_1(x), \quad \frac{1}{T}\int_0^T u(x,t)\eta_2(t)\,dt = \Phi_2(x), \quad x \in \mathbb{R}^n \tag{3}$$

of determining the triple (u, F_1, F_2), where Φ_1, Φ_2 are given functions from $S_{\gamma,(a)}(\mathbb{R}^n)$, $F_0 \in S_{\gamma,(a)}(\bar{Q})$, η_1, η_2 are given functions from $C^2[0,T]$.

Definition 1. *The triple* $(u, F_1, F_2) \in S_{\gamma,(a)}(\bar{Q}) \times S_{\gamma,(a)}(\mathbb{R}^n) \times S_{\gamma,(a)}(\mathbb{R}^n)$ *is called a solution of the problem* (1)–(3) *if it satisfies equation* (1) *in* Q *and conditions* (2), (3).

Definition 2. *The vector-function* $(G_0(x,t), G_1(x,t), G_2(x,t))$ *is called a Green's vector-function of the Cauchy problem* (1), (2) *if under rather regular* F_0, F_1, F_2 *the function*

$$u(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y, t-\tau) F_0(y,\tau)\,dy + \sum_{j=1,2} \int_{\mathbb{R}^n} G_j(x-y,t) F_j(y)\,dy, \quad (x,t) \in \bar{Q}, \tag{4}$$

is a classical (from $C_{2,\beta}(\bar{Q})$*) solution of this problem.*

Such Green's vector-function exists (see, for example, [4,17,25,31], namely

$$G_j(x,t) = \int_0^t f_{j-\beta}(\tau) G_0(x, t-\tau)\,d\tau, \quad (x,t) \in Q, \quad j = 1,2.$$

Moreover, $G_j(\cdot,t) \in L_1(\mathbb{R}^n)$ for each $t > 0$, $j = 0,1,2$, and the following bounds hold [17]

$$|G_0(x,t)| \le C t^{-\beta\frac{n}{2}} e^{-c(|x|t^{-\frac{\beta}{2}})^{\frac{2}{2-\beta}}} \Psi_n(|x|t^{-\frac{\beta}{2}}),$$

where

$$\Psi_n(z) = \begin{cases} 1, & n = 1, \\ |ln|z||, & n = 2, \\ |z|^{2-n}, & n \ge 3 \end{cases}$$

for $|z| < 1$, $\Psi_n(z) = \Psi_n(1)$ for $|z| > 1$, $c < (2-\beta)\left(\frac{\beta^\beta}{4}\right)^{\frac{1}{2-\beta}}$.

The expressions for components of the Green's vector-function through the H-function of Fox [6] are written in [4,17,31].

Remark. *The following results spread to the case of replacing the Laplace operator with an elliptic differential operator of the second order with constant coefficients or some coefficients dependent on spatial variables, multiplication by which functions from* $S_{\gamma,(a)}(\mathbb{R}^n)$ *belong to* $S_{\gamma,(a)}(\mathbb{R}^n)$. *The existence and properties of the Green's vector-function for such equations is obtained in [17].*

We denote

$$(G_j\varphi)(x,t) = \int_{\mathbb{R}^n} G_j(x-y,t)\varphi(y)\,dy, \quad (x,t)\in Q, \quad j=0,1,2.$$

Lemma 1. *If $\gamma \ge 1-\frac{\beta}{2}$, $0 < aT^{\frac{\beta}{2\gamma}} < c$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, then there exist positive numbers C, a' ($a' = c_\gamma a$ with $c_\gamma = 1$ if $\gamma \ge 1$) such that for all $k\in\mathbb{N}$, $k\ne 1$, the following bounds hold*

$$\|(G_0\varphi)(\cdot,t)\|_{k,(a')} \le Ct^{\beta-1}\|\varphi\|_{k,(a)}, \quad \|(G_j\varphi)(\cdot,t)\|_{k,(a')} \le Ct^{j-1}\|\varphi\|_{k,(a)}, \quad j=1,2.$$

Proof. Lemma can be proved by the scheme of [15, Lemma 1]. □

Theorem 1. *Assume that $\gamma \ge 1$, $0 < aT^{\frac{\beta}{2\gamma}} \le c$, $F_0 \in S_{\gamma,(a)}(\bar{Q})$, $F_1, F_2 \in S_{\gamma,(a)}(\mathbb{R}^n)$. Then there exists the unique solution $u \in S_{\gamma,(a)}(\bar{Q})$ of the Cauchy problem* (1), (2)*. It is defined by* (4)*.*

Proof. It follows from [17, 28, 31] the existence of a unique classical (from $C_{2,\beta}(\bar{Q})$) solution u of the Cauchy problem for F_1, F_2 bounded and from $L_1(\mathbb{R}^n)$, $F_0(x,t)$ continuous, bounded and locally Hölder with respect to spatial variables $x\in\mathbb{R}^n$ for every $t\in(0,T]$. The representation of the solution in the form (4) was obtained.

Then under conditions of Theorem 1, defined by (4) function u belongs to $S_{\gamma,(a)}(\bar{Q})$. Indeed, for each $k\in\mathbb{N}$, $k\ge 2$, $t\in[0,T]$, by Lemma 1 we get

$$\left\| \int_0^t d\tau \int_{\mathbb{R}^n} G_0(\cdot - y, t-\tau)F_0(y,\tau)\,dy \right\|_{k,(a)} \le C\int_0^t (t-\tau)^{\beta-1}\|F_0(\cdot,\tau)\|_{k,(a)}\,d\tau \le \frac{Ct^\beta}{\beta}\max_{\tau\in[0,T]}\|F_0(\cdot,\tau)\|_{k,(a)},$$

$$\left\| \int_{\mathbb{R}^n} G_j(\cdot - y,t)F_j(y)\,dy \right\|_{k,(a)} \le Ct^{j-1}\|F_j\|_{k,(a)}, \quad j=1,2.$$

□

2 Solutions of the inverse problem

We study the problem (1)–(3) under the assumptions

$$\begin{aligned} &\gamma \ge 1, \quad 0 < aT^{\frac{\beta}{2\gamma}} \le c, \quad F_0 \in S_{\gamma,(a)}(\bar{Q}), \\ &\Phi_j \in S_{\gamma,(a)}(\mathbb{R}^n), \quad \eta_j \in C^2[0,T], \quad \eta_j(T)=0, \quad j=1,2. \end{aligned} \tag{A}$$

Let u be a solution of the problem (1), (2). By using (3) and (2), we get

$$\int_0^T \Delta u(x,t)\eta_j(t)\,dt = \Delta\int_0^T u(x,t)\eta_j(t)\,dt = T\Delta\Phi_j(x),$$

$$\begin{aligned} \int_0^T D_t^\beta u(x,t)\eta_j(t)\,dt &= \int_0^T [f_{2-\beta}(t) * u_{tt}(x,t)]\eta_j(t)\,dt = \int_0^T\left(\int_0^t f_{2-\beta}(t-s)u_{ss}(x,s)\,ds\right)\eta_j(t)\,dt \\ &= \int_0^T u_{ss}(x,s)\left(\int_s^T f_{2-\beta}(t-s)\eta_j(t)\,dt\right)ds \\ &= \int_0^T u_{ss}(x,s)\left(\int_0^{T-s} f_{2-\beta}(\tau)\eta_j(\tau+s)\,d\tau\right)ds \\ &= -F_2(x)\int_0^T f_{2-\beta}(t)\eta_j(t)\,dt - \int_0^T u_s(x,s)\left(\int_0^{T-s} f_{2-\beta}(\tau)\eta_j'(\tau+s)\,d\tau\right)ds \\ &= -F_2(x)\int_0^T f_{2-\beta}(t)\eta_j(t)\,dt + F_1(x)\left(\int_0^T f_{2-\beta}(\tau)\eta_j'(\tau)\,d\tau\right) \\ &\quad + \int_0^T u(x,t)\left[(f_{2-\beta}\widehat{*}\eta_j'')(t) - f_{2-\beta}(T-t)\eta_j'(T)\right]dt, \quad j=1,2. \end{aligned}$$

Here

$$(f_{2-\beta}\widehat{*}\eta_j'')(t) = \int_t^T f_{2-\beta}(s-t)\eta_j''(s)\,ds = \int_0^{T-t} f_{2-\beta}(\tau)\eta_j''(t+\tau)\,d\tau, \quad j=1,2.$$

Then from equation (1) we obtain the system

$$\begin{aligned}
&F_1(x)\int_0^T f_{2-\beta}(\tau)\eta_1'(\tau)\,d\tau - F_2(x)\int_0^T f_{2-\beta}(t)\eta_1(t)\,dt\\
&\quad = T\Delta\Phi_1(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_1'')(t) - f_{2-\beta}(T-t)\eta_1'(T)\Big]dt + \int_0^T F_0(x,t)\eta_1(t)\,dt,\\
&F_1(x)\int_0^T f_{2-\beta}(\tau)\eta_2'(\tau)\,d\tau - F_2(x)\int_0^T f_{2-\beta}(t)\eta_2(t)\,dt\\
&\quad = T\Delta\Phi_2(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_2'')(t) - f_{2-\beta}(T-t)\eta_2'(T)\Big]dt + \int_0^T F_0(x,t)\eta_2(t)\,dt.
\end{aligned}$$

From here, denoting

$$(f,\eta) = \int_0^T f(t)\eta(t)dt,$$

under the assumption

$$d(T) := (f_{2-\beta},\eta_2')\,(f_{2-\beta},\eta_1) - (f_{2-\beta},\eta_1')\,(f_{2-\beta},\eta_2) \neq 0 \tag{5}$$

we find

$$\begin{aligned}
d(T)F_1(x) = &-\Big\{T\Delta\Phi_1(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_1'')(t) - f_{2-\beta}(T-t)\eta_1'(T)\Big]dt\\
&\qquad + \int_0^T F_0(x,t)\eta_1(t)\,dt\Big\}(f_{2-\beta},\eta_2)\\
&+\Big\{T\Delta\Phi_2(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_2'')(t) - f_{2-\beta}(T-t)\eta_2'(T)\Big]dt\\
&\qquad + \int_0^T F_0(x,t)\eta_2(t)\,dt\Big\}(f_{2-\beta},\eta_1),\\
d(T)F_2(x) = &\Big\{T\Delta\Phi_2(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_2'')(t) - f_{2-\beta}(T-t)\eta_2'(T)\Big]dt\\
&\qquad + \int_0^T F_0(x,t)\eta_2(t)\,dt\Big\}(f_{2-\beta},\eta_1')\\
&-\Big\{T\Delta\Phi_1(x) - \int_0^T u(x,t)\Big[(f_{2-\beta}\widehat{*}\eta_1'')(t) - f_{2-\beta}(T-t)\eta_1'(T)\Big]dt\\
&\qquad + \int_0^T F_0(x,t)\eta_1(t)\,dt\Big\}(f_{2-\beta},\eta_2'),
\end{aligned}$$

that is,

$$\begin{aligned}
F_1(x) &= \frac{1}{d(T)}\int_0^T u(x,t)w_1(t,\eta_1,\eta_2)\,dt + v_1(x),\\
F_2(x) &= \frac{1}{d(T)}\int_0^T u(x,t)w_2(t,\eta_1,\eta_2)\,dt + v_2(x), \quad x\in\mathbb{R}^n,
\end{aligned} \tag{6}$$

where

$$\begin{aligned} w_1(t,\eta_1,\eta_2) := {} & (f_{2-\beta},\eta_2)\Big[(f_{2-\beta}\widehat{*}\eta_1'')(t) - f_{2-\beta}(T-t)\eta_1'(T)\Big] \\ & - (f_{2-\beta},\eta_1)\Big[(f_{2-\beta}\widehat{*}\eta_2'')(t) - f_{2-\beta}(T-t)\eta_2'(T)\Big], \\ w_2(t,\eta_1,\eta_2) := {} & (f_{2-\beta},\eta_2')\Big[(f_{2-\beta}\widehat{*}\eta_1'')(t) - f_{2-\beta}(T-t)\eta_1'(T)\Big] \\ & - (f_{2-\beta},\eta_1')\Big[(f_{2-\beta}\widehat{*}\eta_2'')(t) - f_{2-\beta}(T-t)\eta_2'(T)\Big], \\ v_1(x) = {} & \frac{1}{d(T)}\int_0^T \Big\{F_0(x,t)\big[\eta_2(t)(f_{2-\beta},\eta_1) - \eta_1(t)(f_{2-\beta},\eta_2)\big] \\ & + T\big[\Delta\Phi_2(x)(f_{2-\beta},\eta_1) - \Delta\Phi_1(x)(f_{2-\beta},\eta_2)\big]\Big\}dt, \\ v_2(x) = {} & \frac{1}{d(T)}\int_0^T \Big\{F_0(x,t)\,\big[\eta_2(t)(f_{2-\beta},\eta_1') - \eta_1(t)(f_{2-\beta},\eta_2')\big] \\ & + T\big[\Delta\Phi_2(x)\,(f_{2-\beta},\eta_1') - \Delta\Phi_1(x)\,(f_{2-\beta},\eta_2')\big]\Big\}dt, \qquad x\in\mathbb{R}^n. \end{aligned}$$

Note that the condition (5) is performed, in particular, if the functions $\eta_1(t),\eta_2(t)$ are positive and $\eta_1'(t)\eta_2'(t)\le 0$, $t\in[0,T]$.

In assumptions (A), $w_j(\cdot,\eta_1,\eta_2)\in C[0,T]$ and for $u\in S_{\gamma,(a)}(\bar{Q})$ we have $v_j, F_j\in S_{\gamma,(a)}(\mathbb{R}^n)$ for $j=1,2$.

Substituting the expressions (6) in (4), we get

$$\begin{aligned} u(x,t) = {} & \frac{1}{d(T)}\int_0^T ds\int_{\mathbb{R}^n}\Big\{G_1(x-y,t)u(y,s)\,w_1(s,\eta_1,\eta_2) \\ & + G_2(x-y,t)u(y,s)\,w_2(s,\eta_1,\eta_2)\Big\}\,dy + u_0(x,t), \quad (x,t)\in\bar{Q}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} u_0(x,t) = {} & \int_0^t d\tau\int_{\mathbb{R}^n} G_0(x-y,t-\tau)F_0(y,\tau)\,dy \\ & + \int_{\mathbb{R}^n}\big[G_1(x-y,t)v_1(y) + G_2(x-y,t)v_2(y)\big]\,dy, \quad (x,t)\in\bar{Q}. \end{aligned} \tag{8}$$

In view of Lemma 1 and preliminary assessments, we obtain $u_0\in S_{\gamma,(a)}(\bar{Q})$ and

$$\|u_0(\cdot,t)\|_{k,(a)} \le C_0(T)\Big[T^\beta \max_{\tau\in[0,T]}\|F_0(\cdot,\tau)\|_{k,(a)} + \|\Delta\Phi_1\|_{k,(a)} + \|\Delta\Phi_2\|_{k,(a)}\Big], \quad t\in[0,T]. \tag{9}$$

Lemma 2. *In assumptions* (A) *and* (5) *the triple* $(u,F_1,F_2)\in S_{\gamma,(a)}(\bar{Q})\times S_{\gamma,(a)}(\mathbb{R}^n)\times S_{\gamma,(a)}(\mathbb{R}^n)$ *is the solution of the problem* (1) – (3) *if and only if* u *is the solution of the equation* (7) *in the space* $S_{\gamma,(a)}(\bar{Q})$, F_1,F_2 *are defined by* (6).

Proof. It was shown that the solution $u\in S_{\gamma,(a)}(\bar{Q})$ of the problem (1) – (3) satisfies the equation (7) and F_1,F_2 are defined by (6). Vice versa, let $u\in S_{\gamma,(a)}(\bar{Q})$ be the solution of the equation (7). Because (7) is the same as (4) if one substitute the expressions (6) in (4) instead of $F_1(x)$ and $F_2(x)$, then by Theorem 1 the function u satisfies the problem (1), (2). We will show that u satisfies the conditions (3) if F_1,F_2 are defined by (6).

Suppose this is not the case, and let

$$\frac{1}{T}\int_0^T u(x,t)\eta_j(t)dt = \Phi_j^*(x), \quad x \in \mathbb{R}^n, \quad j = 1,2.$$

We introduce the notations

$$H_j(x) = \frac{1}{d(T)}\int_0^T u(x,s)\, w_j(s,\eta_1,\eta_2)\, ds,$$
$$H_j^*(x) = \frac{1}{d(T)}\int_0^T u^*(x,s)\, w_j(s,\eta_1,\eta_2)\, ds, \quad j = 1,2,$$

where u^* is the solution of the equation (7) in which the expressions for v_1, v_2 in (8) has Φ_j^* instead of Φ_j, $j = 1,2$. Then (6) implies

$$\begin{aligned} B_1(x) := H_1(x) - H_1^*(x) + \frac{T}{d(T)}\int_0^T \Big\{ \big[\Delta\Phi_2(x) - \Delta\Phi_2^*(x)\big]\ (f_{2-\beta}, \eta_1) \\ - \big[\Delta\Phi_1(x) - \Delta\Phi_1^*(x)\big]\ (f_{2-\beta}, \eta_2) \Big\}\, dt = 0, \\ B_2(x) := H_2(x) - H_2^*(x) + \frac{T}{d(T)}\int_0^T \Big\{ \big[\Delta\Phi_2(x) - \Delta\Phi_2^*(x)\big]\ (f_{2-\beta}, \eta_1') \\ - \big[\Delta\Phi_1(x) - \Delta\Phi_1^*(x)\big]\ (f_{2-\beta}, \eta_2') \Big\}\, dt = 0, \quad x \in \mathbb{R}^n. \end{aligned} \tag{10}$$

It follows from (7) and (8) that

$$u(x,t) - u^*(x,t) = -\frac{1}{d(T)}\int_{\mathbb{R}^n} \Big\{ G_1(x-y,t)B_1(y) + G_2(x-y,t)B_2(y) \Big\}\, dy.$$

On the basis of (10) we obtain $u(x,t) - u^*(x,t) = 0$, $(x,t) \in \bar{Q}$. Then, by definition, $\Phi_j(x) - \Phi_j^*(x) = 0$, $x \in \mathbb{R}^n$, $j = 1,2$. Lemma brought. □

Theorem 2. *Assume that* (A), (5) *hold and*

$$\widehat{C}(T) := \frac{1}{|d(T)|}\int_0^T \big[|w_1(s,\eta_1,\eta_2)| + T|w_2(s,\eta_1,\eta_2)|\big]\, ds \tag{B}$$

is a monotonically increasing function.

Then there exists the unique solution $(u, F_1, F_2) \in S_{\gamma,(a)}(\bar{Q}) \times S_{\gamma,(a)}(\mathbb{R}^n) \times S_{\gamma,(a)}(\mathbb{R}^n)$ *of the problem* (1)–(3).

Proof. By Lemma 2 it is sufficient to prove (at least, local in time) solvability of the linear integral equation (7) and the uniqueness of the inverse problem. In fact, to find the functions F_1 and F_2 we solve the equation (7) at least with a fixed sufficiently small time value $T > 0$. Then by Theorem 1, we may find the solution of the Cauchy problem (1), (2) in $S_{\gamma,(a)}(\bar{Q})$ with every finite $T > 0$ such that

$$aT^{\frac{\beta}{2\gamma}} < c = (2-\beta)\left(\frac{\beta^\beta}{4}\right)^{\frac{1}{2-\beta}},$$

using formula (4).

We define

$$(\mathbb{K}v)(x,t) = \frac{1}{d(T)} \int_0^T ds \int_{\mathbb{R}^n} \Big\{ G_1(x-y,t)v(y,s)\, w_1(s,\eta_1,\eta_2) \\ + G_2(x-y,t)v(y,s)\, w_2(s,\eta_1,\eta_2) \Big\}\, dy + u_0(x,t), \quad (x,t) \in Q,\ v \in S_{\gamma,(a)}(\bar{Q}).$$

It was shown (see (9)), that $u_0 \in S_{k,(a)}(\bar{Q})$ for all $k \in \mathbb{N}$, $k \neq 1$ and $\|u_0\|_{k,(a)} < +\infty$. By Lemma 1, for all $v \in S_{\gamma,(a)}(\bar{Q})$, we have

$$\|\mathbb{K}v\|_{k,(a)} \leq \int_0^T \Big\{ \Big\| \int_{\mathbb{R}^n} G_1(x-y,t)v(y,s)\, dy \Big\|_{k,(a)} \Big| \frac{w_1(s,\eta_1,\eta_2)}{d(T)} \Big| \\ + \Big\| \int_{\mathbb{R}^n} G_2(x-y,t)v(y,s)\, dy \Big\|_{k,(a)} \Big| \frac{w_2(s,\eta_1,\eta_2)}{d(T)} \Big| \Big\}\, ds \\ \leq \widehat{C}(T)\|v\|_{k,(a)} + \|u_0\|_{k,(a)}.$$

By assumption (B), choosing $R > 2\|u_0\|_{k,(a)}$, we find T_0 such that $\widehat{C}(T_0) < \frac{1}{2}$ and obtain

$$\|\mathbb{K}v\|_{k,(a)} < \frac{1}{2}\|v\|_{k,(a)} + \|u_0\|_{k,(a)} \leq \frac{R}{2} + \frac{R}{2} = R, \quad \forall v \in S_{\gamma,(a)}(\bar{Q}).$$

Similarly, we get

$$\|\mathbb{K}v_1 - \mathbb{K}v_2\|_{k,(a)} < \frac{1}{2}\|v_1 - v_2\|_{k,(a)}, \quad \forall v_1, v_2 \in S_{\gamma,(a)}(\bar{Q}).$$

So, $\mathbb{K}$ is a contraction for some $T_0 > 0$ and we obtain the existence of the solution of the linear integral equation (7) in $S_{\gamma,(a)}(\bar{Q})$ with $T = T_0$, and by Lemma 2 the existence of the solution of the inverse problem (1)–(3).

Let us show the uniqueness of the solution of this problem. If (u_1, F_{11}, F_{21}), (u_2, F_{12}, F_{22}) are two solutions of the problem (1)–(3), $u = u_1 - u_2$, $F_1 = F_{11} - F_{12}$, $F_2 = F_{21} - F_{22}$, then the triple (u, F_1, F_2) satisfies the problem

$$D_t^\beta u - \Delta u = 0, \quad (x,t) \in Q, \\ u(x,0) = F_1(x), \qquad \frac{\partial}{\partial t}u(x,0) = F_2(x), \quad x \in \mathbb{R}^n, \\ \int_0^T u(x,t)\eta_1(t)dt = 0, \qquad \int_0^T u(x,t)\eta_2(t)dt = 0, \quad x \in \mathbb{R}^n.$$

By Lemma 2, every solution $u(x,t)$ of this problem is a solution of the equation (7) with $u_0(x,t) = 0$, $(x,t) \in \bar{Q}$,

$$F_1(x) = \frac{1}{d(T)} \int_0^T u(x,t)\, w_1(t,\eta_1,\eta_2)\, dt, \\ F_2(x) = \frac{1}{d(T)} \int_0^T u(x,t)\, w_2(t,\eta_1,\eta_2)\, dt, \quad x \in \mathbb{R}^n. \tag{11}$$

According to the above, there exists $T_0 > 0$ such that the function $u(x,t) = 0$, $x \in \mathbb{R}^n$, $t \in [0, T_0]$, is the unique solution of the obtained equation. Then (11) implies $F_j = 0$ in $S_{\gamma,(a)}(\mathbb{R}^n)$, $j = 1, 2$. □

The obtained in Theorem 2 result can be strengthened.

We define the weighted norms $\|v\|_{k,(a),\sigma} = \sup\limits_{t\in(0,T]} e^{-\sigma t} \|v(\cdot,t)\|_{k,(a)}$ in $S_{\gamma,(a)}(\bar{Q})$. We have the equivalence of norms $\|v\|_{k,(a),\sigma}$ and $\|v\|_{k,(a)}$, namely

$$e^{-\sigma T}\|v\|_{k,(a)} \le \|v\|_{k,(a),\sigma} \le \|v\|_{k,(a)}.$$

Lemma 3. *If $\gamma \ge 1$, $0 < aT^{\frac{\beta}{2\gamma}} < c$, $\sigma > 0$, $\varphi \in S_{\gamma,(a)}(\bar{Q})$, $b \in C[0,T]$, then there exist positive numbers $C_\sigma, C_{\sigma,1}, C_{\sigma,2}$ such that for all $k \in \mathbb{N}$, $k \neq 1$, $t \in (0,T]$ the following bounds*

$$\left\| \int_0^t (G_0\varphi)(\cdot, t-\tau)\, d\tau \right\|_{k,(a),\sigma} = \sup_{t\in(0,T]} e^{-\sigma t} \left\| \int_0^t (G_0\varphi)(\cdot, t-\tau)\, d\tau \right\|_{k,(a)} \le C_\sigma \|\varphi\|_{k,(a),\sigma},$$

$$\begin{aligned}\left\| \int_0^T (G_j\varphi)(x,t,s) b(s)\, ds \right\|_{k,(a),\sigma} &= \sup_{t\in(0,T]} e^{-\sigma t} \left\| \int_0^T \left(\int_{\mathbb{R}^n} G_j(x-y,t)\varphi(y,s)\, dy \right) b(s)\, ds \right\|_{k,(a)} \\ &\le C_{\sigma,j} T^{j-1} \|\varphi\|_{k,(a),\sigma}, \quad j = 1,2,\end{aligned}$$

hold and $C_\sigma \to 0$ as $\sigma \to +\infty$.

Proof. By using Lemma 1, we get

$$\begin{aligned}
&\sup_{t\in(0,T]} e^{-\sigma t} \left\| \int_0^t (G_0\varphi)(\cdot, t-\tau)\, d\tau \right\|_{k,(a)} \\
&\quad = \sup_{t\in(0,T]} \left\| \int_0^t e^{-\sigma(t-\tau)} \left(\int_{\mathbb{R}^n} G_0(x-y,t-\tau) e^{-\sigma\tau} \varphi(y,\tau)\, dy \right) d\tau \right\|_{k,(a)} \\
&\quad \le C \sup_{t\in(0,T]} \int_0^t (t-\tau)^{\beta-1} e^{-\sigma(t-\tau)}\, d\tau \sup_{\tau\in(0,T]} e^{-\sigma\tau} \|\varphi(\cdot,\tau)\|_{k,(a)} \\
&\quad = C \sup_{t\in(0,T]} \int_0^t (t-\tau)^{\beta-1} e^{-\sigma(t-\tau)}\, d\tau \|\varphi\|_{k,(a),\sigma} \\
&\quad = C \sup_{t\in(0,T]} \int_0^t \tau^{\beta-1} e^{-\sigma\tau}\, d\tau \|\varphi\|_{k,(a),\sigma} = C_\sigma \|\varphi\|_{k,(a),\sigma},
\end{aligned}$$

$$\begin{aligned}
&\left\| \int_0^T (G_j\varphi)\,(x,t,s) b(s)\, ds \right\|_{k,(a),\sigma} \\
&\quad = \sup_{t\in(0,T]} \left\| \int_0^T e^{-\sigma(t-s)} \left(\int_{\mathbb{R}^n} G_j(x-y,t) b(s) e^{-\sigma s} \varphi(y,s)\, dy \right) ds \right\|_{k,(a)} \\
&\quad \le C \sup_{t\in(0,T]} \left[t^{j-1} \int_0^T e^{-\sigma(t-s)} |b(s)|\, ds \right] \sup_{s\in(0,T]} \left[e^{-\sigma s} \|\varphi(\cdot,s)\|_{k,(a)} \right] \\
&\quad = C \sup_{t\in(0,T]} \left[t^{j-1} e^{-\sigma t} \right] \int_0^T e^{\sigma s} |b(s)|\, ds \|\varphi\|_{k,(a),\sigma} \\
&\quad \le C T^{j-1} \int_0^T e^{\sigma s} |b(s)|\, ds \|\varphi\|_{k,(a),\sigma} = C_{\sigma,j} \|\varphi\|_{k,(a),\sigma}, \quad j = 1,2.
\end{aligned}$$

Here

$$C_\sigma = C \int_0^T \tau^{\beta-1} e^{-\sigma\tau}\, d\tau, \qquad C_{\sigma,j} = C T^{j-1} \int_0^T e^{\sigma\tau} |b(\tau)|\, d\tau, \quad j = 1,2.$$

□

Corollary. *Assume that* (A), (5) *hold. As in the proof of Theorem 2, but using Lemma 3, we obtain*

$$\begin{aligned}\|\mathbb{K}v\|_{k,(a),\sigma} &\leq \frac{C}{|d(T)|}\|v\|_{k,(a),\sigma}\int_0^T e^{\sigma s}\Big\{|w_1(s,\eta_1,\eta_2)| + T|w_2(s,\eta_1,\eta_2)|\Big\}\,ds + \|u_0\|_{k,(a),\sigma} \\ &\leq \widehat{C}(T)e^{\sigma T}\|v\|_{k,(a),\sigma} + \|u_0\|_{k,(a),\sigma}.\end{aligned}$$

If the condition (B) is not fulfilled, but the function $\widehat{C}(T)$ is bounded by a constant independent of T, or if the function $\widehat{C}(T)e^{\sigma T}$, at least for sufficiently large $\sigma = \sigma_0$, is monotonically increasing, then, according to proof of Theorem 2, there exist $T_0 \in (0, T]$ such that for all $T \in (0, T_0]$ there exists a unique solution of the linear integral equation (7). Consequently, there exists a unique solution $(u, F_1, F_2) \in S_{\gamma,(a)}(\bar{Q}) \times S_{\gamma,(a)}(\mathbb{R}^n) \times S_{\gamma,(a)}(\mathbb{R}^n)$ of the problem (1) – (3).

Conclusions. Sufficient conditions for the unique classical solvability of the inverse problem of determining the initial values (from Schwartz-type spaces of functions rapidly decreasing to zero at infinity) of the solution for a time-fractional diffusion-wave equation are found. The solving of the problem is reduced to solving the linear Fredholm integral equation of the second kind with integrable kernel.

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Знайдено достатні умови однозначної класичної розв'язності оберненої задачі відновлення двох функцій у початкових умовах задачі Коші для дифузійно-хвильового рівняння з дробовою похідною Капуто-Джрбашяна-Нерсесіана та правою частиною зі значеннями в просторах гладких функцій типу Шварца, що швидко спадають до нуля на нескінченності.

Ми використовуємо дві інтегральні за часом умови перевизначення

$$\frac{1}{T}\int_0^T u(x,t)\eta_1(t)dt = \Phi_1(x), \quad \frac{1}{T}\int_0^T u(x,t)\eta_2(t)dt = \Phi_2(x), \quad x \in \mathbb{R}^n,$$

де u — розв'язок задачі Коші для такого рівняння, Φ_1, Φ_2 — задані функції з простору типу Шварца, η_1, η_2 — задані функції з $C^2[0,T]$.

Використовуємо метод вектор-функції Гріна. Шукані початкові дані виражаються через розв'язок деякого лінійного інтегрального рівняння Фредгольма другого роду в просторі неперервних функцій зі значеннями в просторах типу Шварца.

Ключові слова і фрази: функціональний простір типу Шварца, дробова похідна, задача Коші, обернена задача, вектор-функція Гріна.