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On semitopological simple inverse ω-semigroups with compact maximal subgroups

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We describe the structure of (0-)simple inverse Hausdorff semitopological ω-semigroups with compact maximal subgroups. In particular, we show that if S is a simple inverse Hausdorff semitopological ω-semigroup with compact maximal subgroups, then S is topologically isomorphic to the Bruck-Reilly extension $\big(\mathbf{BR}(T,\theta),\tau_{\mathbf{BR}}^{\oplus}\big)$ of a finite semilattice $T=\big[E;G_\alpha,\varphi_{\alpha,\beta}\big]$ of compact groups G_α in the class of topological inverse semigroups, where $\tau_{\mathbf{BR}}^{\oplus}$ is the sum direct topology on $\mathbf{BR}(T,\theta)$. Also, we prove that every Hausdorff locally compact shift-continuous topology on a simple inverse Hausdorff semitopological ω-semigroup with compact maximal subgroups with adjoined zero is either compact or the zero is an isolated point.

Key words and phrases: bicyclic semigroup, simple inverse ω-semigroup, semitopological semigroup, locally compact, topological semigroup, compact maximal subgroup, adjoining zero, compact ideal.

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Introduction

We shall follow the terminology of [8, 10, 11, 14, 22, 31]. We denote by ω the set of all non-negative integers, and by $\mathbb{N}$ the set of all positive integers. All topological spaces, considered in this paper, are Hausdorff, if the otherwise is not stated explicitly. If A is a subset of a topological space X, then by $\operatorname{cl}_X(A)$ and $\operatorname{int}_X(A)$ we denote the closure and interior of A in X, respectively.

Let $\mathfrak{h}\colon S\to T$ be a map of sets. Then for any $s\in S$ and $A\subseteq S$ by $(s)\mathfrak{h}$ and $(A)\mathfrak{h}$ we denote the images of s and A, respectively, under the map $\mathfrak{h}$. Also, for any $t\in T$ and $B\subseteq T$ we denote by $(t)\mathfrak{h}^{-1}$ and $(B)\mathfrak{h}^{-1}$ the full preimages of t and B, respectively, under the map $\mathfrak{h}$.

A semigroup S is called *inverse* if for any element $x\in S$ there exists a unique $x^{-1}\in S$ such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$. The element x^{-1} is called the *inverse of* $x\in S$. If S is an inverse semigroup, then the function $\operatorname{inv}\colon S\to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e\preccurlyeq f$ if and only if $ef=fe=e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A *chain* is a linearly ordered

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semilattice.

A semigroup S is said to be *simple* (0-*simple*) if S has no proper two-sided ideals (if S has the zero $\mathbf{0}$ and $\{\mathbf{0}\}$ is the unique proper two-sided ideal of S). A semigroup S is called an *ω-semigroup* if the band $E(S)$ is order isomorphic to $(\omega, \geqslant)$. Also, an inverse semigroup S is 0-simple ω-semigroup if S is 0-simple and the subset of non-zero idempotents $E(S) \setminus \{\mathbf{0}\}$ is order isomorphic to $(\omega, \geqslant)$.

If S is an inverse semigroup, then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S (see [36]).

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence, simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence (see [10]).

Using the construction of the bicyclic monoid, R.H. Bruck built the construction of isomorphic embedding of any (inverse) semigroup into a simple inverse monoid (see [7] and [11, Section 8.5]). Subsequently, N.R. Reilly [30] and R.J. Warne [37] generalized Bruck's construction to describe the structure of bisimple regular ω-semigroups in the following way.

Construction 1 ([30, 37])**.** *Let S be a monoid with the unit element 1_S and let $\theta\colon S \to H_{1_s}$ be a homomorphism from S into the group of units $H(1_s)$ of S. On the set $\mathbf{BR}(S,\theta) = \omega \times S \times \omega$ we define the semigroup operation by the formula*

$$(i,s,j) \cdot (k.t,l) = \big(i+k-\min\{j,k\}, (s)\theta^{k-\min\{j,k\}}(t)\theta^{j-\min\{j,k\}}, j+l-\min\{j,k\}\big),$$

where $i,j,k,l \in \omega$, $s,t \in S$ and θ^0 is the identity map on S. Then $\mathbf{BR}(S,\theta)$ with such defined semigroup operation is called the Bruck-Reilly extension *of S.*

In the sequel, we assume that S is a monoid.

For arbitrary $i,j \in \omega$ and a non-empty subset A of the semigroup S we define the subset $A_{i,j}$ of $\mathbf{BR}(S,\theta)$ by $A_{i,j} = \{(i,s,j)\colon s \in A\}$.

We observe that if S is a trivial monoid then $\mathbf{BR}(S,\theta)$ is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$ and in case when θ is an annihilating homomorphism, i.e. $(s)\theta = 1_S$, then $\mathbf{BR}(S) = \mathbf{BR}(S,\theta)$ is called the *Bruck semigroup over monoid S* (see [15]). Also N.R. Reilly and R.J. Warne proved that every bisimple regular ω-semigroup is isomorphic to the Bruck-Reilly extension of some group [30, 37].

We need the following assertion, which is a simple generalization of [27, Lemma 1.2] and follows from [29, Theorem XI.1.1].

Proposition 1. *Let S be an arbitrary monoid and $\theta\colon S \to H_S(1)$ be a homomorphism from S into the group of units $H_S(1)$ of S. Then a map $\eta\colon \mathbf{BR}(S,\theta) \to \mathscr{C}(p,q)$, defined by the formula $\eta(i,s,j) = q^i p^j$, is a homomorphism and hence the relation $\eta^\natural$ on $\mathbf{BR}(S,\theta)$, defined in the following way*

$$(i,s,j)\eta^\natural(m,t,n) \qquad \textit{if and only if} \qquad i = m \quad \textit{and} \quad j = n,$$

is a congruence.

We need the following well-known construction.

Construction 2 ([29]). *Let E be a semilattice. To each $\alpha \in E$ associate a semigroup S_α and assume that $S_\alpha \cap S_\beta = \varnothing$ if $\alpha \neq \beta$. For each pair $\beta \preccurlyeq \alpha$, let $\varphi_{\alpha,\beta}\colon S_\alpha \to S_\beta$ be a homomorphism, and assume that the following conditions hold:*

(1) $\varphi_{\alpha,\alpha}\colon S_\alpha \to S_\alpha$ is the identity map of S_α for any $\alpha \in S_\alpha$;

(2) if $\gamma \preccurlyeq \beta \preccurlyeq \alpha$ in E, then $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

On the set $S = \bigcup\limits_{\alpha\in E} S_\alpha$ define a semigroup operation by the formula

$$a * b = \big((a)\varphi_{\alpha,\alpha\beta}\big)\big((b)\varphi_{\beta,\alpha\beta}\big)$$

if $a \in S_\alpha$, $b \in S_\beta$, and denote $S = \big[E; S_\alpha, \varphi_{\alpha,\beta}\big]$. The semigroup $\big[E; S_\alpha, \varphi_{\alpha,\beta}\big]$ is called a (strong) semilattice of semigroups S_α.

Well-known Clifford's Theorem states that an inverse semigroup S is Clifford, i.e. $E(S)$ is contained in the center of S, if and only if S is a semilattice of groups (see [29, Theorem II.2.6]).

In [21], B.P. Kochin showed that every simple inverse ω-semigroup is isomorphic to the Bruck-Reilly extension $\mathbf{BR}(S,\theta)$ of a finite chain of groups $S = \big[E; G_\alpha, \varphi_{\alpha,\beta}\big]$.

A continuous map $f\colon X \to Y$ from a topological space X into a topological space Y is called:

- *quotient* if the set $(U)f^{-1}$ is open in X if and only if U is open in Y (see [25] and [14, Section 2.4]);
- *hereditarily quotient* (or *pseudoopen*) if for every $B \subset Y$ the restriction $f|_B\colon (B)f^{-1} \to B$ of f is a quotient map (see [24] and [14, Section 2.4]);
- *open* if $(U)f$ is open in Y for every open subset U in X;
- *closed* if $(F)f$ is closed in Y for every closed subset F in X;
- *perfect* if X is Hausdorff, f is a closed map and all fibers $(y)f^{-1}$ are compact subsets of X (see [35]).

Every perfect map is closed, every closed map and every hereditarily quotient map are quotient [14]. Moreover, a continuous map $f\colon X \to Y$ from a topological space X onto a topological space Y is hereditarily quotient if and only if for every $y \in Y$ and every open subset U in X which contains $(y)f^{-1}$ we have

$$y \in \operatorname{int}_Y(f(U))$$

(see [14, 2.4.F]).

A (*semi*)*topological semigroup* is a topological space with a (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S, τ) is a topological semigroup;
- an *inverse semigroup* topology if (S, τ) is a topological inverse semigroup;
- a *shift-continuous* topology if (S, τ) is a semitopological semigroup;
- an *inverse shift-continuous* topology if (S, τ) is a semitopological semigroup with continuous inversion.

We observe that if $S = [E; G_\alpha, \varphi_{\alpha,\beta}]$ is a semitopological Clifford semigroup, then all homomorphisms $\varphi_{\alpha,\beta}$ are continuous [5].

It is well-known [6, 13] that the bicyclic monoid $\mathscr{C}(p,q)$ admits only the discrete semigroup (shift-continuous) Hausdorff topology. Semigroup and shift-continuous T_1-topologies on $\mathscr{C}(p,q)$ are studied in [9]. Topologizations of Bruck semigroups and Bruck–Reilly extensions, their topological properties and applications established in [15,16,18,20,28,32–34].

In the paper [17], it is proved that every Hausdorff locally compact shift-continuous topology on the bicyclic monoid with adjoined zero is either compact or discrete. This result was extended by S. Bardyla onto a polycyclic monoid [2] and graph inverse semigroups [3], and by T. Mokrytskyi onto the monoid of order isomorphisms between principal filters of $\mathbb{N}^n$ with adjoined zero [26]. In [4], S. Bardyla proved that a Hausdorff locally compact semitopological McAlister semigroup $\mathcal{M}_1$ is either compact or discrete. However, this dichotomy does not hold for the McAlister semigroup $\mathcal{M}_2$. Moreover, $\mathcal{M}_2$ admits continuum many different Hausdorff locally compact inverse semigroup topologies [4]. Also, in [19], it is proved that the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}^0$ with adjoined zero admits distinct $\mathfrak{c}$-many shift-continuous topologies, however every Hausdorff locally compact semigroup topology on $\mathscr{C}_{\mathbb{Z}}^0$ is discrete. Algebraic properties on a group G, such that if the discrete group G has these properties, then every locally compact shift continuous topology on G with adjoined zero is either compact or discrete, are studied in [23].

In this paper, we describe the structure of (0-)simple inverse Hausdorff semitopological ω-semigroups with compact maximal subgroups. In particular, we show that if S is a simple inverse Hausdorff semitopological ω-semigroups with compact maximal subgroups, then S is topologically isomorphic to the Bruck-Reilly extension $\left(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$ of a finite semilattice $T = [E; G_\alpha, \varphi_{\alpha,\beta}]$ of compact groups G_α in the class of topological inverse semigroups, where $\tau_{\mathbf{BR}}^{\oplus}$ is the sum direct topology on $\mathbf{BR}(T,\theta)$. Also we prove that every Hausdorff locally compact shift-continuous topology on the simple inverse Hausdorff semitopological ω-semigroup with compact maximal subgroups with adjoined zero is either compact or the zero is an isolated point.

1 On simple inverse semitopological ω-semigroups with compact maximal subgroups

We need the following simple lemma.

Lemma 1. *Let $S = [E; G_\alpha, \varphi_{\alpha,\beta}]$ be a semitopological semigroup which is a semilattice of groups G_α. If S is a topological sum of topological groups G_α, then S is a topological inverse semigroup.*

Proof. Since G_α is a topological group for any $\alpha \in E$ and S is a Clifford inverse semigroup, the inversion is continuous in S.

Fix arbitrary $a, b \in S$ such that $a \in G_\alpha$ and $b \in G_\beta$ for some $\alpha, \beta \in E$. The assumptions of the lemma imply that G_γ is an open-and-closed subset of S for any $\gamma \in E$. Since $G_{\alpha\beta}$ is a topological group, for any open neighbourhood $U\left((a)\varphi_{\alpha,\alpha\beta}(b)\varphi_{\beta,\alpha\beta}\right) \subseteq G_{\alpha\beta}$ of the point $(a)\varphi_{\alpha,\alpha\beta}(b)\varphi_{\beta,\alpha\beta}$ in S there exist open neighbourhoods $V\left((a)\varphi_{\alpha,\alpha\beta}\right) \subseteq G_{\alpha\beta}$ and $V\left((b)\varphi_{\beta,\alpha\beta}\right) \subseteq G_{\alpha\beta}$ of the points $(a)\varphi_{\alpha,\alpha\beta}$ and $(b)\varphi_{\beta,\alpha\beta}$ in S, respectively, such that

$$V\left((a)\varphi_{\alpha,\alpha\beta}\right) \cdot V\left((b)\varphi_{\beta,\alpha\beta}\right) \subseteq U\left((a)\varphi_{\alpha,\alpha\beta}(b)\varphi_{\beta,\alpha\beta}\right).$$

Since homomorphisms $\varphi_{\alpha,\alpha\beta}\colon G_\alpha \to G_{\alpha\beta}$ and $\varphi_{\beta,\alpha\beta}\colon G_\beta \to G_{\alpha\beta}$ are continuous, and G_γ is an open-and-closed subset of S for any $\gamma \in E$, we have that there exist open neighbourhoods $W(a) \subseteq G_\alpha$ and $W(b) \subseteq G_\beta$ of the points a and b in S, respectively, such that

$$(W(a))\varphi_{\alpha,\alpha\beta} \subseteq V\left((a)\varphi_{\alpha,\alpha\beta}\right) \qquad \text{and} \qquad (W(b))\varphi_{\beta,\alpha\beta} \subseteq V\left((b)\varphi_{\beta,\alpha\beta}\right).$$

The above inclusions imply that

$$W(a) * W(b) \subseteq V\left((a)\varphi_{\alpha,\alpha\beta}\right) \cdot V\left((b)\varphi_{\beta,\alpha\beta}\right) \subseteq U\left((a)\varphi_{\alpha,\alpha\beta}(b)\varphi_{\beta,\alpha\beta}\right),$$

hence, the semigroup operation in S is continuous. □

Proposition 2. *Let $S = \left[E; G_\alpha, \varphi_{\alpha,\beta}\right]$ be a Hausdorff semitopological semigroup which is a finite semilattice of compact groups G_α. Then S is a compact topological inverse semigroup.*

Proof. Since the semilattice E is finite, S is a compact as the union of finitely many compact subsets G_α. Also finiteness of E and Hausdorffness of S imply that G_α is open-and-closed subset of S. Next we apply Lemma 1. □

Definition 1. *Let $\mathfrak{STS}$ be a some class of semitopological semigroups and $(S, \tau_S) \in \mathfrak{STS}$. If $\tau_{\mathbf{BR}}$ is a topology on $\mathbf{BR}(S,\theta)$ such that $(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}}) \in \mathfrak{STS}$ and for some $i \in \omega$ the subsemigroup $S_{i,i}$ with the topology restricted from $(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}})$ is topologically isomorphic to (S, τ_S) under the map $\xi_i\colon S_{i,i} \ni (i,s,i) \mapsto s \in S$, then $(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}})$ is called a* topological Bruck-Reilly extension *of (S, τ_S) in the class $\mathfrak{STS}$.*

Proposition 3. *Every Hausdorff semitopological simple inverse ω-semigroup S is topologically isomorphic to a topological Bruck-Reilly extension $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ of a Hausdorff semitopological semigroup $T = \left[E; G_\alpha, \varphi_{\alpha,\beta}\right]$ which is a finite semilattice of semitopological groups G_α in the class of semitopological semigroups. Moreover, if S is locally compact, then T is a locally compact semitopological semigroup.*

Proof. By Kochin's Theorem (see [21]) every simple inverse ω-semigroup S is (algebraically) isomorphic to the Bruck-Reilly extension of semigroup $T = \left[E; G_\alpha, \varphi_{\alpha,\beta}\right]$ which is a finite semilattice of groups G_α. Then $T_{1,1}$ is a submonoid of $\mathbf{BR}(T,\theta)$. Let τ_1 be the topology induced from $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ onto $T_{1,1}$. By Definition 1 the semitopological semigroup $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ is a topological Bruck-Reilly extension of the semitopological semigroup $(T_{1,1}, \tau_1)$. Moreover, by [18, Proposition 2.4] for any $i, j \in \omega$ the subsemigroups $T_{i,i}$ and $T_{j,j}$ with the induced from $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ topologies are topologically isomorphic by the mapping $f^{i,i}_{j,j}\colon T_{i,i} \to T_{j,j}$, defined as follows $x \mapsto (j, 1_S, i) \cdot x \cdot (i, 1_S, j)$.

Also, [18, Proposition 2.4] implies that for any $i \in \omega$ the following sets $(i, 1_S, i) \cdot \mathbf{BR}(T,\theta)$ and $\mathbf{BR}(T,\theta) \cdot (i, 1_S, i)$ are retracts of $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$, hence, by [14, 1.5.C] they are closed subsets in the topological space $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$. Then

$$T_{1,1} = \mathbf{BR}(T,\theta) \setminus ((1, 1_S, 1) \cdot \mathbf{BR}(T,\theta) \cup \mathbf{BR}(T,\theta) \cdot (1, 1_S, 1))$$

is an open subset of $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$. This implies the last statement, because by [14, Theorem 3.3.8] an open subspace of a locally compact space is locally compact as well. □

Definition 2. *Let $\mathscr{B}_S$ be a base of the topology τ_S on a semitopological semigroup S. The topology $\tau_{\mathbf{BR}}^{\oplus}$ on $\mathbf{BR}(S,\theta)$ generated by the base $\mathscr{B}_{\mathbf{BR}}^{\oplus} = \{U_{i,j} \colon U \in \mathscr{B}_S,\ i,j \in \omega\}$ is called a sum direct topology on $\mathbf{BR}(S,\theta)$.*

The following statement is proved in [15, 20].

Proposition 4. *Let (S, τ_S) be a semitopological semigroup. Then $\left(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$ is a semitopological semigroup, i.e. $\left(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$ is a topological Bruck-Reilly extension of (S, τ_S) in the class of semitopological semigroups. Moreover, if (S, τ_S) satisfies one of the following conditions: it is metrizable, Hausdorff, a semitopological semigroup with the continuous inversion, a topological semigroup, a topological inverse semigroup, then so is $\left(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$, and $\left(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$ is a topological Bruck-Reilly extension of (S, τ_S) in the corresponding class of semitopological semigroups.*

The following statement is a consequence of [20, Theorem 8].

Corollary 1. *Let (S, τ_S) be a Hausdorff compact semitopological semigroup. If $(\mathbf{BR}(S,\theta), \tau_{\mathbf{BR}})$ is a topological Bruck-Reilly extension of (S, τ_S) in the class of Hausdorff semitopological semigroups, then $\tau_{\mathbf{BR}}$ coincides with the sum direct topology $\tau_{\mathbf{BR}}^{\oplus}$ on $\mathbf{BR}(S,\theta)$.*

Theorem 1. *Let T be a compact Hausdorff topological semigroup and $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ be a topological Bruck-Reilly extension of T in the class of Hausdorff semitopological semigroups. Then $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ is a Hausdorff topological semigroup. Moreover, if T is a topological inverse semigroup, then so is $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$.*

Proof. By Corollary 1, $\tau_{\mathbf{BR}}$ coincides with the sum direct topology $\tau_{\mathbf{BR}}^{\oplus}$ on $\mathbf{BR}(T,\theta)$.

Fix arbitrary $(i,s,j), (k,t,l) \in \mathbf{BR}(T,\theta)$. Then we have that

$$(i,s,j) \cdot (k,t,l) = \begin{cases} \left(i-j+k, (s)\theta^{k-j} \cdot t, l\right), & \text{if } j < k, \\ \left(i, s \cdot t, l\right), & \text{if } j = k, \\ \left(i, s \cdot (t)\theta^{j-k}, j-k+l\right), & \text{if } j > k. \end{cases}$$

Next we consider the following cases.

Case 1. Suppose that $j < k$. Then for any open neighbourhood $U((s)\theta^{k-j} \cdot t)$ of the point $(s)\theta^{k-j} \cdot t$ in T there exist open neighbourhoods $V((s)\theta^{k-j})$ and $V(t)$ of the points $(s)\theta^{k-j}$ and t in T, respectively, such that $V((s)\theta^{k-j}) \cdot V(t) \subseteq U((s)\theta^{k-j} \cdot t)$, because T is a topological semigroup. By [18, Proposition 2.4] the homomorphism $\theta \colon T \to H(1_T)$ is continuous. Hence there exists an open neighbourhood $O(s)$ of the point s in T such that $(O(s))\theta^{k-j} \subseteq V((s)\theta^{k-j})$.

Since $j < k$, $O(s)_{i,j} \subseteq T_{i,j}$, $V(t)_{k,l} \subseteq T_{k,l}$, and $U((s)\theta^{k-j} \cdot t)_{i-j+k,l} \subseteq T_{i-j+k,l}$, the semigroup operation in $\mathbf{BR}(T,\theta)$ implies that

$$O(s)_{i,j} \cdot V(t)_{k,l} \subseteq U((s)\theta^{k-j} \cdot t)_{i-j+k,l}.$$

Case 2. Suppose that $j = k$. Since T is a topological semigroup, for any open neighbourhood $U(s \cdot t)$ of the point $s \cdot t$ in the space T there exist open neighbourhoods $V(s)$ and $V(t)$ of the points s and t in T, respectively, such that $V(s) \cdot V(t) \subseteq U(s \cdot t)$. Since $j = k$, $V(s)_{i,j} \subseteq T_{i,j}$, $V(t)_{k,l} \subseteq T_{k,l}$, and $U(s \cdot t)_{i,l} \subseteq T_{i,l}$, by the semigroup operation of $\mathbf{BR}(T,\theta)$ we obtain that

$$V(s)_{i,j} \cdot V(t)_{k,l} \subseteq U(s \cdot t)_{i,l}.$$

Case 3. Suppose that $j > k$. Since T is a topological semigroup, for any open neighbourhood $U(s \cdot (t)\theta^{j-k})$ of the point $s \cdot (t)\theta^{j-k}$ in the space T there exist open neighbourhoods $V(s)$ and $V((t)\theta^{j-k})$ of s and $(t)\theta^{j-k}$ in T, respectively, such that $V(s) \cdot V((t)\theta^{j-k}) \subseteq U(s \cdot (t)\theta^{j-k})$. By [18, Proposition 2.4] the homomorphism $\theta\colon T \to H(1_T)$ is continuous. Hence, there exists an open neighbourhood $O(t)$ of t in the topological space T such that $(O(t))\theta^{j-k} \subseteq V((t)\theta^{j-k})$. Since $j > k$, $V(s)_{i,j} \subseteq T_{i,j}$, $O(t)_{k,l} \subseteq T_{k,l}$, and $U(s \cdot (t)\theta^{j-k})_{i,j-k+l} \subseteq T_{i,j-k+l}$, by the semigroup operation of $\mathbf{BR}(T,\theta)$ we get that

$$V(s)_{i,j} \cdot O(t)_{k,l} \subseteq U(s \cdot (t)\theta^{j-k})_{i,j-k+l}.$$

The above three cases imply that the semigroup operation is continuous in $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$.

If T is an inverse semigroup, then $(i,s,j)^{-1} = (j,s^{-1},i)$ for any $(i,s,j) \in \mathbf{BR}(T,\theta)$. Since T is an inverse topological semigroup, for any open neighbourhood $U(s^{-1})$ of s^{-1} in T there exists an open neighbourhood $V(s)$ of s in T such that $(V(s))^{-1} \subseteq U(s^{-1})$. Since $V(s)_{i,j} \subseteq T_{i,j}$ and $U(s^{-1})_{j,i} \subseteq T_{j,i}$, the semigroup operation in $\mathbf{BR}(T,\theta)$ implies that $(V(s)_{i,j})^{-1} \subseteq U(s^{-1})_{j,i}$. Hence, the inversion is continuous in $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$. □

The main result of this section is the following theorem.

Theorem 2. *Let S be a Hausdorff semitopological simple inverse ω-semigroup such that every maximal subgroup of S is compact. Then S is topologically isomorphic to the Bruck-Reilly extension $\left(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}}^{\oplus}\right)$ of a finite semilattice $T = \left[E; G_\alpha, \varphi_{\alpha,\beta}\right]$ of compact groups G_α in the class of topological inverse semigroups. Moreover, the space of S is locally compact.*

Proof. The first statement of the theorem follows from Proposition 3 and Theorem 1. The second statement is a consequence of [14, Theorem 3.3.12]. □

The following example shows that the statement of Theorem 2 is not true when a Hausdorff locally compact semitopological simple inverse ω-semigroup S contains non-compact maximal subgroup.

Example 1 ([18, Example 4.7])**.** *Let $\mathbb{Z}(+)$ be the additive group of integers and $0_{\mathbb{Z}}$ be the neutral element of $\mathbb{Z}(+)$. We define a topology τ_{cf} on $\mathbf{BR}(\mathbb{Z}(+),\theta)$ in the following way. Let (i,g,j) be an isolated point of $(\mathbf{BR}(\mathbb{Z}(+),\theta), \tau_{\mathrm{cf}})$ in the following cases:*

(i) $g \neq 0_{\mathbb{Z}}$ *and* $i,j \in \omega$*;*

(ii) $i = 0$ *or* $j = 0$*.*

The family

$$\mathscr{B}_{cf}(i,0_{\mathbb{Z}},j)=\left\{(UF)^0_{i-1,j-1}=(\mathbb{Z}(+)\setminus F)_{i-1,j-1}\cup\{(i,0_{\mathbb{Z}},j)\}\colon F \text{ is a finite subset of } \mathbb{Z}(+)\right\}$$

is a base of the topology τ_{cf} on $\mathbf{BR}(\mathbb{Z}(+),\theta)$ at the point $(i,0_{\mathbb{Z}},j)$ for all $i,j\in\omega$. Then $(\mathbf{BR}(\mathbb{Z}(+),\theta),\tau_{\mathrm{cf}})$ is a Hausdorff locally compact semitopological inverse semigroup with continuous inversion.

2 On adjoining zero to a simple inverse locally compact semitopological ω-semigroup with compact maximal subgroups

Throughout the section, we denote by $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ the Bruck-Reilly semigroup $\mathbf{BR}(S,\theta)$ with an adjoined zero $\mathbf{0}$ (see [10, Section 1.1]).

Proposition 5. *Let $\tau^{\mathbf{0}}_{\mathbf{BR}}$ be a Hausdorff topology on $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ such that the set $S_{i,j}$ is open in $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$ for all $i,j\in\omega$. Then $\eta^\natural$ is a closed congruence on $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$.*

Proof. Fix arbitrary non-$\eta^\natural$-equivalent non-zero elements (i,s,j) and (m,t,n) of the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$. Then $S_{i,j}$ and $S_{m,n}$ are open disjoint neighbourhoods of the points (i,s,j) and (m,t,n) in the space $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$, respectively, such that $\eta^\natural\cap\big(S_{i,j}\times S_{m,n}\big)=\varnothing$. Since the topology $\tau^{\mathbf{0}}_{\mathbf{BR}}$ is Hausdorff, there exist disjoint open neighbourhoods $U(i,s,j)$ and $U(\mathbf{0})$ of (i,s,j) and $\mathbf{0}$ in $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$, respectively. This implies that $U(i,s,j)\times U(\mathbf{0})$ is an open neighbourhood of the ordered pair $((i,s,j),\mathbf{0})$ in $\mathbf{BR}^{\mathbf{0}}(S,\theta)\times\mathbf{BR}^{\mathbf{0}}(S,\theta)$ with the product topology which does not intersect the congruence $\eta^\natural$ of the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$. Hence, $\eta^\natural$ is a closed congruence on the semigroup $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$. □

We put $\mathscr{C}^0=\mathscr{C}(p,q)\sqcup\{0\}$ to be the bicyclic semigroup with adjoined zero. Obviously that the congruence $\eta^\natural$ on the Bruck-Reilly extension $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ of a semigroup S generates the natural homomorphism $\eta\colon\mathbf{BR}^{\mathbf{0}}(S,\theta)\to\mathscr{C}^0$.

Lemma 2. *Let $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$ be a semitopological semigroup with a compact (left, right) ideal. If the natural homomorphism $\eta\colon\mathbf{BR}^{\mathbf{0}}(S,\theta)\to\mathscr{C}^0$ is a quotient map, then η is an open map.*

Proof. Let us suppose that $\mathscr{C}^0$ admits a topology such that the natural homomorphism $\eta\colon\mathbf{BR}^{\mathbf{0}}(S,\theta)\to\mathscr{C}^0$ is a quotient map.

If U is an open subset of $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$ such that $U\not\ni\mathbf{0}$, then $\eta(U)$ is an open subset of $\mathscr{C}^0$, because by [13, Proposition 1] the bicyclic monoid $\mathscr{C}(p,q)$ is a discrete open subset of the space $\mathscr{C}^0$.

Suppose $U\ni\mathbf{0}$ is an open subset of $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$. Put $U^*=\eta^{-1}(\eta(U))$. Then $U^*=\eta^{-1}(\eta(U^*))$. Since $\eta\colon\mathbf{BR}^{\mathbf{0}}(S,\theta)\to\mathscr{C}^0$ is a natural homomorphism, we have

$$U^*=\bigcup\left\{G_{i,j}\colon G_{i,j}\cap U\neq\varnothing\right\}\cup\{\mathbf{0}\}.$$

By [20, Theorem 8] the restriction of the topology $\tau^{\mathbf{0}}_{\mathbf{BR}}$ on the semigroup $\mathbf{BR}(S,\theta)$ coincides with the sum direct topology $\tau^{\oplus}_{\mathbf{BR}}$ on $\mathbf{BR}(S,\theta)$. This implies that U^* is an open subset of the space $\big(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\big)$. Since η is a quotient map and $U^*=\eta^{-1}(\eta(U^*))$, we conclude that $\eta(U)$ is an open subset of the space $\mathscr{C}^0$. □

The following example from [17] shows that the semigroup $\mathscr{C}^0$ admits a shift-continuous compact Hausdorff topology.

Example 2 ([17]). *On the semigroup $\mathscr{C}^0$ we define a topology τ_{Ac} in the following way:*

(i) *every element of the bicyclic monoid $\mathscr{C}(p,q)$ is an isolated point in the space $(\mathscr{C}^0, \tau_{Ac})$;*

(ii) *the family $\mathscr{B}(0) = \{U \subseteq \mathscr{C}^0 \colon U \ni 0 \text{ and } \mathscr{C}(p,q) \setminus U \text{ is finite}\}$ determines a base of the topology τ_{Ac} at zero $0 \in \mathscr{C}^0$,*

i.e. τ_{Ac} is the topology of the Alexandroff one-point compactification of the discrete space $\mathscr{C}(p,q)$ with the remainder $\{0\}$. Then $(\mathscr{C}^0, \tau_{Ac})$ is a Hausdorff compact semitopological semigroup.

Lemma 3. *Let $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ be a Hausdorff semitopological semigroup with a compact subsemigroup $S_{i,i}$ for some $i \in \omega$. Then $S_{i,j}$ is an open-and-closed subspace of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ for any $i, j \in \omega$.*

Proof. Since $(i, 1_S, i)$ is an idempotent of $\mathbf{BR^0}(S,\theta)$ for any $i \in \omega$, the subsets $(i, 1_S, i) \cdot \mathbf{BR^0}(T,\theta)$ and $\mathbf{BR^0}(T,\theta) \cdot (i, 1_S, i)$ are retracts of $\left(\mathbf{BR^0}(T,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Hence by [14, 1.5.C] they are closed subsets in the topological space $\left(\mathbf{BR^0}(T,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Then

$$T_{k,k} = \mathbf{BR^0}(T,\theta) \setminus \left((k+1, 1_S, k+1) \cdot \mathbf{BR^0}(T,\theta) \cup \mathbf{BR}(T,\theta) \cdot (k+1, 1_S, k+1)\right)$$

is an open subset of $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ for any $k \in \omega$.

Since the subsemigroup $S_{i,i}$ of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is compact for some $i \in \omega$ and the subspaces $S_{i,j}$, $i, j \in \omega$, of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ are homeomorphic due to [18, Proposition 2.4(iv)], $S_{i,j}$ are compact subspaces of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Then for any $i, j \leqslant k$ the subspace $S_{i,j}$ is open-and-closed in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. □

Proposition 6. *Let $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ be a Hausdorff locally compact semitopological semigroup with a compact subsemigroup $S_{i,i}$ for some $i \in \omega$. Then the quotient semigroup $\mathbf{BR^0}(S,\theta)/\eta^{\natural}$ with the quotient topology is topologically isomorphic to the semigroup $\mathscr{C}^0$ with either the topology τ_{Ac} or the discrete topology.*

Proof. By Lemma 3, $S_{i,j}$ is an open-and-closed subspace of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ for any $i, j \in \omega$. Hence, by Proposition 5, $\eta^{\natural}$ is a closed congruence on $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Then the quotient semigroup $\mathbf{BR^0}(S,\theta)/\eta^{\natural}$ with the quotient topology is a Hausdorff space. Lemma 2 implies that $\eta \colon \mathbf{BR^0}(S,\theta) \to \mathscr{C}^0$ is an open map. Hence, by [14, Theorem 3.3.15], the quotient semigroup $\mathbf{BR^0}(S,\theta)/\eta^{\natural}$ with the quotient topology is a locally compact space. Since $\mathbf{BR^0}(S,\theta)/\eta^{\natural}$ is isomorphic to the semigroup $\mathscr{C}^0$, [17, Theorem 1] implies the statement of the proposition. □

Throughout the section, if the otherwise is not stated explicitly, we assume that $\tau^{\mathbf{0}}_{\mathbf{BR}}$ is a Hausdorff locally compact shift-continuous topology on the semigroup $\mathbf{BR^0}(S,\theta)$ such that the following conditions hold:

(i) the subsemigroup $S_{i,i}$ of $\mathbf{BR^0}(S,\theta)$ with the restriction topology from $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is a compact semitopological semigroup for some $i \in \omega$ (hence, by [18, Proposition 2.4] for all $i \in \omega$);

(ii) $\mathbf{0}$ is a non-isolated point of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$.

Let $\mathscr{P} = \{P_\alpha \colon \alpha \in \mathscr{I}\}$ be an infinite family of nonempty subsets of a set X. We shall say that a set $A \subseteq X$ *intersects almost all* subsets of $\mathscr{P}$ if $A \cap P_\alpha = \varnothing$ for finitely many $P_\alpha \in \mathscr{P}$.

Lemma 4. *Every open neighbourhood $U_{\mathbf{0}}$ of zero $\mathbf{0}$ in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ intersects almost all subsets $S_{i,j}$, $i,j \in \omega$, of $\mathbf{BR}(S,\theta)$.*

Proof. Suppose to the contrary that there exists an open neighbourhood $U_{\mathbf{0}}$ of zero in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ such that $U_{\mathbf{0}} \cap S_{i,j} = \varnothing$ for infinitely many $S_{i,j}$, $i,j \in \omega$. Then by Lemma 2 the quotient natural homomorphism $\eta \colon \mathbf{BR}^{\mathbf{0}}(S,\theta) \to \mathscr{C}^0$ is an open map, and hence the quotient semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)/\eta^\natural$, equipped with the quotient topology, is neither compact nor discrete, which contradicts Proposition 6. □

For an arbitrary subset A of $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ and any $i,j \in \omega$ we denote $[A]_{i,j} = A \cap S_{i,j}$.

Lemma 5. *For every open neighbourhood $U_{\mathbf{0}}$ of $\mathbf{0}$ in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ and any $i_0 \in \omega$ the sets*

$$\{j \in \omega \colon S_{i_0,j} \nsubseteq U_{\mathbf{0}}\} \qquad \text{and} \qquad \{j \in \omega \colon S_{j,i_0} \nsubseteq U_{\mathbf{0}}\}$$

are finite.

Proof. Suppose to the contrary that there exists an open neighbourhood $U_{\mathbf{0}}$ of zero in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ and $i_0 \in \omega$ such that $\{j \in \omega \colon S_{i_0,j} \subseteq U_{\mathbf{0}}\}$ is infinite. Since $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is a locally compact space, we can take a regular open neighbourhood $U_{\mathbf{0}}$ of the zero with compact closure.

We consider the following two cases:

(i) there exists $j_0 \in \omega$ such that $[U_{\mathbf{0}}]_{i_0,j} \neq S_{i_0,j}$ for all $j \geqslant j_0$;

(ii) for every $k \in \mathbb{N}$ there exists a positive integer $n > k$ such that $[U_{\mathbf{0}}]_{i_0,n} = S_{i_0,n}$.

Let the case (i) holds. Since every subset $S_{i,j}$, $i,j \in \omega$, of $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is compact, the separate continuity of the semigroup operation in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ and Lemma 4 imply that without loss of generality we may assume that $j_0 = 0$. By Lemma 3, every subset $S_{i,j}$ is open-and-compact in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Hence, the set

$$\mathcal{S}^0_{i_0}(U_{\mathbf{0}}) = \{\mathbf{0}\} \cup \bigcup_{j \in \omega} \left[\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})\right]_{i_0,j}$$

is compact. By Lemma 3, the family $\mathscr{U}_{i_0} = \left\{\{U_{\mathbf{0}}\}, \{S_{i_0,j} \colon j \in \omega\}\right\}$ is an open cover of the compactum $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$. Hence, there exists $j_1 \in \mathbb{N}$ such that

$$[U_{\mathbf{0}}]_{i_0,n} = \left[\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})\right]_{i_0,n}$$

for all integers $n \geqslant j_1$. Since the right shift

$$\rho_{(1,1_S,0)} \colon \mathbf{BR}^{\mathbf{0}}(S,\theta) \ni x \mapsto x \cdot (1,1_S,0) \in \mathbf{BR}^{\mathbf{0}}(S,\theta),$$

is continuous in $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$, the full preimage $V_{\mathbf{0}} = \rho^{-1}_{(1,1_S,0)}(U_{\mathbf{0}})$ is an open neighbourhood of the zero. By Lemma 3, the family $\mathscr{V}_{i_0} = \left\{\{V_{\mathbf{0}}\}, \{S_{i_0,j} \colon j \in \omega\}\right\}$ is an open cover of the compactum $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$. Hence, there exists a positive integer $j_2 \geqslant j_1$ such that

$$[V_{\mathbf{0}}]_{i_0,n} = [U_{\mathbf{0}}]_{i_0,n} = \left[\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})\right]_{i_0,n} \tag{1}$$

for all integers $n \geqslant j_2$. Indeed, since $(i_0, s, j) \cdot (1, 1_S, 0) = (i_0, s, j-1)$ for all $j \in \mathbb{N}$ and any $s \in S$, we obtain that the equalities (1) hold for all integers $n \geqslant j_2$.

Put $\widetilde{U}_{\mathbf{0}} = U_{\mathbf{0}} \setminus \left(S_{i_0,0} \cup \cdots \cup S_{i_0,j_2-1}\right)$. By Lemma 3, $\widetilde{U}_{\mathbf{0}}$ is an open neighbourhood of zero in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ such that

$$\left[\widetilde{U}_{\mathbf{0}}\right]_{i_0,n} = [U_{\mathbf{0}}]_{i_0,n} = \left[\mathrm{cl}_{\mathbf{BR^0}(S,\theta)}(U_{\mathbf{0}})\right]_{i_0,n}$$

for all integers $n \geqslant j_2$. Since the space $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is locally compact, without loss of generality we may assume that the neighbourhood $U_{\mathbf{0}}$ is a regular open set. This implies that $\widetilde{U}_{\mathbf{0}}$ is a regular open set as well. Hence, there exist distinct $s, t \in S$ such that $(i_0, s, n) \notin [U_{\mathbf{0}}]_{i_0,n}$ and $(i_0, t, n) \in [U_{\mathbf{0}}]_{i_0,n}$ for all integers $n \geqslant j_2$. But we have that

$$(i_0, s \cdot ((t)\theta)^{-1}, i_0+1) \cdot (i_0, t, n) = (i_0, s \cdot ((t)\theta)^{-1} \cdot (t)\theta, n+1) = (i_0, s, n+1).$$

Let $W_{\mathbf{0}} = (\widetilde{U}_{\mathbf{0}})\lambda^{-1}_{(i_0, s\cdot((t)\theta)^{-1}, i_0+1)}$, where $\lambda_{(i_0, s\cdot((t)\theta)^{-1}, i_0+1)}$ is the left shift on the element $(i_0, s \cdot ((t)\theta)^{-1}, i_0+1)$ in the semigroup $\mathbf{BR^0}(S,\theta)$. Then we have that

$$\left[\widetilde{U}_{\mathbf{0}}\right]_{i_0,n} \setminus \left[W_{\mathbf{0}}\right]_{i_0,n} \neq \varnothing \qquad \text{and} \qquad \left[W_{\mathbf{0}}\right]_{i_0,n} \setminus \left[\widetilde{U}_{\mathbf{0}}\right]_{i_0,n} \neq \varnothing$$

for all integers $n \geqslant j_2+1$. By Lemma 3, the family $\mathscr{W}_{i_0} = \left\{\{W_0\}, \{S_{i_0,j} \colon j \in \omega\}\right\}$ is an open cover of $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$ which has no a finite subcover. This contradicts the compactness of $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$. Hence, the set $\left\{j \in \omega \colon S_{i_0,j} \nsubseteq U_{\mathbf{0}}\right\}$ is finite.

Let the case (ii) holds. Then there are infinitely many $j \in \omega$ such that $[U_{\mathbf{0}}]_{i_0,j} = S_{i_0,j}$, but $[U_{\mathbf{0}}]_{i_0,j-1} \neq S_{i_0,j-1}$. Since every subset $S_{i,j}$, $i, j \in \omega$, of $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is compact, Lemma 3 implies that every subset $S_{i,j}$ is open-and-compact in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$. Hence, the set

$$\mathcal{S}^0_{i_0}(U_{\mathbf{0}}) = \{\mathbf{0}\} \cup \bigcup_{j \in \omega} \left[\mathrm{cl}_{\mathbf{BR^0}(S,\theta)}(U_{\mathbf{0}})\right]_{i_0,j}$$

is compact. Let $V_{\mathbf{0}} = (U_{\mathbf{0}})\rho^{-1}_{(1,1_S,0)}$, where $\rho_{(1,1_S,0)}$ is the right shift on the element $(1, 1_S, 0)$ in the semigroup $\mathbf{BR^0}(S,\theta)$. By Lemma 3, the family $\mathscr{V}_{i_0} = \left\{\{V_0\}, \{S_{i_0,j} \colon j \in \omega\}\right\}$ is an open cover of the compactum $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$. Then the continuity of the right shift $\rho_{(1,1_S,0)}$ in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ and the equality $(i_0, s, j) \cdot (1, 1_S, 0) = (i_0, s, j-1)$ imply that $\left[V_{\mathbf{0}}\right]_{i_0,j} \neq S_{i_0,j}$ for infnitely many $j \in \omega$. Also, the equality $(i_0, s, j) \cdot (1, 1_S, 0) = (i_0, s, j-1)$ and the assumption of the case (ii) imply that $\left[U_{\mathbf{0}}\right]_{i_0,j} \setminus \left[V_{\mathbf{0}}\right]_{i_0,j} \neq \varnothing$ for infinitely many $j \in \omega$. Hence, the open cover $\mathscr{V}_{i_0}$ of $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$ does not have finite subcovers, which contradicts the compactness of $\mathcal{S}^0_{i_0}(U_{\mathbf{0}})$. Hence, the set $\left\{j \in \omega \colon S_{i_0,j} \nsubseteq U_{\mathbf{0}}\right\}$ is finite.

The proof of the statement that the set $\left\{j \in \omega \colon S_{j,i_0} \nsubseteq U_{\mathbf{0}}\right\}$ is finite is similar. □

Lemma 6. *For every open neighbourhood $U_{\mathbf{0}}$ of $\mathbf{0}$ in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ the set*

$$N_{U_{\mathbf{0}}} = \left\{(i,j) \in \omega \times \omega \colon S_{i,j} \subseteq U_{\mathbf{0}}\right\}$$

is finite.

Proof. Suppose to the contrary that there exists an open neighbourhood $U_{\mathbf{0}}$ of zero in $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ such that the set $N_{U_{\mathbf{0}}}$ is infinite. Since $\left(\mathbf{BR^0}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ is a locally compact space, without loss of generality we may assume that the closure $\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})$ of the neighbourhood $U_{\mathbf{0}}$ is compact and the neighbourhood $U_{\mathbf{0}}$ is regular open. By Lemma 5, for every $k \in \mathbb{N}$ there exists $(i,j) \in N_{U_{\mathbf{0}}}$ such that $i > k$ and $j > k$.

Using induction, we define an infinite sequence $\{(i_n, j_n)\}_{n\in\omega}$ of elements of the set $N_{U_{\mathbf{0}}}$ in the following way. By the assumption, there exists the smallest $i_0 \in \omega$ such that $S_{i_0,j} \nsubseteq U_{\mathbf{0}}$, $j \in \omega$. By Lemma 5, there exists $j_0 = \max\{j \in \omega \colon S_{i_0,j} \nsubseteq U_{\mathbf{0}}\}$.

At $(k+1)$th step of induction we define pair $(i_{k+1,jk+1}) \in N_{U_{\mathbf{0}}}$ as follows. Let i_{k+1} be the smallest integer which is greater than i_k such that $S_{i_k,j} \nsubseteq U_{\mathbf{0}}$, $j \in \omega$. By Lemma 5, there exists $j_{k+1} = \max\{j \in \omega \colon S_{i_{k+1},j} \nsubseteq U_{\mathbf{0}}\}$. Our assumption and Lemma 5 imply that the ordered pair (i_{k+1}, j_{k+1}) belongs to $N_{U_{\mathbf{0}}}$.

By the separate continuity of the semigroup operation in $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}})$, there exists an open neighbourhood $V_{\mathbf{0}} \subseteq U_{\mathbf{0}}$ of zero in $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}})$ such that $V_{\mathbf{0}} \cdot (1, 1_S, 0) \subseteq U_{\mathbf{0}}$. The construction of the sequence $\{(i_n, j_n)\}_{n\in\omega}$ implies that

$$[V_{\mathbf{0}}]_{i_n,j_n} \subseteq [U_{\mathbf{0}}]_{i_n,j_n} \neq S_{i_n,j_n} \qquad \text{and} \qquad [U_{\mathbf{0}}]_{i_n,j_n+1} = S_{i_n,j_n+1}$$

for each $(i_n, j_n) \in N_{U_{\mathbf{0}}}$. By Lemma 3, the family $\mathscr{V} = \{\{V_0\}, \{S_{i,j} \colon i,j \in \omega\}\}$ is an open cover of the compact set $\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})$. The continuity of the right shift $\rho_{(1,1_S,0)}$ in $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\mathbf{0}}_{\mathbf{BR}})$ implies that $[V_{\mathbf{0}}]_{i_n,j_n+1} \neq S_{i_n,j_n+1}$ for infinitely many ordered pairs $(i_n, j_{n+1}) \in N_{U_{\mathbf{0}}}$. Hence, we obtain that $[U_{\mathbf{0}}]_{i_n,j_n+1} \setminus [V_{\mathbf{0}}]_{i_n,j_n+1} \neq \varnothing$ for infinitely many $(i_n, j_{n+1}) \in N_{U_{\mathbf{0}}}$. The above arguments guarantee that the cover $\mathscr{V}$ has no finite subcovers, which contradicts the compactness of $\operatorname{cl}_{\mathbf{BR}^{\mathbf{0}}(S,\theta)}(U_{\mathbf{0}})$. The obtained contradiction implies the statement of the lemma. □

Example 3. *Let (S, τ_S) be a Hausdorff semitopological monoid, $\theta \colon S \to H(1_S)$ be a continuous homomorphism and $\mathscr{B}_S(s)$ be a base of the topology τ_S at a point $s \in S$.*

On the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ we define a topology $\tau^{\oplus}_{\mathbf{BR}}$ in the following way:

(i) for any non-zero element $(i,s,j) \in S_{i,j}$ of the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ the family

$$\mathscr{B}^{\oplus}_{\mathbf{BR}}(i,s,j) = \{U_{i,j} \colon U \in \mathscr{B}_S(s)\}$$

is a base of the topology $\tau^{\oplus}_{\mathbf{BR}}$ at the point $(i,s,j) \in \mathbf{BR}^{\mathbf{0}}(S,\theta)$;

(ii) zero $\mathbf{0} \in \mathbf{BR}^{\mathbf{0}}(S,\theta)$ is an isolated point in $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\oplus}_{\mathbf{BR}})$.

The semigroup operation in $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\oplus}_{\mathbf{BR}})$ is separately continuous (see [20]). Moreover, if (S, τ_S) be a topological monoid, then so is $(\mathbf{BR}^{\mathbf{0}}(S,\theta), \tau^{\oplus}_{\mathbf{BR}})$ (see [15]).

In the following example, we extend the construction, proposed in [18, Example 3.4], onto compact Bruck-Reilly extensions of compact semitopological monoids in the class of Hausdorff semitopological semigroups with adjoined zero.

Example 4. *Let (S, τ_S) be a Hausdorff compact semitopological monoid, $\theta \colon S \to H(1_S)$ be a continuous homomorphism and $\mathscr{B}_S(s)$ be a base of the topology τ_S at a point $s \in S$. On the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ we define a topology $\tau^{\mathbf{Ac}}_{\mathbf{BR}}$ in the following way:*

(i) for any non-zero element $(i,s,j) \in S_{i,j}$ of the semigroup $\mathbf{BR}^{\mathbf{0}}(S,\theta)$ the family

$$\mathscr{B}^{\mathbf{Ac}}_{\mathbf{BR}}(i,s,j) = \{U_{i,j} \colon U \in \mathscr{B}_S(s)\}$$

is a base of the topology $\tau^{\mathbf{Ac}}_{\mathbf{BR}}$ at the point $(i,s,j) \in \mathbf{BR}^{\mathbf{0}}(S,\theta)$;

(ii) *the family* $\mathscr{B}^{\mathbf{Ac}}_{\mathbf{BR}}(\mathbf{0}) = \left\{U_{(i_1,j_1),\dots,(i_k,j_k)} \colon (i_1,j_1),\dots,(i_k,j_k) \in \omega\times\omega\right\}$, *where*

$$U_{(i_1,j_1),\dots,(i_k,j_k)} = \mathbf{BR}^{\mathbf{0}}(S,\theta) \setminus \left(S_{i_1,j_1} \cup \dots \cup S_{i_k,j_k}\right),$$

is a base of the topology $\tau^{\mathbf{Ac}}_{\mathbf{BR}}$ *at zero* $\mathbf{0} \in \mathbf{BR}^{\mathbf{0}}(S,\theta)$.

Obviously that $\tau^{\mathbf{Ac}}_{\mathbf{BR}}$ *is the topology of the Alexandroff one-point compactification of the Hausdorff locally compact space* $\bigoplus\{S_{i,j} \colon i,j\in\omega\}$ *with the remainder* $\{\mathbf{0}\}$ *(here for any* $i,j\in\omega$ *the space* $S_{i,j}$ *is homeomorphic to the compact semigroup* (S,τ_S) *by the map* $(i,s,j)\mapsto s$*). Simple routine verifications show that the semigroup operation in* $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{Ac}}_{\mathbf{BR}}\right)$ *is separately continuous.*

Lemmas 3 and 6 imply the following dichotomy for locally compact Bruck-Reilly extensions of compact semitopological monoids in the class of Hausdorff semitopological semigroups with adjoined zero.

Theorem 3. *Let* $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *be a Hausdorff locally compact semitopological semigroup with a compact subsemigroup* $S_{i,i}$ *for some* $i\in\omega$. *Then* $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *is topologically isomorphic either to* $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\oplus}_{\mathbf{BR}}\right)$ *or to* $\left(\mathbf{BR}^{\mathbf{0}}(S,\theta),\tau^{\mathbf{Ac}}_{\mathbf{BR}}\right)$.

The following theorem describes the structure of inverse 0-simple ω-semigroups.

Theorem 4. *Every inverse* 0*-simple* ω*-semigroup is isomorphic to an inverse simple* ω*-semigroup with adjoined zero.*

Proof. Suppose that S is an inverse 0-simple ω-semigroup and $\mathbf{0}$ is zero of S. We shall show that $S\setminus\{\mathbf{0}\}$ is an inverse subsemigroup of S. Since S is an inverse semigroup, we have that $x^{-1}\in S\setminus\{\mathbf{0}\}$ for a non-zero element x from S.

Suppose to the contrary that there exist $x,y\in S\setminus\{\mathbf{0}\}$ such that $x\cdot y=\mathbf{0}$. If x^{-1} and y^{-1} are inverse elements of x and y in S, then $x^{-1}\neq\mathbf{0}\neq y^{-1}$. Then $x^{-1}\cdot x$ and $y\cdot y^{-1}$ are non-zero idempotents of S. Since S is an inverse 0-simple ω-semigroup, we conclude that $(x^{-1}\cdot x)\cdot(y\cdot y^{-1})\neq\mathbf{0}$, but $(x^{-1}\cdot x)\cdot(y\cdot y^{-1}) = x^{-1}\cdot(x\cdot y)\cdot y^{-1} = x^{-1}\cdot\mathbf{0}\cdot y^{-1}=\mathbf{0}$, a contradiction. □

The Kochin's Theorem [21] and Theorem 4 imply the following assertion.

Theorem 5. *Every inverse* 0*-simple* ω*-semigroup* S *is isomorphic to the Bruck-Reilly extension* $\mathbf{BR}^{\mathbf{0}}(T,\theta)$ *of a finite chain of groups* $T=\left[E;G_\alpha,\varphi_{\alpha,\beta}\right]$ *with adjoined zero.*

The main result of this section is the following theorem.

Theorem 6. *Let* S *be a Hausdorff semitopological* 0*-simple* ω*-semigroup such that every maximal subgroup of* S *is compact. Then* S *is topologically isomorphic to the topological Bruck-Reilly extension* $\left(\mathbf{BR}^{\mathbf{0}}(T,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *of a finite semilattice* $T=\left[E;G_\alpha,\varphi_{\alpha,\beta}\right]$ *of compact groups* G_α *in the class of Hausdorff topological inverse semigroups with adjoined zero such that the topology* $\tau^{\mathbf{0}}_{\mathbf{BR}}$ *induces on* $\mathbf{BR}^{\mathbf{0}}(T,\theta)$ *the sum direct topology* $\tau^{\oplus}_{\mathbf{BR}}$. *Moreover, if the space* S *is locally compact, then either the space* $\left(\mathbf{BR}^{\mathbf{0}}(T,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *is compact or any* $\mathscr{H}$*-class in* $\left(\mathbf{BR}^{\mathbf{0}}(T,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *is open-and-compact.*

Proof. The first statement of the theorem follows from Theorems 2 and 5. Next, using Theorem 3, we obtain the second statement. □

Remark 1. *We observe that the Bruck-Reilly extension* $\mathbf{BR^0}(T,\theta)$ *of a finite semilattice* $T=[E;G_\alpha,\varphi_{\alpha,\beta}]$ *of groups* G_α *with adjoined zero has two types of* $\mathscr{H}$*-classes: the first is a singleton and it consists of zero* $\mathbf{0}$*, and other classes are of the form* $(G_\alpha)_{i,j}$*,* $i,j\in\omega$*.*

Since the bicyclic monoid $\mathscr{C}(p,q)$ does not embed into any Hausdorff compact topological semigroup [1], Theorem 6 implies the following corollary.

Corollary 2. *Let* S *be a Hausdorff topological* 0*-simple inverse* ω*-semigroup such that every maximal subgroup of* S *is compact. Then* S *is topologically isomorphic to the topological Bruck-Reilly extension* $\left(\mathbf{BR^0}(T,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *of a finite semilattice* $T=[E;G_\alpha,\varphi_{\alpha,\beta}]$ *of compact groups* G_α *in the class of Hausdorff topological inverse semigroups with adjoined zero and any* $\mathscr{H}$*-class in* $\left(\mathbf{BR^0}(T,\theta),\tau^{\mathbf{0}}_{\mathbf{BR}}\right)$ *is open-and-compact.*

3 On closures of simple inverse semitopological ω-semigroup with compact maximal subgroups

We need the following lemma which is a simple generalization of [13, Lemma I.1(i)].

Lemma 7. *Let* $\mathbf{BR}(T,\theta)$ *be the Bruck-Reilly extension of a monoid* T*. Then for arbitrary* T_{i_1,j_1} *and* T_{i_2,j_2} *of* $\mathbf{BR}(T,\theta)$*,* $i_1,j_1,i_2,j_2\in\omega$*, there exist finitely many subsets* $T_{i,j}$ *in* $\mathbf{BR}(T,\theta)$*,* $i,j\in\omega$*, such that* $T_{i_1,j_1}\cdot T_{i,j}\subseteq T_{i_2,j_2}$ $(T_{i,j}\cdot T_{i_1,j_1}\subseteq T_{i_2,j_2})$*.*

Proof. The definitions of the semigroup operations of the Bruck-Reilly extension $\mathbf{BR}(S,\theta)$ and the bicyclic monoid $\mathscr{C}(p,q)$ imply that if $(i_a,s_a,j_a)\cdot(i_x,s_x,j_x)=(i_b,s_b,j_b)$ in $\mathbf{BR}(S,\theta)$, then $(i_a,j_a)\cdot(i_x,j_x)=(i_b,j_b)$ in $\mathscr{C}(p,q)$. By [13, Lemma I.1(i)], every equation of the form $ax=b$ $(xa=b)$ in $\mathscr{C}(p,q)$ has finitely many solutions, which implies the statement of the lemma. ☐

The following proposition generalizes [13, Theorem I.3] and the corresponding proposition from [17].

Proposition 7. *Let* T *be a compact Hausdorff topological semigroup and* $(\mathbf{BR}(T,\theta),\tau_{\mathbf{BR}})$ *be a topological Bruck-Reilly extension of* T *in the class of Hausdorff semitopological semigroups. If* $(\mathbf{BR}(T,\theta),\tau_{\mathbf{BR}})$ *is a dense subsemigroup of a Hausdorff semitopological monoid* S *and* $I=S\setminus\mathbf{BR}(T,\theta)\neq\varnothing$*, then* I *is a two-sided ideal of the semigroup* S*.*

Proof. Fix an arbitrary element $y\in I$. If $(i,s,j)\cdot y=z\notin I$ for some $(i,s,j)\in\mathbf{BR}(T,\theta)$, then $z=(k,t,l)\in\mathbf{BR}(T,\theta)$ for some $t\in T$ and $k,l\in\omega$. By Theorem 2, there exists an open neighbourhood $U(y)$ of the point y in the space S such that $\{(i,s,j)\}\cdot U(y)\subseteq T_{k,l}$. Since T is a compact Hausdorff topological semigroup, Theorem 2 implies that the topology $\tau_{\mathbf{BR}}$ coincides with the sum direct topology $\tau^{\oplus}_{\mathbf{BR}}$. By [18, Proposition 2.4], all subsets of the form $T_{n,m}$, $n,m\in\omega$, are compact. Hence the neighbourhood $U(y)$ intersects infinitely many sets of the form $T_{n,m}$, $n,m\in\omega$. Then the semigroup operation of $\mathbf{BR}(T,\theta)$ implies that $\{(i,s,j)\}\cdot U(y)\nsubseteq T_{k,l}$, which contradicts Lemma 7. The obtained contradiction implies that $(i,s,j)\cdot y\in I$. The proof of the statement that $y\cdot(i,s,j)\in I$ for all $(i,s,j)\in\mathbf{BR}(T,\theta)$ and $y\in I$ is similar.

Suppose to the contrary that $xy = w = (k,t,l) \in \mathbf{BR}(T,\theta)$ for some $x,y \in I$. Theorem 2 and the separate continuity of the semigroup operation in S imply that there exist open neighbourhoods $U(x)$ and $U(y)$ of the points x and y in S, respectively, such that $\{x\} \cdot U(y) \subseteq T_{k,l}$ and $U(x) \cdot \{y\} \subseteq T_{k,l}$. By [18, Proposition 2.4], all subsets of the form $T_{n,m}$, $n,m \in \omega$, are compact. Hence, the neighbourhood $U(y)$ intersects infinitely many sets of the form $T_{n,m}$, $n,m \in \omega$, therefore both inclusions $\{x\} \cdot U(y) \subseteq T_{k,l}$ and $U(x) \cdot \{y\} \subseteq T_{k,l}$ contradict mentioned above Lemma 7. The obtained contradiction implies that $xy \in I$. □

For an arbitrary ideal I of a semigroup S the binary relation $\mathfrak{C}_I = \{(t,t) \colon t \in S\} \cup (I \times I)$ on S is a congruence and $\mathfrak{C}_I$ is called the *Rees congruence* on S [10]. Also the quotient semigroup $S/\mathfrak{C}_I$ is called the *Rees-quotient semigroup* and denoted by S/I.

We need the following trivial lemma, which follows from the separate continuity of the semigroup operation in semitopological semigroups.

Lemma 8. *Let S be a Hausdorff semitopological semigroup and I be a compact ideal in S. Then the Rees-quotient semigroup S/I with the quotient topology is a Hausdorff semitopological semigroup.*

Theorem 7. *Let T be a compact Hausdorff semitopological semigroup and $(\mathbf{BR}(T,\theta), \tau_{\mathbf{BR}})$ be a topological Bruck-Reilly extension of T in the class of Hausdorff semitopological semigroups. Let $\mathbf{BR}_I(T,\theta) = \mathbf{BR}(T,\theta) \sqcup I$ and τ be a Hausdorff locally compact shift-continuous topology on $\mathbf{BR}_I(T,\theta)$, where I is a compact ideal of $\mathbf{BR}_I(T,\theta)$. Then either $(\mathbf{BR}_I(T,\theta), \tau)$ is a compact semitopological semigroup or the ideal I is open.*

Proof. Suppose that the ideal I is not open. By Lemma 8, the Rees-quotient semigroup $\mathbf{BR}_I(T,\theta)/I$, endowed with the quotient topology $\tau_{\mathbf{q}}$, is a semitopological semigroup. Let $\pi \colon \mathbf{BR}_I(T,\theta) \to \mathbf{BR}_I(T,\theta)/I$ be the natural homomorphism, which is a quotient map. It is obvious that the Rees-quotient semigroup $\mathbf{BR}_I(T,\theta)/I$ is isomorphic to the Bruck-Reilly extension with adjoined zero $\mathbf{BR}^{\mathbf{0}}(T,\theta)$ and the image $(I)\pi$ is zero of the semigroup $\mathbf{BR}^{\mathbf{0}}(T,\theta)$.

We show that the natural homomorphism $\pi \colon \mathbf{BR}_I(T,\theta) \to \mathbf{BR}_I(T,\theta)/I$ is a hereditarily quotient map. In particular, we show that for every open neighbourhood $U(I)$ of the compact ideal I in $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$ the image $(U(I))\pi$ is an open neighbourhood of $\mathbf{0}$ in $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$. Indeed, $\mathbf{BR}_I(T,\theta)/I \setminus U(I)$ is a closed subset of $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$. Also, since the restriction $\pi\restriction_{\mathbf{BR}(T,\theta)} \colon \mathbf{BR}(T,\theta) \to (\mathbf{BR}(T,\theta))\pi$ of the natural homomorphism $\pi \colon \mathbf{BR}_I(T,\theta) \to \mathbf{BR}_I(T,\theta)/I$ is one-to-one, we get that $(\mathbf{BR}_I(T,\theta)/I \setminus U(I))\pi$ is a closed subset of $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$. Hence, $(U(I))\pi$ is an open neighbourhood of $\mathbf{0}$ of the semigroup $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$. This implies that the natural homomorphism $\pi \colon \mathbf{BR}_I(T,\theta) \to \mathbf{BR}_I(T,\theta)/I$ is a hereditarily quotient map.

Since I is a compact ideal of the semitopological semigroup $(\mathbf{BR}_I(T,\theta), \tau)$, the preimage $(y)\pi^{-1}$ is a compact subset of $(\mathbf{BR}_I(T,\theta), \tau)$ for every $y \in \mathbf{BR}_I(T,\theta)/I$. By the Din'N'e T'ong's Theorem, the image of a locally compact Hausdorff space under a hereditary quotient map with compact fibers into a Hausdorff space is locally compact (see [12] or [14, 3.7.E]). Hence, the space $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$ is Hausdorff and locally compact. Since the ideal I is not open, by Theorem 6 the semitopological semigroup $(\mathbf{BR}_I(T,\theta)/I, \tau_{\mathbf{q}})$ is topologically isomorphic to $\left(\mathbf{BR}^{\mathbf{0}}(T,\theta), \tau_{\mathbf{BR}}^{\mathbf{Ac}}\right)$, and hence, it is compact.

Next, we show that the space $(\mathbf{BR}_I(T,\theta),\tau)$ is compact. Let $\mathscr{U} = \{U_\alpha \colon \alpha \in \mathscr{I}\}$ be any open cover of $(\mathbf{BR}_I(T,\theta),\tau)$. Since the ideal I is compact, it can be covered by some finite subfamily $\mathscr{U}' = \{U_{\alpha_1},\ldots,U_{\alpha_k}\}$ of $\mathscr{U}$. Put $U = U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. Then $\mathbf{BR}_I(T,\theta) \setminus U$ is a closed subset of $(\mathbf{BR}_I(T,\theta),\tau)$. Since the restriction $\pi{\restriction}_{\mathbf{BR}(T,\theta)} \colon \mathbf{BR}(T,\theta) \to (\mathbf{BR}(T,\theta))\pi$ of the natural homomorphism $\pi\colon \mathbf{BR}_I(T,\theta) \to \mathbf{BR}_I(T,\theta)/I$ is one-to-one, the image $(\mathbf{BR}_I(T,\theta) \setminus U)\pi$ is a closed subset of the space $(\mathbf{BR}_I(T,\theta)/I,\tau_{\mathbf{q}})$. Hence, the image $(\mathbf{BR}_I(T,\theta) \setminus U)\pi$ is compact, because the semitopological semigroup $(\mathbf{BR}_I(T,\theta)/I,\tau_{\mathbf{q}})$ is compact. Therefore, the set $\mathbf{BR}_I(T,\theta) \setminus U$ is compact, and hence, there exists a finite subfamily $\mathscr{U}''$ of $\mathscr{U}$, which is an open cover of $\mathbf{BR}_I(T,\theta) \setminus U$. Then $\mathscr{U}' \cup \mathscr{U}''$ is a finite cover of the space $(\mathbf{BR}_I(T,\theta),\tau)(\mathbf{BR}_I(T,\theta),\tau)$. Hence, the space $(\mathbf{BR}_I(T,\theta),\tau)$ is compact too. ☐

Theorem 7 implies the following assertion.

Theorem 8. *Let S be a Hausdorff semitopological simple inverse ω-semigroup such that every maximal subgroup of S is compact. Let $S_I = S \sqcup I$, τ be a Hausdorff locally compact shift-continuous topology on S_I, and I be a compact ideal of S_I. Then either (S_I,τ) is a compact semitopological semigroup or the ideal I is open.*

Since every Bruck-Reilly extension of a monoid contains an isomorphic copy of the bicyclic monoid $\mathscr{C}(p,q)$ and compact topological semigroups do not contain the semigroup $\mathscr{C}(p,q)$, Theorem 7 implies the following corollary.

Corollary 3. *Let T be a compact Hausdorff topological semigroup and $(\mathbf{BR}(T,\theta),\tau_{\mathbf{BR}})$ be a topological Bruck-Reilly extension of T in the class of Hausdorff semitopological semigroups. Let $\mathbf{BR}_I(T,\theta) = \mathbf{BR}(T,\theta) \sqcup I$ and τ be a Hausdorff locally compact shift-continuous topology on $\mathbf{BR}_I(T,\theta)$, where I is a compact ideal of $(\mathbf{BR}_I(T,\theta),\tau)$. Then the ideal I is open in $(\mathbf{BR}_I(T,\theta),\tau)$.*

Corollary 3 implies the following result.

Corollary 4. *Let S be a Hausdorff semitopological simple inverse ω-semigroup such that every maximal subgroup of S is compact. Let $S_I = S \sqcup I$, τ be a Hausdorff locally compact semigroup topology on S_I, and I be a compact ideal of S_I. Then the ideal I is open in (S_I,τ).*

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Гутік О.В., Максимик К.М. *Про напівтопологічні прості інверсні ω-напівгрупи з компактними максимальними підгрупами* // Карпатські матем. публ. — 2025. — Т.17, №1. — С. 110–127.

Описано структуру (0-)простих інверсних гаусдорфових напівтопологічних ω-напівгруп з компактними максимальними підгрупами. Зокрема, доведено, що якщо S — проста інверсна гаусдорфова напівтопологічна ω-напівгрупа з компактними максимальними підгрупами, то S є топологічно ізоморфною розширенню Брука-Рейлі $\left(\mathbf{BR}(T,\theta),\tau_{\mathbf{BR}}^{\oplus}\right)$ скінченної напіґратки $T=\left[E;G_\alpha,\varphi_{\alpha,\beta}\right]$ компактних груп G_α у класі топологічних інверсних напівгруп, де $\tau_{\mathbf{BR}}^{\oplus}$ — це топологія прямої суми на $\mathbf{BR}(T,\theta)$. Також доведено, що кожна гаусдорфова локально компактна трансляційно-неперервна топологія на простій інверсній гаусдорфовій напівтопологічній ω-напівгрупі з компактними максимальними підгрупами та приєднаним нулем є або компактною, або нуль є ізольованою точкою.

Ключові слова і фрази: біциклічна напівгрупа, проста інверсна ω-напівгрупа, напівтопологічна напівгрупа, локально компактний, топологічна напівгрупа, компактна максимальна підгрупа, приєднаний нуль, компактний ідеал.