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Epsilon-omega order on the set of natural numbers

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The limit of the sequence $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$, denoted ε_0, is the smallest epsilon number, i.e. ordinal number α such that $\omega^\alpha = \alpha$. While all the epsilon numbers with countable indices are known to be order types of countable sets, it is a highly counter-intuitive property. The goal of this paper is to provide better insight into the structure of epsilon ordinals by constructing $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\omega$ orders on the set of natural numbers. The presented construction can be easily extended to even larger epsilon numbers.

Key words and phrases: ordinal number, epsilon-omega, epsilon number, well-ordered set.

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Introduction

Ordinal numbers were constructed by Georg Cantor in 1883 [1]. They are used to describe the order types of well-ordered sets. Every such set is order-isomorphic to exactly one ordinal number. Each set consisting of ordinals is also well-ordered, in particular the set $\{\beta : \beta < \alpha\}$ has order type α.

Ordinal 0 is the order type of the empty set. Natural numbers $1, 2, 3, \ldots$ describe order types of finite well-ordered sets. The order type of natural numbers is denoted ω. It is the smallest countable ordinal. Examples of larger ordinals are $\omega + 1, \omega \cdot 2, \omega \cdot 3, \omega^2, \omega^\omega$, ω^{ω^ω} and ε_0, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. The ε_0 is the smallest ordinal α such that $\omega^\alpha = \alpha$. Ordinals satisfying this property are called epsilon numbers. The next such ordinal is $\varepsilon_1 = \sup\{\varepsilon_0, \varepsilon_0^{\varepsilon_0}, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}, \ldots\}$, followed by $\varepsilon_2 = \sup\{\varepsilon_1, \varepsilon_1^{\varepsilon_1}, \varepsilon_1^{\varepsilon_1^{\varepsilon_1}}, \ldots\}$ etc. The supremum of the set $\{\varepsilon_n : n \in \mathbb{N}\}$ is also an epsilon number, denoted ε_ω.

All the ordinals mentioned above are countable, i.e. they describe order types of countable sets. It follows from the fact that ordinal arithmetic (described in Section 1) differs from the cardinal arithmetic.

In particular, ε_0 can be described as a countable union of countable sets. Also, it is the order type of the set of finite rooted trees (see [4]). Nonetheless, the most straightforward way to visualise its countability would be to construct an example ε_0 order on the set of positive integers $\mathbb{N}_1$. Such example is presented in Section 2.

In Section 3, we construct $\varepsilon_1, \varepsilon_2, \ldots$ orders on the set $\mathbb{N}_1$, by generalizing the method from Section 2. Finally, Section 4 describes an example ε_ω order on the set of positive integers. We stop there, pointing out that the construction can be continued for even larger epsilon ordinals.

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1 Basic definitions

In this section we recall basic notions and properties of ordinal numbers (see [2,5]).

1.1 Successor and limit ordinals

If there exists β such that α is the smallest ordinal larger than β, we say that α is a successor ordinal and write $\alpha = \beta + 1$. Otherwise α is a limit ordinal, equal to a supremum of all ordinals smaller than itself. Zero is an exception, considered neither a successor nor a limit ordinal.

1.2 Ordinal arithmetic

Let $(A, <_A)$ and $(B, <_B)$ be sets of order types α and β, respectively. We briefly recall the notions of addition, multiplication and exponentiation of ordinal numbers.

Addition

We define an order on the disjoint union $A \sqcup B$. For each $x, y \in A \sqcup B$ we write $x < y$ if one of the following conditions is met:

- $x, y \in A$ and $x <_A y$,
- $x, y \in B$ and $x <_B y$,
- $x \in A$ and $y \in B$.

Then the set $(A \sqcup B, <)$ is well-ordered and has order type $\alpha + \beta$.

Remark 1. *The ordinal addition is **left-cancellative**. That is, for any ordinals α, β and γ, if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.*

Multiplication

We define an order on $A \times B$ as follows. For any $(x_A, x_B), (y_A, y_B) \in A \times B$ we say that $(x_A, x_B) < (y_A, y_B)$ if either of the conditions is met:

- $x_B <_B y_B$,
- $x_B = y_B$ and $x_A <_A y_B$.

Then the set $(A \times B, <)$ is well-ordered and has order type $\alpha \cdot \beta$.

Remark 2. *Ordinal multiplication is **left-distributive** over addition. That is, for any ordinals α, β and γ, we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.*

Exponentiation

Let 0_A be the smallest element of A. We denote A^B as the set of all functions $f : B \to A$ with finite support, i.e. there is only a finite number of $x \in B$ such that $f(x) \neq 0_A$. We order them lexicographically, i.e. for $f, g \in A^B$ we say that $f < g$ if for the largest x such that $f(x) \neq g(x)$, we have $f(x) < g(x)$. Then the set $(A^B, <)$ is well-ordered and has order type α^β.

For an ordinal number $\alpha > 0$, exponentiation α^γ can also be defined inductively (over γ):

- $\alpha^0 = 1$,
- $\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha$,
- $\alpha^\gamma = \sup \{\alpha^\delta : \delta < \gamma\}$ if γ is a limit ordinal.

1.3 Initial segment

Let A be a non-empty well-ordered set and let $a \in A$. We say that the set $\{x \in A : x < a\}$ is an initial segment of A corresponding to an element a. It has a smaller order type than A.

1.4 Induced order

Let A, B be non-empty sets. Additionally, we assume that B is a well-ordered set with order type β. Given a bijection $f : A \to B$, we can define a β order on A as follows

$$x < y \text{ in } A \iff f(x) < f(y) \text{ in } B.$$

We will say that such order is induced on A by f.

2 Construction of the epsilon-zero order

We are going to write $\mathbb{N}$ for the set of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{N}_1$ for the set of positive natural numbers. Both of these sets have a natural order type ω.

Consider the set $\mathbb{N}^{\mathbb{N}_1}$ of all functions $f : \mathbb{N}_1 \to \mathbb{N}$ with finite support, ordered lexicographically. By definition of ordinal exponentiation, $\mathbb{N}^{\mathbb{N}_1}$ has order type ω^ω. Each function $f \in \mathbb{N}^{\mathbb{N}_1}$ can be expressed as a sequence $(f(1), f(2), f(3), \ldots)$. Therefore $\mathbb{N}^{\mathbb{N}_1}$ consists of all sequences of natural numbers with only a finite number of non-zero terms.

We say that $(x_1, x_2, \ldots) < (y_1, y_2, \ldots)$ in $\mathbb{N}^{\mathbb{N}_1}$ if and only if for the largest n such that $x_n \neq y_n$, we have $x_n < y_n$, i.e.

$$\begin{aligned}(0,0,0,\ldots) &< (1,0,0,\ldots) < (2,0,0,\ldots) < \ldots < (0,1,0,\ldots)\\ &< (1,1,0,\ldots) < (2,1,0,\ldots) < (3,1,0,\ldots) < \ldots < (0,2,0,\ldots)\\ &< (1,2,0,\ldots) < \ldots < (0,3,0,\ldots) < \ldots < (0,0,1,0,\ldots) < \ldots .\end{aligned}$$

Now let $(p_1, p_2, p_3, \ldots) = (2, 3, 5, \ldots)$ be an increasing sequence of all prime numbers. Each $n \in \mathbb{N}_1$ has unique prime factorization

$$n = \prod_{i=1}^{\infty} p_i^{q_i},$$

where $q(n) = (q_1, q_2, \ldots)$ is an element of $\mathbb{N}^{\mathbb{N}_1}$. Conversely, each sequence in $\mathbb{N}^{\mathbb{N}_1}$ serves as the sequence of exponents of prime factorization for a unique $n \in \mathbb{N}_1$.

Remark 3. *The one-to-one correspondence between the set of natural numbers and a given countable set of mathematical objects is called Gödel numbering. The method was used by K. Gödel in [3].*

The bijection $q : \mathbb{N}_1 \to \mathbb{N}^{\mathbb{N}_1}$ will be called the prime factorization function. The function q induces ω^ω ordering on $\mathbb{N}_1$. We will denote the set $\mathbb{N}_1$ with such order as $\mathbb{N}_2$, namely

$$1 < p_1 < p_1^2 < \ldots < p_2 < p_2 \cdot p_1 < p_2 \cdot p_1^2 < \ldots < p_2^2 < p_2^2 \cdot p_1 < \ldots < p_2^3 < \ldots < p_3 < \ldots .$$

Proposition 1. *To compare two distinct numbers $k, n \in \mathbb{N}_2$, first we identify the largest prime p_i that has different exponents in prime factorization of k and n. Then we compare these exponents. The number with a larger exponent is larger in $\mathbb{N}_2$.*

Now let us consider the set $\mathbb{N}^{\mathbb{N}_2}$ with lexicographic order. It has order type ω^{ω^ω}. Since the set $\mathbb{N}^{\mathbb{N}_2}$ consists of exactly the same elements as the set $\mathbb{N}^{\mathbb{N}_1}$, the prime factorization function q serves also as a bijection of $\mathbb{N}_1$ and $\mathbb{N}^{\mathbb{N}_2}$. Hence, it induces ω^{ω^ω} ordering on $\mathbb{N}_1$. We denote $\mathbb{N}_1$ with this ordering as $\mathbb{N}_3$, i.e.

$$
\begin{aligned}
1 < p_1 < p_1^2 < \ldots < p_{p_1} < p_{p_1} \cdot p_1 < p_{p_1} \cdot p_1^2 < \ldots \\
< p_{p_1}^2 < p_{p_1}^2 \cdot p_1 < \ldots < p_{p_1}^3 < \ldots < p_{p_1{}^2} < \ldots
\end{aligned}
$$

We will use the notation ${}^n\omega := \omega^{\omega^{\cdot^{\cdot^{\omega}}}}$, with n copies of ω on the right side of the equation (for example, ${}^3\omega = \omega^{\omega^\omega}$). Inductively, for each natural $n \geq 3$, we denote by $\mathbb{N}_{n+1}$ the set $\mathbb{N}_1$ with order type ${}^{n+1}\omega$, induced by the prime factorization function $q : \mathbb{N}_1 \to \mathbb{N}^{\mathbb{N}_n}$.

Proposition 2. *The procedure of comparing two numbers $k, m \in \mathbb{N}_{n+1}$ is as follows.*

1. *Find the largest index i (according to the order in $\mathbb{N}_n$) such that the exponents $q_i(k)$ and $q_i(m)$ in prime factorization of k and n are different.*
2. *Compare these exponents (with natural order in $\mathbb{N}$). The number with the larger exponent is the larger number in $\mathbb{N}_{n+1}$.*

Lemma 1. *Let A be an initial segment of $\mathbb{N}_n$ with order type α, corresponding to an element a. We denote by $P(A)$ the set of all positive integers m such that $q_i(m) = 0$ for all $i \notin A$. Then $P(A)$ is an initial segment of $\mathbb{N}_{n+1}$ with order type ω^α, corresponding to an element p_a.*

Proof. The set $P(A) \subset \mathbb{N}_{n+1}$ corresponds to the set $F_A \subset \mathbb{N}^{\mathbb{N}_n}$ of functions whose support is a finite subset of A. Naturally, F_A is order-isomorphic to the set of finite-support functions from A to $\mathbb{N}$, which has order type ω^α. Hence, it is also the order type of $P(A)$.

The element p_a, according to its prime factorization, corresponds to the function $i \mapsto q_i(a)$ from $\mathbb{N}^{\mathbb{N}_n}$, where

$$
q_i(a) = \begin{cases} 1, & \text{for } \ i = a, \\ 0, & \text{for } \ i \neq a. \end{cases}
$$

Observe that F_A is the set of all functions smaller than the above function, hence it is an initial segment of $\mathbb{N}^{\mathbb{N}_n}$. Therefore $P(A)$ is an initial segment of $\mathbb{N}_{n+1}$, corresponding to p_a. $\square$

Lemma 2. *Let $n \geq 1$ and A be an initial segment of $\mathbb{N}_n$ and $\mathbb{N}_{n+1}$. Then A is an initial segment of $\mathbb{N}_m$ for all $m \geq n$.*

Proof. By Lemma 1, $P(A)$ is an initial segment of $\mathbb{N}_{n+1}$ and $\mathbb{N}_{n+2}$, with an order type ω^α. Since the set $\{p_i : i \in A\}$ is a subset of $P(A)$ with α order, we obtain $\omega^\alpha \geq \alpha$.

Both A and $P(A)$ are initial in $\mathbb{N}_{n+1}$, hence $A \subset P(A)$, which is initial in $\mathbb{N}_{n+2}$. Therefore A is an initial segment of $\mathbb{N}_{n+2}$. This allows us to repeat the same reasoning for $n+1$ instead of n and by induction, this finishes the proof. $\square$

We denote $P_1 := P(\{1\})$ and $P_n := P(P_{n-1})$ for $n \geq 2$. Note that $\{1\}$ is an initial segment of both $\mathbb{N}_1$ and $\mathbb{N}_2$, with order type 1. By Lemma 2, it appears as an initial segment in all $\mathbb{N}_n$, $n \geq 1$. Hence, by Lemma 1, $P_1 = \{1, p_1, p_1^2, p_1^3, \ldots\}$ is an initial segment of all $\mathbb{N}_n$ for $n \geq 2$. It has order type $\omega^1 = \omega$. By induction, P_k is an initial segment of all $\mathbb{N}_n$ such that $n \geq k+1$, with order type ${}^k\omega$.

2.1 Epsilon-zero order

Note that $P_1, P_2, P_3, \dots$ is a sequence of well-ordered sets such that P_k is an initial segment of P_{k+1} for each $k \geq 1$. Hence, the set

$$\mathbb{N}_\omega := \bigcup_{k=1}^{\infty} P_k$$

is also well-ordered. Since each P_k has order type ${}^k\omega$, then $\mathbb{N}_\omega$ has order type ε_0, i.e. the supremum of all ${}^k\omega$.

To show that $\mathbb{N}_\omega$ consists of all positive integers, we need to define recursive prime factorization.

Definition 1. *Let $n > 1$ be a natural number. Rewrite it using the prime factorization as $n = p_{n_1}^{q_{n_1}} \cdot p_{n_2}^{q_{n_2}} \cdot \dots \cdot p_{n_k}^{q_{n_k}}$ with $q_{n_i} > 0$ for each $1 \leq i \leq k$.*

Now consider the indices n_i. If some of them are greater than 1, rewrite them using prime factorization again. Continue until all of the obtained subindices are equal to 1. This procedure will be called recursive prime factorization.

Since for each prime number p_i we have $i < p_i$, the indices decrease in each step of the above algorithm. It means that for any given n, the procedure eventually terminates.

Definition 2. *For $n > 1$, let s_n denote the number of necessary steps of recursive prime factorization for a given n. We also define $s_1 := 1$.*

For example,

$$52 \overset{I}{=} 2^2 \cdot 13 = p_1^2 \cdot p_6 \overset{II}{=} p_1^2 \cdot p_{2\cdot 3} = p_1^2 \cdot p_{p_1 \cdot p_2} \overset{III}{=} p_1^2 \cdot p_{p_1 \cdot p_{p_1}},$$

hence $s_{52} = 3$.

Then, by definition of the sets P_k, for each $n \geq 1$ we have $n \in P_{s_n} \subset \mathbb{N}_\omega$, which was to be shown. Moreover, $k = s_n$ is the smallest index such that $n \in P_k$.

Proposition 3. *To compare two numbers $m, n \in \mathbb{N}_\omega$, we start with comparing s_m and s_n:*

- *if $s_m < s_n$ (in natural order), then $m < n$ in $\mathbb{N}_\omega$;*
- *if $s_m = s_n = k$, compare m and n in $\mathbb{N}_{k+1}$.*

Remark 4. *To obtain the successor of a given n in any of the $\mathbb{N}_2, \mathbb{N}_3, \dots, \mathbb{N}_\omega$ orders, we need to replace the prime factorization exponent $q_1(n)$ (corresponding to the smallest prime $p_1 = 2$) with its successor in $\mathbb{N}_0$, which is $q_1(n) + 1$. Hence the successor of n in any of these orders is equal to $2n$.*

3 Between epsilon-zero and epsilon-omega

Using similar methods as in the previous section, we define ε_n order on the set of positive natural numbers, for any given $n \in \mathbb{N}_1$. We will use the following notation:

- ${}^2\varepsilon_n = \varepsilon_n^{\varepsilon_n}$, ${}^3\varepsilon_n = \varepsilon_n^{\varepsilon_n^{\varepsilon_n}}$, etc.,
- $\mathbb{N}_{k,0}$ – the set of natural numbers with order type ε_k and initial element 0,
- $\mathbb{N}_{k,m}$ – the set of positive natural numbers with ${}^m\varepsilon_k$ order and initial element 1.

We are going to construct the sets $\mathbb{N}_{k,m}$ for each $k \in \mathbb{N}$ and $m \in \mathbb{N}_1$. We start with:

- $\mathbb{N}_{0,1} := \mathbb{N}_\omega$,
- $\mathbb{N}_{0,0} := \{0\} \sqcup \mathbb{N}_\omega$, i.e. the set $\mathbb{N}_\omega$ with 0 added as an initial element.

Proposition 4. *Suppose that for a given $k \in \mathbb{N}$ and $m \in \mathbb{N}_1$ we have constructed the sets $\mathbb{N}_{k,0}$ and $\mathbb{N}_{k,m}$. Then the set $(\mathbb{N}_{k,0})^{\mathbb{N}_{k,m}}$ of functions $f : \mathbb{N}_{k,m} \to \mathbb{N}_{k,0}$ with finite support has order type $\varepsilon_k^{({}^m\varepsilon_k)} = {}^{m+1}\varepsilon_k$. Moreover, the prime factorization function $q : \mathbb{N}_1 \to \mathbb{N}_{k,0}^{\mathbb{N}_{k,m}}$ induces ${}^{m+1}\varepsilon_k$ order on the set $\mathbb{N}_1$. The resulting set will be denoted as $\mathbb{N}_{k,m+1}$. Since 0 is initial element of $\mathbb{N}_{k,0}$, the initial element of $\mathbb{N}_{k,m+1}$ is equal to $p_1^0 \cdot p_2^0 \cdot \ldots = 1$.*

Having defined the sets $\mathbb{N}_{0,0}$ and $\mathbb{N}_{0,1}$ with order type ε_0, by Proposition 4 we construct the set $\mathbb{N}_{0,2}$ with $\varepsilon_0^{\varepsilon_0}$ order. By induction, for each natural $n \geq 2$ we obtain the set $\mathbb{N}_{0,n+1}$ with order type ${}^{n+1}\varepsilon_0$.

Proposition 5. *To compare two distinct numbers $k, n \in \mathbb{N}_{0,2}$, we find the largest index i in $\mathbb{N}_{0,1}$ order, such that the prime factorization exponents $q_i(k)$ and $q_i(n)$ are different. If $q_i(k) > q_i(n)$ in $\mathbb{N}_{0,0}$, then $k > n$ in $\mathbb{N}_{0,2}$.*

What distinguishes the order in $\mathbb{N}_{0,2}$ from $\mathbb{N}_1, \mathbb{N}_2, \ldots, \mathbb{N}_\omega$ orders, is that it uses recursive prime factorization not only for the i indices of prime numbers, but also for the exponents. In $\mathbb{N}_{0,m}$ with $m > 2$ the order is similar, but with more layers of recursion. Namely, the i indices are compared in $\mathbb{N}_{0,m-1}$ instead of $\mathbb{N}_{0,1}$.

As in the previous section, the initial segments play a key role in the next step the construction. Therefore, we need the following two lemmas.

Lemma 3. *For a given $k \in \mathbb{N}$ and $n \in \mathbb{N}_1$, let A be an initial segment of $\mathbb{N}_{k,n}$ with order type α, corresponding to the element a.*

We denote by $P^{(k)}(A)$ the set of all positive integers m such that $q_i(m) = 0$ for all $i \notin A$. Then $P^{(k)}(A)$ is an initial segment of $\mathbb{N}_{k,n+1}$ with order type $(\varepsilon_k)^\alpha$, corresponding to the element p_a.

Lemma 4. *Let $n \geq 1$ and A be an initial segment of $\mathbb{N}_{k,n}$ and $\mathbb{N}_{k,n+1}$. Then A is an initial segment of $\mathbb{N}_{k,m}$ for all $m \geq n$.*

Proof. The proofs of Lemmas 3 and 4 are analogous to the proofs of Lemmas 1 and 2, respectively. Only small and straightforward modifications are necessary, i.e. replacing ω with ε_k, and each $\mathbb{N}_i$ with $\mathbb{N}_{k,i}$. □

Suppose that for a given $k \in \mathbb{N}$ we have constructed the sets $\mathbb{N}_{k,0}$ and $\mathbb{N}_{k,1}$ (hence, by Proposition 4, we obtain the sets $\mathbb{N}_{k,m}$ for all $m \in \mathbb{N}_1$). Under this assumption, let us introduce the notation $P_1^{(k)} := P^{(k)}(\{1\})$ and $P_n^{(k)} := P^{(k)}(P_{n-1}^{(k)})$ for all $n \geq 2$. Since $\{1\}$ has order type 1 and is an initial segment of $\mathbb{N}_{k,m}$ for all $m \geq 1$, then, by Lemma 3, $P_1^{(k)}$ is initial in $\mathbb{N}_{k,m}$ for all $m \geq 2$. By Lemma 4, it has order type $\varepsilon_k^1 = \varepsilon_k$. Inductively, $P_n^{(k)}$ has order type ${}^n\varepsilon_k$ and is an initial segment of $\mathbb{N}_{k,m}$ for all $m \geq n+1$.

Hence $P_1^{(k)}, P_2^{(k)}, \ldots$ is a sequence of well-ordered sets with each $P_n^{(k)}$ being the initial segment of $P_{n+1}^{(k)}$.

Definition 3. *We define the set*

$$\mathbb{N}_{k,\omega} := \bigcup_{n=1}^{\infty} P_n^{(k)}.$$

The order type of $\mathbb{N}_{k,\omega}$ is a supremum of order types of the sets $P_n^{(k)}$. Each $P_n^{(k)}$ has ${}^n\varepsilon_k$ order, hence $\mathbb{N}_{k,\omega}$ has order type ε_{k+1}.

Note that the beginning element of $\mathbb{N}_{k,\omega}$ is 1. More importantly, by definition, each set $P_n^{(k)}$ consists of exactly the same elements as the set P_n from the previous section. Hence, $\mathbb{N}_{k,\omega}$ (just like $\mathbb{N}_\omega$) consists of all positive integers and its order is strongly related to recursive prime factorization (see Definition 1).

Since we have already defined the sets $\mathbb{N}_{0,m}$ for all $m \in \mathbb{N}_1$, we can construct the set $\mathbb{N}_{0,\omega}$ with order type ε_1.

Proposition 6. *Having constructed the set $\mathbb{N}_{k,\omega}$ for a given $k \in \mathbb{N}$, we define*

- $\mathbb{N}_{k+1,1} := \mathbb{N}_{k,\omega}$,
- $\mathbb{N}_{k+1,0} := \{0\} \sqcup \mathbb{N}_{k,\omega}$, *i.e. the set $\mathbb{N}_{k,\omega}$ with 0 added as an initial element.*

Hence we obtain the sets $\mathbb{N}_{1,0}$ and $\mathbb{N}_{1,1}$. Proposition 4 allows us to construct the sets $\mathbb{N}_{1,m}$ for all $m \geq 2$. Then, by Definition 3, we obtain the set $\mathbb{N}_{1,\omega}$, and by Proposition 6, we can define $\mathbb{N}_{2,0}$ and $\mathbb{N}_{2,1}$. By repeating this procedure inductively, we obtain the sets $\mathbb{N}_{k,m}$ for all $k, m \in \mathbb{N}$.

3.1 Description of the orders

Recall that for each $m \geq 1$ and $k \geq 0$ the set $\mathbb{N}_{k,m}$ has order type ${}^m\varepsilon_k$. In particular, note that each $\mathbb{N}_{k,1}$ has order type ε_k.

All of the defined orders are based on recursive prime factorization (see Definition 1) and the number of needed steps of recursion s_n (see Definition 2).

Let r, t be distinct positive natural numbers.

Proposition 7. *Let $k \in \mathbb{N}$ and $m \geq 2$. To compare r and t in $\mathbb{N}_{k,m}$:*

1) *find the largest index $i \in \mathbb{N}_{k,m-1}$ such that the prime factorization exponents $q_i(r)$ and $q_i(t)$ are different,*
2) *compare these exponents according to $\mathbb{N}_{k,0}$ order.*

The number with the larger exponent is larger in $\mathbb{N}_{k,m}$.

Remark 5. *Since $P_1^{(k)} \subset P_2^{(k)} \subset \ldots \subset P_{m-1}^{(k)}$ are initial segments of $\mathbb{N}_{k,m}$ and each $P_n^{(k)}$ consists exactly of the same elements as the set P_n from the previous section, we can use another criterion of comparison. Namely,*

$$s_r < s_t < m \text{ in } \mathbb{N}_1 \implies r < t \text{ in } \mathbb{N}_{k,m}.$$

Proposition 8. *Let $k \in \mathbb{N}_1$. To compare $r, t \in \mathbb{N}_{k,1}$:*

- *if $s_r < s_t$ (in natural order), then $r < t$ in $\mathbb{N}_{k,1}$;*
- *if $s_r = s_t = n$, compare r and t in $\mathbb{N}_{k-1,n+1}$.*

Remark 6. *The order in* $\mathbb{N}_{k,0}$ *is the same as in* $\mathbb{N}_{k,1}$ *with* 0 *added as the initial element.*

Remark 7. *Let* $k \geq 1$. *The successor of a given* n *in any of the* $\mathbb{N}_{k-1,2}, \mathbb{N}_{k-1,3}, \ldots,$ *and* $\mathbb{N}_{k,1}$ *orders is obtained by replacing the prime factorization exponent* $q_1(n)$ *(corresponding to the prime* $p_1 = 2$*) with its successor in* $\mathbb{N}_{k-1,0}$.

Corollary 1. *Since in each* $\mathbb{N}_{k,0}$ *the initial element* 0 *is followed by* 1*, then by Remark 7, the successor of every odd number* $n \geq 1$ *is equal to* $2n$ *in* $\mathbb{N}_{k,m}$ *for all* $k, m \geq 0$.

For even numbers, the successors vary between different $\mathbb{N}_{k,m}$. For example, the successor of $256 = 2^8 = 2^{2^3}$:

- in $\mathbb{N}_{0,1}$ is equal to $2 \cdot 256 = 2^9$,
- in $\mathbb{N}_{1,1}$ is equal to $2^{2 \cdot 8} = 2^{16}$,
- in $\mathbb{N}_{2,1}, \mathbb{N}_{3,1}, \ldots$ is equal to $2^{2^{2 \cdot 3}} = 2^{2^6} = 2^{64}$.

4 Epsilon-omega order (and beyond)

In this section, we present the ε_ω order on the set of positive integers. The construction is based on $\mathbb{N}_{k,m}$ orders from the previous section and once again focuses on their initial segments, but requires a slightly different approach.

Recall that for a given $n \geq 1$, the sets $P_n^{(k)}$ consist of exactly the same elements for all $k \geq \mathbb{N}$.

Definition 4. *We define the following sets:*

- $B_1 := P_1^{(1)}$,
- $B_k := P_k^{(k)} \setminus P_{k-1}^{(k)}$ *for* $k \geq 2$.

Remark 8. *Note that for each* $k \in \mathbb{N}_1$*, we have*

$$B_k = \{n \in \mathbb{N}_{k,1} \mid s_n = k\}.$$

Therefore each positive natural number n *belongs to the set* B_{s_n} *and is not included in* B_k *for all* $k \neq s_n$.

Let us calculate the order type β_k of the set B_k for each $k \in \mathbb{N}_1$. Obviously, $\beta_1 = \varepsilon_1$. Now let $k \geq 2$. Since $P_k^{(k)} = P_{k-1}^{(k)} \sqcup B_k$ with $P_{k-1}^{(k)}$ as the initial segment, we have

$$^{k-1}\varepsilon_k + \beta_k = {}^k\varepsilon_k. \tag{1}$$

Note that by left-distributivity of ordinal multiplication over addition (see Remark 2) and the fact that $1 + \omega = \omega$ (adding a new initial element does not affect the ω order type), the following equality

$$^{k-1}\varepsilon_k + {}^{k-1}\varepsilon_k \cdot \omega = {}^{k-1}\varepsilon_k \cdot (1 + \omega) = {}^{k-1}\varepsilon_k \cdot \omega \tag{2}$$

holds. Since $^{k-1}\varepsilon_k \cdot \omega < {}^k\varepsilon_k$, each set with $^k\varepsilon_k$ order has an initial segment with order type $^{k-1}\varepsilon_k \cdot \omega$. Hence, and by (2), we get

$$^{k-1}\varepsilon_k + {}^k\varepsilon_k = {}^k\varepsilon_k. \tag{3}$$

Similarly,

$$^{k-1}\varepsilon_{k-1} + {}^k\varepsilon_k = {}^k\varepsilon_k. \tag{4}$$

From equations (1) and (3) and by left-cancellativity of ordinal addition (see Remark 1), we obtain $\beta_k = {}^k\varepsilon_k$.

4.1 The epsilon-omega order

We define the set

$$\mathbb{N}_{\omega,1} := \bigsqcup_{k=1}^{\infty} B_k.$$

In this set:

- B_1 is an initial segment,
- for each $k \in \mathbb{N}_1$, the segment B_k is followed by the segment B_{k+1}.

Remark 9. *The order type of $\mathbb{N}_{\omega,1}$ is equal to the supremum of order types of the sets $B_1 \sqcup \ldots \sqcup B_k$ with $k \in \mathbb{N}_1$.*

The set B_1 has ε_1 order, $B_1 \sqcup B_2$ has $\varepsilon_1 + {}^2\varepsilon_2 = {}^2\varepsilon_2$ order, and inductively, for all $k \geq 1$ the set $B_1 \sqcup \ldots \sqcup B_k$ has order type ${}^k\varepsilon_k$. Note that for each $k \in \mathbb{N}_1$ we have $\varepsilon_k < {}^k\varepsilon_k < \varepsilon_{k+1}$. Therefore

$$\sup\{\varepsilon_1, {}^2\varepsilon_2, {}^3\varepsilon_3, \ldots\} = \sup\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\} = \varepsilon_\omega.$$

Hence the order type of $\mathbb{N}_{\omega,1}$ is ε_ω.

Proposition 9. *By Remarks* 8 *and* 9*, the set $\mathbb{N}_{\omega,1}$ consists of all positive integers and has order type ε_ω. To compare two numbers m, n in $\mathbb{N}_{\omega,1}$, we determine the values of s_m and s_n. Then:*

- *if $s_m < s_n$ (in natural $\mathbb{N}_1$ order), then we have $m < n$ in $\mathbb{N}_{\omega,1}$,*
- *if $s_m = s_n = k$, then we compare m and n in $\mathbb{N}_{k+1,1}$. The larger number in $\mathbb{N}_{k+1,1}$ is the larger number in $\mathbb{N}_{\omega,1}$.*

Remark 10. *The successor of a given $n \in \mathbb{N}_{\omega,1}$ is equal to its successor in $\mathbb{N}_{s_n+1,1}$ (see Remark 7).*

We hope that all of the presented orders provide a good illustration not only of the countability of the corresponding ordinals, but also of their complexity and recursivity. Moreover, note that for any given countable set X, any bijection from X to the set of positive integers can be used to induce ε_k order on X, where $0 \leq k \leq \omega$, by having $\mathbb{N}_{k,1}$ as the codomain of that bijection.

While we stop at the example of ε_ω order, we would like to emphasise that using the presented methods it is possible to arrange the set of positive integers in a way isomorphic to even larger epsilon ordinals.

4.2 Beyond epsilon-omega

Observe that starting with $\mathbb{N}_{\omega,1}$ and $\mathbb{N}_{\omega,0} := \{0\} \sqcup \mathbb{N}_{\omega,1}$ and reapplying the ideas from Section 3, one could easily obtain $\varepsilon_{\omega+1}, \varepsilon_{\omega+2}, \ldots$ orders on the set $\mathbb{N}_1$ and then, by methods from Section 4, construct the $\varepsilon_{\omega\cdot 2}$ order on $\mathbb{N}_1$, and so on.

In general, Section 3 shows how to construct $\varepsilon_{\alpha+1}$ order on $\mathbb{N}_1$ using the previously defined ε_α order on this set. On the other hand, having defined $\varepsilon_{\alpha_1}, \varepsilon_{\alpha_2}, \ldots$ orders on $\mathbb{N}_1$, with $\alpha_1 < \alpha_2 < \ldots < \beta$ and $\beta = \sup\{\alpha_n : n \in \mathbb{N}_1\}$, one can define ε_β order on $\mathbb{N}_1$ by adjusting the methods from Section 4.

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Границя послідовності $\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots$, що позначається як ε_0, є найменшим епсилон-числом, тобто таким порядковим числом α, що $\omega^{\alpha} = \alpha$. Хоча відомо, що всі епсилон-числа зі зліченними індексами є порядковими типами зліченних множин, ця властивість часто сприймається як контрінтуїтивна. Метою даної роботи є глибше розкриття структури епсилон-ординалів шляхом побудови відповідних порядків $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\omega}$ на множині натуральних чисел. Представлений метод може бути легко узагальнений для побудови ще більших епсилон-чисел.

Ключові слова і фрази: порядкове число, епсилон-омега, епсилон-число, цілком впорядкована множина.