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On Banach spaces of normalized Bloch mappings

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Applying the theory of tensor products of Banach spaces, we study the Banach spaces of normalized Bloch maps from $\mathbb{D}$ (the complex unit open disc) into X^* (the dual of a complex Banach space X) that can be represented canonically as the dual of the completion of the tensor product $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$, where $\mathrm{lin}(\Gamma(\mathbb{D}))$ is the space of X-valued Bloch molecules on $\mathbb{D}$ and α is a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. We show that the normalized spaces of Bloch maps, p-summing Bloch maps and Bloch maps that factor through a Hilbert space admit such a representation. On the converse problem, we characterize when a Banach normalized Bloch space $B(\mathbb{D}, X^*)$ is isometrically isomorphic to $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ for some Bloch cross-norm α, in terms of the compactness of its unit ball with respect to the weak* Bloch topology.

Key words and phrases: vector-valued Bloch mapping, tensor product, p-summing operator, duality.

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1 Introduction

The use of the tensor product of Banach spaces to study spaces of vector-valued holomorphic mappings is well known (see, for example, [2, 11–13]). Recently, some tensorial tools have been applied to address the problem of the duality of some distinguished ideals of vector-valued Bloch mappings in [3, 4, 8, 9].

Let $\mathbb{D}$ be the complex unit open disc, X be a complex Banach space and $\mathcal{H}(\mathbb{D}, X)$ be the space of all holomorphic maps from $\mathbb{D}$ into X. A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be Bloch if there exists a constant $c \geq 0$ such that $(1 - |z|^2)\, \|f'(z)\| \leq c$ for all $z \in \mathbb{D}$.

The normalized Bloch space $\widehat{\mathcal{B}}(\mathbb{D}, X)$ is the Banach space of all mappings $f \in \mathcal{H}(\mathbb{D}, X)$ for which $f(0) = 0$ satisfying

$$\rho_{\mathcal{B}}(f) := \sup\left\{(1 - |z|^2)\big\|f'(z)\big\| : z \in \mathbb{D}\right\} < \infty$$

equipped with the Bloch norm $\rho_{\mathcal{B}}$. In particular, $\widehat{\mathcal{B}}(\mathbb{D}) := \widehat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. The monographs by J.M. Anderson [1] and K. Zhu [16] provide a comprehensive study of these known function spaces.

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Following [8], a Banach ideal of normalized Bloch maps is an assignment $[\mathcal{I}^{\widehat{\mathcal{B}}}, \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}]$ that associates with every complex Banach space X, a set $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X)$ and a function $\|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \colon \mathcal{I}^{\widehat{\mathcal{B}}} \to \mathbb{R}$ satisfying the properties:

(i) $(\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X), \|\cdot\|_{\mathcal{I}^{\widehat{\mathcal{B}}}})$ is a Banach space and $\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \geq \rho_{\mathcal{B}}(f)$ for all $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$,

(ii) for each $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$, the map $g \cdot x \colon z \in \mathbb{D} \mapsto g(z)x \in X$ belongs to $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ and $\|g \cdot x\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} = \rho_{\mathcal{B}}(g)\,\|x\|$,

(iii) *the ideal property*: if $f \in \mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$, $h \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function such that $h(0) = 0$ and $T \in \mathcal{L}(X, Y)$, where Y is a complex Banach space, then $T \circ f \circ h$ is in $\mathcal{I}^{\widehat{\mathcal{B}}}(\mathbb{D}, Y)$ and $\|T \circ f \circ h\|_{\mathcal{I}^{\widehat{\mathcal{B}}}} \leq \|T\|\,\|f\|_{\mathcal{I}^{\widehat{\mathcal{B}}}}$.

If the ideal property (iii) is removed, then the concept of Banach space of normalized Bloch mappings arises, which has a linear counterpart in the notion of Banach space of operators presented by J.R. Holub in [7].

Definition 1. *A Banach space of normalized Bloch mappings (or simply, a Banach normalized Bloch space) from $\mathbb{D}$ into X is a linear space $B(\mathbb{D}, X) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X)$ endowed with a norm $\|\cdot\|_B$ satisfying:*

(i) $(B(\mathbb{D}, X), \|\cdot\|_B)$ is a Banach space and $\|f\|_B \geq \rho_{\mathcal{B}}(f)$ for every $f \in B(\mathbb{D}, X)$,

(ii) for every $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$, we have $g \cdot x \in B(\mathbb{D}, X)$ and $\|g \cdot x\|_B = \rho_{\mathcal{B}}(g)\|x\|$.

To describe the content of this paper, we need to introduce some notation and recall certain concepts and results from [4, 8].

Notation. Given complex Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from X to Y under the operator canonical norm. In particular, the space $\mathcal{L}(X, \mathbb{C})$ is denoted by X^*. The symbol κ_X stands for the canonical injection from X into X^{**}. For $x \in X$ and $x^* \in X^*$, we will sometimes write $\langle x^*, x\rangle = x^*(x)$. As usual, B_X and S_X represent the closed unit ball of X and the unit sphere of X, respectively.

Given Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we will write $(X, \|\cdot\|_X) \leq (Y, \|\cdot\|_Y)$ to indicate that $X \subseteq Y$ and $\|x\|_Y \leq \|x\|_X$ for all $x \in X$; and $(X, \|\cdot\|_X) \cong (Y, \|\cdot\|_Y)$ to point out that they are isometrically isomorphic.

Preliminaries. For each $z \in \mathbb{D}$, a Bloch atom of $\mathbb{D}$ is the function $\gamma_z \in \widehat{\mathcal{B}}(\mathbb{D})^*$ given by

$$\gamma_z(f) = f'(z), \qquad f \in \widehat{\mathcal{B}}(\mathbb{D}).$$

The elements of $\mathrm{lin}(\{\gamma_z \colon z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*$ are called Bloch molecules of $\mathbb{D}$. The Bloch-free Banach space over $\mathbb{D}$, denoted by $\mathcal{G}(\mathbb{D})$, is the norm-closed linear hull of $\{\gamma_z \colon z \in \mathbb{D}\} \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*$. The mapping $\Gamma \colon \mathbb{D} \to \mathcal{G}(\mathbb{D})$, defined by $\Gamma(z) = \gamma_z$ for all $z \in \mathbb{D}$, is holomorphic with $\|\gamma_z\| = 1/(1 - |z|^2)$ for all $z \in \mathbb{D}$.

For any $z \in \mathbb{D}$ and $x \in X$, the functional $\gamma_z \otimes x \colon \widehat{\mathcal{B}}(\mathbb{D}, X^*) \to \mathbb{C}$, given by

$$(\gamma_z \otimes x)\,(f) = \langle f'(z), x\rangle, \qquad f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*),$$

is in $\widehat{\mathcal{B}}(\mathbb{D}, X^*)^*$ with $\|\gamma_z \otimes x\| = \|x\|\,/(1 - |z|^2)$. Define the linear space

$$\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X := \mathrm{lin}\left(\{\gamma_z \otimes x \colon z \in \mathbb{D},\ x \in X\}\right) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X^*)^*.$$

The elements of this tensor product space are referred to as X-valued Bloch molecules on $\mathbb{D}$.

We denote by $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ the linear space $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ with the norm α, and by $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$ the completion of $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$. A norm α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ is said to be a Bloch reasonable cross-norm if it satisfies the following properties:

(i) $\alpha(\gamma_z \otimes x) = \|\gamma_z\| \|x\|$ for all $z \in \mathbb{D}$ and $x \in X$,

(ii) for every $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$, the linear functional $g \otimes x^* \colon \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X \to \mathbb{C}$, which is defined by $(g \otimes x^*)(\gamma_z \otimes x) = g'(z)x^*(x)$, is bounded on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ with $\|g \otimes x^*\| \leq \rho_{\mathcal{B}}(g) \|x^*\|$.

For each $z \in \mathbb{D}$, the function $f_z \colon \mathbb{D} \to \mathbb{C}$ defined by

$$f_z(w) = \frac{(1-|z|^2)w}{1-\bar{z}w}, \qquad w \in \mathbb{D},$$

belongs to $\widehat{\mathcal{B}}(\mathbb{D})$ with $\rho_{\mathcal{B}}(f_z) = 1 = (1-|z|^2)f_z'(z)$.

Remark 1. *In [4, Definition 2.5], the inequality $\leq$ is only required in the condition (i) above since the inequality $\geq$ holds always by applying the condition (ii). Indeed, given $z \in \mathbb{D}$ and $x \in X$, take $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$, and it follows that*

$$\begin{aligned}\alpha(\gamma_z \otimes x) &\geq \|f_z \otimes x^*\| \, \alpha(\gamma_z \otimes x) \geq |(f_z \otimes x^*)(\gamma_z \otimes x)| \\ &= |f_z'(z)| \, |x^*(x)| = \frac{\|x\|}{1-|z|^2} = \|\gamma_z\| \, \|x\| \, .\end{aligned}$$

Content. We have divided it into three sections. Inspired by the relation between tensor products and operator spaces studied by R. Schatten [15] and J.R. Holub [7], in Section 2 we introduce and study the space $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ of normalized α-Bloch mappings with respect to a Bloch cross-norm α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, that is, normalized Bloch mappings from $\mathbb{D}$ to X^* that induce a continuous linear functional on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$. We show that several known classes of normalized Bloch maps – namely, Bloch maps, p-summing Bloch maps and Bloch maps that factor through a Hilbert space – are associated to Bloch cross-norms in this way.

Section 3 deals with the duality theory for spaces of normalized α-Bloch mappings and contains the main result of this paper: the space of X^*-valued normalized α-Bloch mappings on $\mathbb{D}$ is canonically isometrically isomorphic to the dual of the space $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$. This section is completed by studying the weak* topology and the weak* Bloch topology on the space $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$.

The main questions we raise in Section 4 can be formulated as follows. When is $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ a Banach space of normalized Bloch mappings? In response to this question, we show that reasonable Bloch cross-norms α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ are justly those for which the cited statement holds.

The canonical identification of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ with the dual space $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ is the basis of our study of the duality for Banach normalized Bloch spaces. On the converse problem, given a Banach space of normalized Bloch mappings, when can it be represented as a space of normalized α-Bloch mappings? In response to this question, we characterize such Banach normalized Bloch spaces by means of the compactness of their unit closed balls with respect to the weak* Bloch topology.

2 Cross-norm-Bloch mappings

The following type of Bloch mappings will permit us to associate canonically a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ with a Banach space of normalized Bloch mappings from $\mathbb{D}$ into X^*. Compare it to the concept of operator with "finite α-norm" between Banach spaces E and F, where α is a cross-norm on $E\otimes F$ that was introduced by R. Schatten in [15, Definition 3.2].

Definition 2. *Let α be a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$. A zero-preserving holomorphic mapping $f\colon \mathbb{D}\to X^*$ is said to be an α-Bloch mapping if there exists a constant $c\geq 0$ such that*

$$\left|\sum_{i=1}^{n}\lambda_i\langle f'(z_i),x_i\rangle\right|\leq c\alpha\left(\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\right)$$

for all $\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$. The infimum of such constants c is denoted by $\rho_\alpha(f)$ and called the α-Bloch norm of f. The set of all α-Bloch maps from $\mathbb{D}$ into X^* is denoted by $\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$.

Remark 2. *Note that if $\gamma=\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ and $f\in\widehat{\mathcal{B}}(\mathbb{D},X^*)$, then*

$$\gamma(f)=\sum_{i=1}^{n}\lambda_i\left\langle f'(z_i),x_i\right\rangle,$$

and therefore $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ if and only if $|\gamma(f)|\leq c\alpha(\gamma)$ for all $\gamma\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$. Moreover, we have

$$\begin{aligned}\rho_\alpha(f)&=\min\left\{c\geq 0\colon\ |\gamma(f)|\leq c\alpha(\gamma),\ \forall\gamma\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X\right\}\\&=\sup\left\{\,|\gamma(f)|:\gamma\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X,\ \alpha(\gamma)\leq 1\right\}.\end{aligned}$$

We now show that every α-Bloch mapping turns out to be a Bloch mapping and this justifies the terminology used in Definition 2.

Lemma 1. *Let α be a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$. Then $(\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*),\rho_\alpha)$ is a normed space such that $(\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*),\rho_\alpha)\leq(\widehat{\mathcal{B}}(\mathbb{D},X^*),\rho_{\mathcal{B}})$.*

Proof. Let $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$. Given $z\in\mathbb{D}$, we have

$$\left|\langle f'(z),x\rangle\right|=|(\gamma_z\otimes x)(f)|\leq\rho_\alpha(f)\alpha\left(\gamma_z\otimes x\right)=\rho_\alpha(f)\|\gamma_z\|\,\|x\|$$

for all $x\in X$. Hence $(1-|z|^2)\|f'(z)\|\leq\rho_\alpha(f)$ for all $z\in\mathbb{D}$, and thus $f\in\widehat{\mathcal{B}}(\mathbb{D},X^*)$ with $\rho_{\mathcal{B}}(f)\leq\rho_\alpha(f)$. Clearly, $\rho_{\mathcal{B}}(f)\geq 0$. If $\rho_\alpha(f)=0$, it follows that $\rho_{\mathcal{B}}(f)=0$, and therefore $f=0$.

Next we use Remark 2. Let $\lambda\in\mathbb{C}$. For any $\gamma\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, we obtain

$$|\gamma(\lambda f)|=|\lambda\gamma(f)|=|\lambda|\,|\gamma(f)|\leq|\lambda|\,\rho_\alpha(f)\alpha(\gamma),$$

and therefore $\lambda f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\rho_\alpha(\lambda f)\leq|\lambda|\,\rho_\alpha(f)$. Moreover, by above-proved, if $\lambda=0$, then $\rho_\alpha(\lambda f)=0=|\lambda|\,\rho_\alpha(f)$, and if $\lambda\neq 0$, we have $\rho_\alpha(f)=\rho_\alpha(\lambda^{-1}(\lambda f))\leq|\lambda|^{-1}\rho_\alpha(\lambda f)$, and hence $|\lambda|\,\rho_\alpha(f)\leq\rho_\alpha(\lambda f)$. This proves that $\rho_\alpha(\lambda f)=|\lambda|\,\rho_\alpha(f)$. Finally, given $g\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$, we have

$$|\gamma(f+g)|=|\gamma(f)+\gamma(g)|\leq|\gamma(f)|+|\gamma(g)|\leq(\rho_\alpha(f)+\rho_\alpha(g))\alpha(\gamma)$$

for all $\gamma\in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, and so $f+g\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\rho_\alpha(f+g)\leq\rho_\alpha(f)+\rho_\alpha(g)$. □

Given $1 \leq p \leq \infty$, let p^* denote the Hölder's conjugate of p given by $p^* = \infty$ if $p = 1$, $p^* = p/(p-1)$ if $1 < p < \infty$ and $p^* = 1$ if $p = \infty$.

The following Bloch variants of the Chevet-Saphar norms on the tensor product of two Banach spaces (see [6, 14]) were introduced in [4].

For each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, the Bloch projective norm π and the p-Chevet-Saphar Bloch norms $d_p^{\mathcal{B}}$ for $1 \leq p \leq \infty$ are defined on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ by

$$
\begin{aligned}
\pi(\gamma) &= \inf\left\{ \sum_{i=1}^{n} \frac{|\lambda_i| \, \|x_i\|}{1-|z_i|^2} \right\}, \\
d_1^{\mathcal{B}}(\gamma) &= \inf\left\{ \left(\sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1 \leq i \leq n} |\lambda_i| \, |g'(z_i)| \right) \right) \left(\sum_{i=1}^{n} \|x_i\| \right) \right\}, \\
d_p^{\mathcal{B}}(\gamma) &= \inf\left\{ \left(\sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^{p^*} \, |g'(z_i)|^{p^*} \right)^{\frac{1}{p^*}} \right) \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \right\}, \qquad 1 < p < \infty, \\
d_\infty^{\mathcal{B}}(\gamma) &= \inf\left\{ \left(\sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i| \, |g'(z_i)| \right) \right) \left(\max_{1 \leq i \leq n} \|x_i\| \right) \right\},
\end{aligned}
$$

where the infimum is taken over all such representations of γ as $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i$ with $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathbb{D}$ and $x_1, \ldots, x_n \in X$.

The following result gathers some properties of such Bloch norms.

Proposition 1 ([4]). *For each $p \in [1, \infty]$, $d_p^{\mathcal{B}}$ is a reasonable Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. Moreover, $\pi(\gamma) = d_1^{\mathcal{B}}(\gamma) = \|\gamma\|$, where*

$$
\|\gamma\| = \sup\left\{ |\gamma(f)| : f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*),\ \rho_{\mathcal{B}}(f) \leq 1 \right\}
$$

for all $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.

Proof. The first assertion of the statement was proved in [4, Theorem 2.6]. The equalities $\pi = d_1^{\mathcal{B}} = \|\cdot\|$ on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ were established in [4, Propositions 2.4 and 2.7]. □

We now identify the space of all Bloch mappings from $\mathbb{D}$ into X^* with the space of all π-Bloch mappings.

Lemma 2. $(\widehat{\mathcal{B}}(\mathbb{D}, X^*), \rho_{\mathcal{B}}) = (\widehat{\mathcal{B}}_\pi(\mathbb{D}, X^*), \rho_\pi)$.

Proof. Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)$. Given $\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, we have

$$
|\gamma(f)| = \left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle \right| \leq \sum_{i=1}^{n} |\lambda_i| \, \|f'(z_i)\| \, \|x_i\| \leq \rho_{\mathcal{B}}(f) \sum_{i=1}^{n} \frac{|\lambda_i| \, \|x_i\|}{1-|z_i|^2},
$$

and taking the infimum over all the representations of γ, we deduce that $|\gamma(f)| \leq \rho_{\mathcal{B}}(f)\pi(\gamma)$. Hence $f \in \widehat{\mathcal{B}}_\pi(\mathbb{D}, X^*)$ and $\rho_\pi(f) \leq \rho_{\mathcal{B}}(f)$. The converse inclusion and inequality follow from Lemma 1. □

Following [4], a zero-preserving map $f \in \mathcal{H}(\mathbb{D}, X)$ is said to be p-summing Bloch for $1 \leq p \leq \infty$, if there exists a constant $c \geq 0$ such that

$$\left(\sum_{i=1}^{n} |\lambda_i|^p \|f'(z_i)\|^p \right)^{\frac{1}{p}} \leq c \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^p \left|g'(z_i)\right|^p \right)^{\frac{1}{p}}, \qquad \text{if} \quad 1 \leq p < \infty,$$

$$\max_{1 \leq i \leq n} |\lambda_i| \|f'(z_i)\| \leq c \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1 \leq i \leq n} |\lambda_i| \left|g'(z_i)\right| \right), \qquad \text{if} \quad p = \infty,$$

for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in \mathbb{D}$. The infimum of such constants c is denoted by $\pi_p^{\mathcal{B}}(f)$ and called the p-summing Bloch norm of f. The set $\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ of all p-summing Bloch mappings from $\mathbb{D}$ into X so that $f(0) = 0$ with the norm $\pi_p^{\mathcal{B}}$ is a Banach space by [4, Proposition 1.2].

Theorem 1. *Let* $1 \leq p \leq \infty$. *Then*

$$(\widehat{\mathcal{B}}_{d_p^{\mathcal{B}}}(\mathbb{D}, X^*), \rho_{d_p^{\mathcal{B}}}) = (\Pi_{p^*}^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*), \pi_{p^*}^{\mathcal{B}}).$$

Proof. Let $f \in \Pi_{p^*}^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*)$ and $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. If $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i$ is a representation of γ, then

$$\begin{aligned} |\gamma(f)| &\leq \sum_{i=1}^{n} |\lambda_i| \left\|f'(z_i)\right\| \|x_i\| \\ &\leq \left(\sum_{i=1}^{n} |\lambda_i|^{p^*} \left\|f'(z_i)\right\|^{p^*} \right)^{\frac{1}{p^*}} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \\ &\leq \pi_{p^*}^{\mathcal{B}}(f) \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^{p^*} \left|g'(z_i)\right|^{p^*} \right)^{\frac{1}{p^*}} \end{aligned}$$

whenever $1 < p < \infty$. For $p = 1$, one has

$$\begin{aligned} |\gamma(f)| &\leq \sum_{i=1}^{n} |\lambda_i| \left\|f'(z_i)\right\| \|x_i\| \\ &\leq \left(\max_{1 \leq i \leq n} |\lambda_i| \left\|f'(z_i)\right\| \right) \sum_{i=1}^{n} \|x_i\| \\ &\leq \pi_{\infty}^{\mathcal{B}}(f) \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1 \leq i \leq n} |\lambda_i| \left|g'(z_i)\right| \right) \sum_{i=1}^{n} \|x_i\|, \end{aligned}$$

and, for $p = \infty$, we obtain

$$\begin{aligned} |\gamma(f)| &\leq \sum_{i=1}^{n} |\lambda_i| \left\|f'(z_i)\right\| \|x_i\| \\ &\leq \left(\max_{1 \leq i \leq n} \|x_i\| \right) \sum_{i=1}^{n} |\lambda_i| \left\|f'(z_i)\right\| \\ &\leq \pi_1^{\mathcal{B}}(f) \left(\max_{1 \leq i \leq n} \|x_i\| \right) \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i| \left|g'(z_i)\right| \right). \end{aligned}$$

Passing to the infimum over all such representations of γ yields that $|\gamma(f)| \leq \pi^{\mathcal{B}}_{p^*}(f)d^{\mathcal{B}}_p(\gamma)$. Hence, $f \in \widehat{\mathcal{B}}_{d^{\mathcal{B}}_p}(\mathbb{D}, X^*)$ and $\rho_{d^{\mathcal{B}}_p}(f) \leq \pi^{\mathcal{B}}_{p^*}(f)$.

Conversely, let $f \in \widehat{\mathcal{B}}_{d^{\mathcal{B}}_p}(\mathbb{D}, X^*)$, $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in \mathbb{D}$. Choose $\varepsilon > 0$. For each $i \in \{1, \ldots, n\}$, we can take $x_i \in X$ with $\|x_i\| \leq 1 + \varepsilon$ such that $\langle f'(z_i), x_i\rangle = \|f'(z_i)\|$. Clearly, the function $T\colon \mathbb{K}^n \to \mathbb{K}$, defined by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^n t_i\lambda_i \left\|f'(z_i)\right\|, \quad \forall (t_1, \ldots, t_n) \in \mathbb{K}^n,$$

belongs to $(\mathbb{K}^n, \|\cdot\|_p)^*$ with

$$\|T\| = \begin{cases} \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left\|f'(z_i)\right\|^{p^*}\right)^{\frac{1}{p^*}}, & \text{if } 1 < p \leq \infty, \\ \max_{1\leq i\leq n} |\lambda_i| \left\|f'(z_i)\right\|, & \text{if } p = 1. \end{cases}$$

For all $(t_1, \ldots, t_n) \in \mathbb{K}^n$ with $\|(t_1, \ldots, t_n)\|_p \leq 1$, we obtain

$$\begin{aligned} |T(t_1, \ldots, t_n)| &= \left|\sum_{i=1}^n \lambda_i \langle f'(z_i), t_ix_i\rangle\right| = \left|\sum_{i=1}^n \lambda_i \langle \gamma_{z_i} \otimes (t_ix_i), f\rangle\right| \\ &\leq \rho_{d^{\mathcal{B}}_p}(f) d^{\mathcal{B}}_p\left(\sum_{i=1}^n \lambda_i\gamma_{z_i} \otimes (t_ix_i)\right). \end{aligned}$$

For $1 < p < \infty$ we get

$$\begin{aligned} |T(t_1, \ldots, t_n)| &\leq \rho_{d^{\mathcal{B}}_p}(f) \left(\sum_{i=1}^n \|t_ix_i\|^p\right)^{\frac{1}{p}} \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left|g'(z_i)\right|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq \rho_{d^{\mathcal{B}}_p}(f)(1+\varepsilon) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left|g'(z_i)\right|^{p^*}\right)^{\frac{1}{p^*}}, \end{aligned}$$

hence one has

$$\left(\sum_{i=1}^n |\lambda_i|^{p^*} \left\|f'(z_i)\right\|^{p^*}\right)^{\frac{1}{p^*}} \leq \rho_{d^{\mathcal{B}}_p}(f)(1+\varepsilon) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left|g'(z_i)\right|^{p^*}\right)^{\frac{1}{p^*}}.$$

By the arbitrariness of ε we get

$$\left(\sum_{i=1}^n |\lambda_i|^{p^*} \left\|f'(z_i)\right\|^{p^*}\right)^{\frac{1}{p^*}} \leq \rho_{d^{\mathcal{B}}_p}(f) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i|^{p^*} \left|g'(z_i)\right|^{p^*}\right)^{\frac{1}{p^*}},$$

and so $f \in \Pi^{\widehat{\mathcal{B}}}_{p^*}(\mathbb{D}, X^*)$ with $\pi^{\mathcal{B}}_{p^*}(f) \leq \rho_{d^{\mathcal{B}}_p}(f)$.

For $p = 1$ we have

$$\begin{aligned} |T(t_1, \ldots, t_n)| &\leq \rho_{d^{\mathcal{B}}_1}(f) \left(\sum_{i=1}^n \|t_ix_i\|\right) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1\leq i\leq n} |\lambda_i| \left|g'(z_i)\right|\right) \\ &\leq \rho_{d^{\mathcal{B}}_1}(f)(1+\varepsilon) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1\leq i\leq n} |\lambda_i| \left|g'(z_i)\right|\right), \end{aligned}$$

which yields

$$\max_{1\le i\le n} |\lambda_i| \left\|f'(z_i)\right\| \le \rho_{d_1^{\mathcal{B}}}(f) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1\le i\le n} |\lambda_i| \left|g'(z_i)\right| \right),$$

and so $f \in \Pi_\infty^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*)$ with $\pi_\infty^{\mathcal{B}}(f) \le \rho_{d_1^{\mathcal{B}}}(f)$.

For $p = \infty$ one has

$$\begin{aligned} |T(t_1,\dots,t_n)| &\le \rho_{d_\infty^{\mathcal{B}}}(f) \left(\max_{1\le i\le n} \|t_i x_i\| \right) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i| \left|g'(z_i)\right| \right) \\ &\le \rho_{d_\infty^{\mathcal{B}}}(f)(1+\varepsilon) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i| \left|g'(z_i)\right| \right), \end{aligned}$$

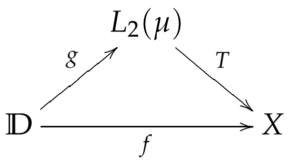
consequently,

$$\sum_{i=1}^n |\lambda_i| \left\|f'(z_i)\right\| \le \rho_{d_\infty^{\mathcal{B}}}(f) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n |\lambda_i| \left|g'(z_i)\right| \right),$$

and so $f \in \Pi_1^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*)$ with $\pi_1^{\mathcal{B}}(f) \le \rho_{d_\infty^{\mathcal{B}}}(f)$. □

We can obtain a similar identification for the space of Bloch maps that admit a factorization through of a Hilbert space. Let us recall the basic facts from [5].

A mapping $f \in \mathcal{H}(\mathbb{D}, X)$ is said to factor through of a Hilbert space if there exists a measure μ, a mapping $g \in \widehat{\mathcal{B}}(\mathbb{D}, L_2(\mu))$ and an operator $T \in \mathcal{L}(L_2(\mu), X)$ such that $f = T \circ g$, that is, the following diagram



commutes. We set $\gamma_2^{\mathcal{B}}(f) = \inf\{\|T\| \rho_{\mathcal{B}}(g)\}$, where the infimum runs over all possible factorizations of f as above. We denote by $(\Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D}, X), \gamma_2^{\mathcal{B}})$ the Banach space of all mappings $f \in \mathcal{H}(\mathbb{D}, X)$ that admit such a factorization.

For $(\lambda_i, z_i)_{i=1}^n \in (\mathbb{C} \times \mathbb{D})^n$ and $(\mu_j, w_j)_{j=1}^m \in (\mathbb{C} \times \mathbb{D})^m$, $n, m \in \mathbb{N}$, we say that $(\lambda_i, z_i)_{i=1}^n$ is Bloch subordinated to $(\mu_j, w_j)_{j=1}^m$, and we write $(\lambda_i, z_i)_{i=1}^n \prec (\mu_j, w_j)_{j=1}^m$, whenever

$$\sum_{i=1}^n |\lambda_i|^2 \left|g'(z_i)\right|^2 \le \sum_{j=1}^m \left|\mu_j\right|^2 \left|g'(w_j)\right|^2$$

for all $g \in \widehat{\mathcal{B}}(\mathbb{D})$. This means that there exists an operator $T \in \mathcal{L}(\ell_2^m, \ell_2^n)$ with $\|T\| \le 1$, represented by an $n \times m$ complex matrix (a_{ij}), such that $\lambda_i \gamma_{z_i} = \sum_{j=1}^m a_{ij} \mu_j \gamma_{w_j}$ for every $1 \le i \le n$.

Moreover, a mapping $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ factors through a Hilbert space if and only if there exists a constant $c \ge 0$ such that

$$\sum_{i=1}^n |\lambda_i|^2 \left\|f'(z_i)\right\|^2 \le c^2 \sum_{j=1}^m \frac{\left|\mu_j\right|^2}{(1-|w_j|^2)^2},$$

whenever $(\lambda_i, z_i)_{i=1}^n \prec (\mu_j, w_j)_{j=1}^m$, $(\lambda_i, z_i)_{i=1}^n \in (\mathbb{C} \times \mathbb{D})^n$, $(\mu_j, w_j)_{j=1}^m \in (\mathbb{C} \times \mathbb{D})^m$, $n, m \in \mathbb{N}$. In this case, $\gamma_2^{\mathcal{B}}(f)$ is the minimum of all constants c satisfying the preceding inequality.

For $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, we set

$$w_2^{\mathcal{B}}(\gamma) = \inf \left\{ \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \frac{|\mu_j|^2}{(1-|w_j|^2)^2} \right)^{\frac{1}{2}} \right\},$$

where the infimum is taken over the representations of γ in the form $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i$ such that $(\lambda_i, z_i)_{i=1}^n \prec (\mu_j, w_j)_{j=1}^m$ with $(\lambda_i, z_i)_{i=1}^n \in (\mathbb{C} \times \mathbb{D})^n$, $(\mu_j, w_j)_{j=1}^m \in (\mathbb{C} \times \mathbb{D})^m$, $n, m \in \mathbb{N}$, and $(x_i)_{i=1}^n \in X^n$. We know that $w_2^{\mathcal{B}}$ defines a reasonable Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.

We now prove that the Banach space of all normalized $w_2^{\mathcal{B}}$-Bloch mappings from $\mathbb{D}$ to X^* can be identified with the Banach space $\Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*)$.

Theorem 2. $(\widehat{\mathcal{B}}_{w_2^{\mathcal{B}}}(\mathbb{D}, X^*), \rho_{w_2^{\mathcal{B}}}) = (\Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*), \gamma_2^{\mathcal{B}})$.

Proof. Let $f \in \Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*)$ and $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. Take a representation $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i$ of γ such that $(\lambda_i, z_i)_{i=1}^n \prec (\mu_j, w_j)_{j=1}^m$, $(\lambda_i, z_i)_{i=1}^n \in (\mathbb{C} \times \mathbb{D})^n$, $(\mu_j, w_j)_{j=1}^m \in (\mathbb{C} \times \mathbb{D})^m$, $n, m \in \mathbb{N}$, and $(x_i)_{i=1}^n \in X^n$. We have

$$\begin{aligned} |\gamma(f)| &\leq \sum_{i=1}^{n} |\lambda_i| \left\| f'(z_i) \right\| \|x_i\| \\ &\leq \left(\sum_{i=1}^{n} |\lambda_i|^2 \left\| f'(z_i) \right\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \\ &\leq \gamma_2^{\mathcal{B}}(f) \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \frac{|\mu_j|^2}{(1-|w_j|^2)^2} \right)^{\frac{1}{2}}, \end{aligned}$$

and therefore $|\gamma(f)| \leq \gamma_2^{\mathcal{B}}(f) w_2^{\mathcal{B}}(\gamma)$ by taking the infimum over all the representations of γ. Hence, $f \in \widehat{\mathcal{B}}_{w_2^{\mathcal{B}}}(\mathbb{D}, X^*)$ and $\rho_{w_2^{\mathcal{B}}}(f) \leq \gamma_2^{\mathcal{B}}(f)$.

Conversely, let $f \in \widehat{\mathcal{B}}_{w_2^{\mathcal{B}}}(\mathbb{D}, X^*)$ and let $(\lambda_i, z_i)_{i=1}^n \in (\mathbb{C} \times \mathbb{D})^n$, $(\mu_j, w_j)_{j=1}^m \in (\mathbb{C} \times \mathbb{D})^m$, $n, m \in \mathbb{N}$, be such that $(\lambda_i, z_i)_{i=1}^n \prec (\mu_j, w_j)_{j=1}^m$. Choose any $\varepsilon > 0$. For each $i \in \{1, \ldots, n\}$, let us choose $x_i \in X$ with $\|x_i\| \leq 1 + \varepsilon$ such that $\langle f'(z_i), x_i \rangle = \|f'(z_i)\|$. Clearly, the function $T \colon \mathbb{K}^n \to \mathbb{K}$, given by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i \lambda_i \left\| f'(z_i) \right\|, \quad \forall (t_1, \ldots, t_n) \in \mathbb{K}^n,$$

is in $(\mathbb{K}^n, \|\cdot\|_2)^*$ with

$$\|T\| = \left(\sum_{i=1}^{n} |\lambda_i|^2 \left\| f'(z_i) \right\|^2 \right)^{\frac{1}{2}}.$$

For all $(t_1, \ldots, t_n) \in \mathbb{K}^n$ with $\|(t_1, \ldots, t_n)\|_2 \leq 1$, we have

$$\begin{aligned} |T(t_1, \ldots, t_n)| = \left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), t_i x_i \right\rangle \right| &= \left| \sum_{i=1}^{n} \lambda_i \left\langle \gamma_{z_i} \otimes (t_i x_i), f \right\rangle \right| \\ &\leq \rho_{w_2^{\mathcal{B}}}(f) w_2^{\mathcal{B}} \left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes (t_i x_i) \right). \end{aligned}$$

Hence, we get

$$|T(t_1,\dots,t_n)| \le \rho_{w_2^{\mathcal{B}}}(f)\left(\sum_{i=1}^{n}\|t_ix_i\|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{m}\frac{|\mu_j|^2}{(1-|w_j|^2)^2}\right)^{\frac{1}{2}}$$
$$\le \rho_{w_2^{\mathcal{B}}}(f)(1+\varepsilon)\left(\sum_{j=1}^{m}\frac{|\mu_j|^2}{(1-|w_j|^2)^2}\right)^{\frac{1}{2}},$$

therefore

$$\left(\sum_{i=1}^{n}|\lambda_i|^2\left\|f'(z_i)\right\|^2\right)^{\frac{1}{2}} \le \rho_{w_2^{\mathcal{B}}}(f)(1+\varepsilon)\left(\sum_{j=1}^{m}\frac{|\mu_j|^2}{(1-|w_j|^2)^2}\right)^{\frac{1}{2}}.$$

Passing to the limit as $\varepsilon\to 0$ yields

$$\left(\sum_{i=1}^{n}|\lambda_i|^2\left\|f'(z_i)\right\|^2\right)^{\frac{1}{2}} \le \rho_{w_2^{\mathcal{B}}}(f)\left(\sum_{j=1}^{m}\frac{|\mu_j|^2}{(1-|w_j|^2)^2}\right)^{\frac{1}{2}},$$

and thus $f\in\Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D},X^*)$ with $\gamma_2^{\mathcal{B}}(f)\le\rho_{w_2^{\mathcal{B}}}(f)$. □

3 Duality for spaces of cross-norm-Bloch mappings

We now establish a canonical identification between the normed space $(\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*),\rho_\alpha)$ and the dual space of $\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$ if α is a Bloch cross-norm on $\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$. Compare this result to [15, Theorem 3.1]. In particular, $(\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*),\rho_\alpha)$ will be a Banach space.

Theorem 3. *Let α be a Bloch cross-norm on $\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$. Then $\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ is isometrically isomorphic to $(\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ via the map $\Lambda\colon\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)\to(\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ defined by*

$$\Lambda(f)(\gamma)=\sum_{i=1}^{n}\lambda_i\left\langle f'(z_i),x_i\right\rangle$$

for all $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\gamma=\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$.

Its inverse satisfies

$$\left\langle(\Lambda^{-1}(\varphi))'(z),x\right\rangle=\varphi(\gamma_z\otimes x)$$

for all $\varphi\in(\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$, $z\in\mathbb{D}$ and $x\in X$.

Proof. Let $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and let $\Lambda_0(f)$ be the linear functional on $\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$ given by

$$\Lambda_0(f)(\gamma)=\sum_{i=1}^{n}\lambda_i\left\langle f'(z_i),x_i\right\rangle$$

for all $\gamma=\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$. Notice that $\Lambda_0(f)\in(\operatorname{lin}(\Gamma(\mathbb{D}))\otimes_\alpha X)^*$ and $\|\Lambda_0(f)\|\le\rho_\alpha(f)$, since

$$|\Lambda_0(f)(\gamma)|=\left|\sum_{i=1}^{n}\lambda_i\left\langle f'(z_i),x_i\right\rangle\right|\le\rho_\alpha(f)\alpha(\gamma)$$

for all $\gamma=\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in\operatorname{lin}(\Gamma(\mathbb{D}))\otimes X$. Hence, there is a unique continuous extension $\Lambda(f)$ of $\Lambda_0(f)$ to $\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$. Moreover, $\Lambda(f)$ is linear and $\|\Lambda(f)\|=\|\Lambda_0(f)\|$. Let $\Lambda\colon\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)\to(\operatorname{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ be the map so defined.

Clearly, the mapping $\Lambda_0\colon \widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)\to(\mathrm{lin}(\Gamma(\mathbb{D}))\otimes_\alpha X)^*$ is linear. If $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\Lambda_0(f)=0$, then $\langle f'(z),x\rangle=\Lambda_0(f)(\gamma_z\otimes x)=0$ for all $z\in\mathbb{D}$ and $x\in X$, hence $f'(z)=0$ for all $z\in\mathbb{D}$ and therefore $f=0$. This proves that Λ_0 is injective.

The map Λ is also linear and injective. Indeed, let $\phi\in\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$ and let $\{\gamma_n\}$ be a sequence in $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes_\alpha X$ such that $\alpha(\gamma_n-\phi)\to 0$ as $n\to\infty$. Given $a,b\in\mathbb{C}$ and $f,g\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$, an easy calculation shows that

$$\Lambda(af+bg)(\gamma_n)=\Lambda_0(af+bg)(\gamma_n)=(a\Lambda_0(f)+b\Lambda_0(g))(\gamma_n)=(a\Lambda(f)+b\Lambda(g))(\gamma_n)$$

for all $n\in\mathbb{N}$. Taking limits with $n\to\infty$, we get $\Lambda(af+bg)(\phi)=(a\Lambda(f)+b\Lambda(g))(\phi)$. Hence, Λ is linear. For the injectivity of Λ, note that if $f\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\Lambda(f)=0$, then $\Lambda_0(f)=0$, which implies that $f=0$ by the injectivity of Λ_0.

We now claim that Λ is a surjective isometry. Indeed, let $\varphi\in(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ and define $g_\varphi\colon\mathbb{D}\to X^*$ by

$$\left\langle g_\varphi(z),x\right\rangle=\varphi(\gamma_z\otimes x),\quad z\in\mathbb{D},\ x\in X.$$

Clearly, $g_\varphi(z)$ is a bounded linear functional on X with $\left\|g_\varphi(z)\right\|=\|\varphi\|\,\|\gamma_z\|$. Reasoning as in the proof of [4, Proposition 2.4], the mapping g_φ is in $\mathcal{H}(\mathbb{D},X^*)$ and there exists a map $f_\varphi\in\widehat{\mathcal{B}}(\mathbb{D},X^*)$ such that $f'_\varphi=g_\varphi$. For any $\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, we have

$$\begin{aligned}\left|\sum_{i=1}^n\lambda_i\left\langle f'_\varphi(z_i),x_i\right\rangle\right|&=\left|\sum_{i=1}^n\lambda_i\left\langle g_\varphi(z_i),x_i\right\rangle\right|\\&=\left|\varphi\left(\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i\right)\right|\le\|\varphi\|\,\alpha\left(\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i\right).\end{aligned}$$

Therefore, $f_\varphi\in\widehat{\mathcal{B}}_\alpha(\mathbb{D},X^*)$ and $\rho_\alpha(f_\varphi)\le\|\varphi\|$. For all $\gamma=\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes_\alpha x_i\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, we get

$$\begin{aligned}\Lambda(f_\varphi)(\gamma)=\Lambda_0(f_\varphi)(\gamma)&=\sum_{i=1}^n\lambda_i\left\langle f'_\varphi(z_i),x_i\right\rangle\\&=\sum_{i=1}^n\lambda_i\varphi(\gamma_{z_i}\otimes x_i)=\varphi\left(\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i\right)=\varphi(\gamma).\end{aligned}$$

Hence, $\Lambda(f_\varphi)=\varphi$ on a dense subspace of $\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$. So, $\Lambda(f_\varphi)=\varphi$. Moreover, $\rho_\alpha(f_\varphi)\le\|\varphi\|=\|\Lambda(f_\varphi)\|$. This completes the proof of our claim.

For the last assertion of the statement, note that

$$\langle(\Lambda^{-1}(\varphi))'(z),x\rangle=\langle f'_\varphi(z),x\rangle=\langle g_\varphi(z),x\rangle=\varphi(\gamma_z\otimes x)$$

for $\varphi\in(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$, $z\in\mathbb{D}$ and $x\in X$. □

Lemma 2 and Theorem 3 provide the following identification.

Corollary 1. $(\widehat{\mathcal{B}}(\mathbb{D},X^*),\rho_{\mathcal{B}})\cong((\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\pi X)^*,\|\cdot\|)$.

From Theorems 1 and 3, we infer the following description for the space of p-summing Bloch mappings from $\mathbb{D}$ into X^*.

Corollary 2. $(\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D},X^*),\pi_p^{\mathcal{B}})\cong((\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{d_{p^*}^{\mathcal{B}}}X)^*,\|\cdot\|)$ *for any* $1\le p\le\infty$.

Similarly, Theorems 2 and 3 permit us to describe the space of Bloch mappings from $\mathbb{D}$ into X^* that factor through a Hilbert space.

Corollary 3. $(\Gamma_2^{\widehat{\mathcal{B}}}(\mathbb{D}, X^*), \gamma_2^{\mathcal{B}}) \cong ((\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_{w_2^{\mathcal{B}}} X)^*, \|\cdot\|)$.

Since $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a dual Banach space by Theorem 3, the following topologies can be considered.

Definition 3. *Let α be a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.*

The weak topology (in short, w^*) on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is the topology induced by the linear space $\kappa_{\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X}(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)$ of linear functionals on $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$.*

The weak Bloch topology (in short, $w^*\mathcal{B}$) on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is the topology induced by the linear space $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ of linear functionals on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$.*

The following facts on $w^*\mathcal{B}$ can be deduced from the theory on topologies induced by families of functions (see, for example, [10, Section 2.4]).

Remark 3. *Let α be a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.*

(i) $w^*\mathcal{B}$ is a locally convex topology on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$, and the dual of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ with respect to this topology is $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. Since the family of functions $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ is separating, then $w^*\mathcal{B}$ is completely regular.

(ii) If $\{f_\nu\}$ is a net in $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ and $f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$, then $\{f_\nu\} \to f$ in the $w^*\mathcal{B}$-topology if and only if $\{\gamma(f_\nu)\} \to \gamma(f)$ for each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.

(iii) If $B(\mathbb{D}, X^*)$ is a linear subspace of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ and $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is equipped with the $w^*\mathcal{B}$-topology, then the relative $w^*\mathcal{B}$-topology of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ on $B(\mathbb{D}, X^*)$ agrees with the topology induced by the linear space $\{\gamma|_{B(\mathbb{D},X^*)} : \gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X\}$ of linear functionals on $B(\mathbb{D}, X^*)$.

Corollary 4. *Let α be a Bloch cross-norm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.*

(i) A net $\{f_\nu\}$ in $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ converges to $f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ in the w^*-topology if and only if $\{\gamma(f_\nu)\}$ converges to $\gamma(f)$ for every $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$.

(ii) On $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$, the $w^*\mathcal{B}$-topology is weaker than the w^*-topology. Moreover, both topologies agree on bounded subsets of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$.

Proof. *(i)* Let $\Lambda\colon \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*) \to (\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ be the identification of Theorem 3. Then we have

$$\begin{aligned}
\{f_\nu\} \to f \text{ in } (\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), w^*) &\Leftrightarrow \{\Lambda(f_\nu)\} \to \Lambda(f) \text{ in } ((\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*, w^*) \\
&\Leftrightarrow \{\langle \kappa_{\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X}(\gamma), \Lambda(f_\nu)\rangle\} \to \langle \kappa_{\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X}(\gamma), \Lambda(f)\rangle, \\
&\qquad\qquad \forall \gamma \in \mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X, \\
&\Leftrightarrow \{\Lambda(f_\nu)(\gamma)\} \to \Lambda(f)(\gamma), \quad \forall \gamma \in \mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X, \\
&\Leftrightarrow \{\gamma(f_\nu)\} \to \gamma(f), \quad \forall \gamma \in \mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X.
\end{aligned}$$

(ii) Let $\{f_\nu\}$ be a net in $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ such that $\{f_\nu\} \to f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ in the w^*-topology. By *(i)*, we have $\{\gamma(f_\nu)\} \to \gamma(f)$ for each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X$. In particular, $\{\gamma(f_\nu)\} \to \gamma(f)$ for each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. This means that $\{f_\nu\} \to f$ in the $w^*\mathcal{B}$-topology. Hence, the identity on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a continuous bijection from the w^*-topology to the $w^*\mathcal{B}$-topology and thus the latter topology is weaker that the former as required. On a bounded subset of $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$, the w^*-topology is compact and the $w^*\mathcal{B}$-topology is Hausdorff, so both topologies must coincide. □

4 Banach spaces of normalized Bloch mappings

We have identified $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ as the dual space $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ in Theorem 3. Our goal now is to address the general duality problem as to when a space of normalized Bloch mappings from $\mathbb{D}$ into X^* is isometrically isomorphic to $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ for some Bloch cross-norm α.

We first determine what are those Bloch cross-norms α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ for which $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a Banach normalized Bloch space.

Theorem 4. *Let α be a Bloch cross-norm on* $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$*. Then $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a Banach normalized Bloch space if and only if α is reasonable.*

Proof. In view of Lemma 1 and Theorem 3, we have that $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a linear subspace of $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$ and $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha)$ satisfies condition *(i)* in Definition 1. Hence, we only need to prove that $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha)$ satisfies condition *(ii)* in Definition 1 if and only if α is reasonable.

Assume that α is reasonable. Let $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$. We have

$$\begin{aligned}\left|\sum_{i=1}^n \lambda_i \left\langle (g\cdot x^*)'(z_i), x_i\right\rangle\right| &= \left|\sum_{i=1}^n \lambda_i g'(z_i)x^*(x_i)\right| \\ &= \left|(g\otimes x^*)\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right)\right| \\ &\le \|g\otimes x^*\|\,\alpha\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right) \\ &\le \rho_{\mathcal{B}}(g)\,\|x^*\|\,\alpha\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right)\end{aligned}$$

for all $\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$. So, $g\cdot x^* \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ and $\rho_\alpha(g\cdot x^*) \le \rho_{\mathcal{B}}(g)\,\|x^*\|$.

For the converse inequality, note that $\rho_{\mathcal{B}}(g)\,\|x^*\| = \rho_{\mathcal{B}}(g\cdot x^*) \le \rho_\alpha(g\cdot x^*)$, where we have used Lemma 1.

Conversely, if $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha)$ enjoys the cited condition *(ii)*, given $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$, one has

$$\begin{aligned}\left|(g\otimes x^*)\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right)\right| &= \left|\sum_{i=1}^n \lambda_i \left\langle (g\cdot x^*)'(z_i), x_i\right\rangle\right| \\ &\le \rho_\alpha(g\cdot x^*)\alpha\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right) \\ &= \rho_{\mathcal{B}}(g)\,\|x^*\|\,\alpha\left(\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i\right)\end{aligned}$$

for all $\sum_{i=1}^n \lambda_i\gamma_{z_i}\otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, and thus α is reasonable. □

The combination of [5, Theorem 5.2], Proposition 1 and Theorem 4 yields the following assertion.

Corollary 5. *$\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a Banach normalized Bloch space for $\alpha = w_2^{\mathcal{B}}$ and $d_p^{\mathcal{B}}$ with $p \in [1, \infty]$.*

We will now deal with the problem of when a Banach normalized Bloch space can be canonically identified with the dual of a tensor product space endowed with a Bloch cross-norm. Our approach is based on the arguments applied to address a similar problem in the setting of operator spaces (see [7, Section 4]).

Lemma 3. *Let $B(\mathbb{D}, X^*)$ be a Banach normalized Bloch space.*

For $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$, define

$$\alpha(\gamma) = \sup \left\{ \left| \sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i \right\rangle \right| : f \in B(\mathbb{D}, X^*),\ \|f\|_B = 1 \right\}$$

and

$$\langle \iota(\gamma), f \rangle = \sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i \right\rangle, \qquad f \in B(\mathbb{D}, X^*).$$

Then α is a reasonable Bloch cross-norm on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$, and ι is a linear isometry from $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ into $B(\mathbb{D}, X^*)^*$.

Proof. Let $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ and $f \in B(\mathbb{D}, X^*)$. Note that $\langle \iota(\gamma), f \rangle = \gamma(f)$. Clearly, $\iota(\gamma)$ is well defined on $B(\mathbb{D}, X^*)$, it is linear and

$$|\langle \iota(\gamma), f \rangle| \leq \rho_{\mathcal{B}}(f)\pi(\gamma) \leq \|f\|_B \pi(\gamma)$$

for all $f \in B(\mathbb{D}, X^*)$. Then $\iota(\gamma)$ is in $B(\mathbb{D}, X^*)^*$ and

$$\|\iota(\gamma)\| := \sup \{ |\langle \iota(\gamma), f \rangle| : f \in B(\mathbb{D}, X^*),\ \|f\|_B = 1 \} \leq \pi(\gamma).$$

Clearly, $\iota\colon \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X \to B(\mathbb{D}, X^*)^*$ is well defined and linear. Moreover, it is injective. Indeed, $\iota(\gamma) = 0$ means that $\langle \iota(\gamma), f \rangle = 0$ for all $f \in B(\mathbb{D}, X^*)$. In particular, we have $\langle \gamma, g \cdot x^* \rangle = \langle \iota(\gamma), g \cdot x^* \rangle = 0$ for all $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$. It follows that

$$x^* \left(\sum_{i=1}^n \lambda_i g'(z_i) x_i \right) = \sum_{i=1}^n \lambda_i g'(z_i) x^*(x_i) = 0$$

for all $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$. Since X^* separates the points of X, this implies that

$$\sum_{i=1}^n \lambda_i g'(z_i) x_i = 0$$

for all $g \in \widehat{\mathcal{B}}(\mathbb{D})$. For each $i \in \{1, \ldots, n\}$, consider the polynomial $q_i\colon \mathbb{D} \to \mathbb{C}$ defined by

$$q_i(z) = \prod_{k=1, k \neq i}^n \frac{z - z_k}{z_i - z_k},$$

and take a polynomial $p_i\colon \mathbb{D} \to \mathbb{C}$ such that $p_i' = q_i$ and $p_i(0) = 0$. Then $p_i \in \widehat{\mathcal{B}}(\mathbb{D})$ with $p_i'(z_k) = \delta_{ik}$ for all $k \in \{1, \ldots, n\}$, where δ_{ik} is the Kronecker's delta.

Hence,

$$0 = \sum_{k=1}^{n} \lambda_k p_i'(z_k) x_k = \sum_{k=1}^{n} \lambda_k \delta_{ik} x_k = \lambda_i x_i$$

for each $i \in \{1, \ldots, n\}$. Therefore, $\gamma(f) = \sum_{i=1}^{n} \langle f'(z_i), \lambda_i x_i \rangle = 0$ for all $f \in B(\mathbb{D}, X^*)$, and so we get $\gamma = 0$.

Define the map α on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ as in the statement. Notice that $\alpha(\gamma) = \|\iota(\gamma)\|$. Then α is a norm on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ and so ι is a linear isometry from $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ into $B(\mathbb{D}, X^*)^*$.

We now prove that α is a reasonable Bloch cross-norm. For any $z \in \mathbb{D}$ and $x \in X$, we have

$$\alpha(\gamma_z \otimes x) = \|\iota(\gamma_z \otimes x)\| \leq \pi(\gamma_z \otimes x) = \frac{\|x\|}{1 - |z|^2}.$$

The converse inequality will follow in light of Remark 1.

Given $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$, for any $g \in S_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $x^* \in S_{X^*}$, we get

$$\begin{aligned}
\left| (g \otimes x^*) \left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right) \right| &= \left| \sum_{i=1}^{n} \lambda_i g'(z_i) x^*(x_i) \right| \\
&= \left| \sum_{i=1}^{n} \lambda_i \left\langle (g \cdot x^*)'(z_i), x_i \right\rangle \right| \\
&\leq \sup \left\{ \left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle \right| : f \in B(\mathbb{D}, X^*),\ \|f\|_B = 1 \right\} \\
&= \alpha\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right),
\end{aligned}$$

and so α is reasonable. □

We are ready to state the main result of this section. Compare it to [7, Theorem 5].

Theorem 5. *Let $B(\mathbb{D}, X^*)$ be a Banach normalized Bloch space. The following statements are equivalent:*

(*i*) *there is a reasonable Bloch cross-norm α on $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ such that*

$$(B(\mathbb{D}, X^*), \|\cdot\|_B) = (\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha),$$

(*ii*) *if f is in $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$ and $\{f_\nu\}$ is a bounded net in $B(\mathbb{D}, X^*)$ converging to f in the weak* Bloch topology of $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$, then $f \in B(\mathbb{D}, X^*)$ and*

$$\|f\|_B \leq \sup\{\|f_\nu\|_B : \nu \in N\}.$$

Proof. $(i) \Rightarrow (ii)$ Let f and $\{f_\nu\}$ be as in (ii). Put $M = \sup\{\|f_\nu\|_B : \nu \in N\}$.

If $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ and $\varepsilon > 0$, then we get

$$\left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle - \sum_{i=1}^{n} \left\langle f'_{\nu_0}(z_i), x_i \right\rangle \right| = \left| \left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right) (f - f_{\nu_0}) \right| < \varepsilon$$

for some $\nu_0 \in N$, and therefore

$$\left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle \right| < \left| \sum_{i=1}^{n} \lambda_i \left\langle f'_{\nu_0}(z_i), x_i \right\rangle \right| + \varepsilon$$
$$\leq \rho_\alpha(f_{\nu_0})\alpha\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right) + \varepsilon$$
$$= \|f_{\nu_0}\|_B \,\alpha\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right) + \varepsilon$$
$$\leq M\alpha\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right) + \varepsilon.$$

Let us take the limit as $\varepsilon \to 0$. We obtain that $f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ and $\rho_\alpha(f) \leq M$. Hence, $f \in B(\mathbb{D}, X^*)$ and $\|f\|_B \leq M$.

$(ii) \Rightarrow (i)$ Take the reasonable Bloch cross-norm α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ and the linear isometry ι from $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ into $B(\mathbb{D}, X^*)^*$ defined in Lemma 3. In order to prove that $(B(\mathbb{D}, X^*), \|\cdot\|_B) = (\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha)$, given $f \in B(\mathbb{D}, X^*)$, the definition of α yields

$$\left| \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle \right| \leq \|f\|_B \,\alpha\left(\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \right)$$

for all $\sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, and thus $f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ with $\rho_\alpha(f) \leq \|f\|_B$.

Conversely, take $f \in \widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ and define the functional $S(f)\colon \iota(\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X) \to \mathbb{C}$ by

$$\langle S(f), \iota(\gamma) \rangle = \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle$$

for $\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. The injectivity of ι guarantees that $S(f)$ is well defined. The linearity of $S(f)$ follows easily. Since

$$|\langle S(f), \iota(\gamma) \rangle| = |\gamma(f)| \leq \rho_\alpha(f)\alpha(\gamma) = \rho_\alpha(f)\|\iota(\gamma)\|$$

for all $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, it follows that $S(f)$ is continuous and $\|S(f)\| \leq \rho_\alpha(f)$. Since $\iota(\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X)$ is a linear subspace of $B(\mathbb{D}, X^*)^*$, the Hahn-Banach theorem gives a functional $\widetilde{S}(f) \in B(\mathbb{D}, X^*)^{**}$ such that

$$\langle \widetilde{S}(f), \iota(\gamma) \rangle = \gamma(f)$$

for all $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, and

$$\|\widetilde{S}(f)\| = \|S(f)\|.$$

Let κ_B be the canonical injection from $B(\mathbb{D}, X^*)$ into $B(\mathbb{D}, X^*)^{**}$. By Goldstein's theorem, there exists a net $\{f_\nu\}$ in $B(\mathbb{D}, X^*)$ for which

$$\sup\{\|f_\nu\|_B \colon \nu \in N\} \leq \|\widetilde{S}(f)\|$$

and $\{\kappa_B(f_\nu)\} \to \widetilde{S}(f)$ in $(B(\mathbb{D}, X^*)^{**}, w^*)$. This means that

$$\langle \kappa_{B^*}(\varphi), \kappa_B(f_\nu) \rangle = \langle \kappa_B(f_\nu), \varphi \rangle \to \langle \kappa_{B^*}(\varphi), \widetilde{S}(f) \rangle = \langle \widetilde{S}(f), \varphi \rangle$$

for every $\varphi \in B(\mathbb{D}, X^*)^*$. Since $\iota(\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X) \subseteq B(\mathbb{D}, X^*)^*$, it follows that for each $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$, we have

$$\{\langle \kappa_B(f_\nu), \iota(\gamma) \rangle\} \to \langle \widetilde{S}(f), \iota(\gamma) \rangle,$$

that is, $\{\gamma(f_\nu)\} \to \gamma(f)$. Hence $\{f_\nu\} \to f$ in $(\widehat{\mathcal{B}}(\mathbb{D}, X^*), w^*\mathcal{B})$ by Remark 3. Then, by hypothesis, $f \in B(\mathbb{D}, X^*)$ and $\|f\|_B \leq \sup\{\|f_\nu\|_B \colon \nu \in N\} \leq \rho_\alpha(f)$. □

Theorem 5 admits the following useful reformulation that can be compared to the result by J.R. Holub (see [7, Corollary, p. 400]).

Corollary 6. *Let* $B(\mathbb{D}, X^*)$ *be a Banach normalized Bloch space. Then the following statements are equivalent:*

(*i*) *there exists a reasonable Bloch cross-norm* α *on* $\operatorname{lin}(\Gamma(\mathbb{D})) \otimes X$ *such that*

$$(B(\mathbb{D}, X^*), \|\cdot\|_B) = (\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha),$$

(*ii*) *the closed unit ball of* $B(\mathbb{D}, X^*)$ *is compact in the weak* Bloch topology of* $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$.

Proof. $(i) \Rightarrow (ii)$ Recall that $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), \rho_\alpha)$ is a dual Banach space by Theorem 3. Then, by (i), Alaoglu's theorem asserts that the closed unit ball of $B(\mathbb{D}, X^*)$ is compact in $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), w^*)$ and therefore in $(\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*), w^*\mathcal{B})$ by Corollary 4. Since $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ is a linear subspace of $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$, this last topology agrees with the relative $w^*\mathcal{B}$-topology of $\widehat{\mathcal{B}}(\mathbb{D}, X^*)$ on $\widehat{\mathcal{B}}_\alpha(\mathbb{D}, X^*)$ by Remark 3, and then (ii) follows easily.

$(ii) \Rightarrow (i)$ Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)$ and let $\{f_\nu\}$ be a bounded net in $B(\mathbb{D}, X^*)$ such that $\{f_\nu\} \to f$ in $(\widehat{\mathcal{B}}(\mathbb{D}, X^*), w^*\mathcal{B})$. Let $M = \sup\{\|f_\nu\|_B \colon \nu \in N\}$. By (ii), the closed unit ball of $B(\mathbb{D}, X^*)$ is closed in $(\widehat{\mathcal{B}}(\mathbb{D}, X^*), w^*\mathcal{B})$. Hence, the limit f/M of the net $\{f_\nu/M\}$ exists in $(\widehat{\mathcal{B}}(\mathbb{D}, X^*), w^*\mathcal{B})$, that is, f/M is in the closed unit ball of $B(\mathbb{D}, X^*)$. Hence, $f \in B(\mathbb{D}, X^*)$ and $\|f\|_B \leq M$. Then the statement (ii) of Theorem 5 holds and we obtain (i). □

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Застосовуючи теорію тензорних добутків банахових просторів, ми досліджуємо банахові простори нормалізованих Блох-образів із $\mathbb{D}$ (відкритого одиничного круга комплексної площини) у X^* (спряжений простір до комплексного банахового простору X), які можуть бути канонічно подані як спряжений простір до поповнення тензорного добутку $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$, де $\mathrm{lin}(\Gamma(D))$ позначає простір X-значних Блох-молекул на $\mathbb{D}$, а α є Блох-кроснормою на $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$. Показано, що простори нормалізованих Блох-образів, p-сумовні Блох-образи, а також Блох-образи, які факторизуються через гільбертів простір, допускають таке подання. У зворотній задачі охарактеризовано умови, за яких банахів простір нормалізованих Блох-образів $B(\mathbb{D}, X^*)$ є ізометрично ізоморфним до $(\mathrm{lin}(\Gamma(\mathbb{D}))\widehat{\otimes}_\alpha X)^*$ для деякої Блох-кроснорми α, зокрема в термінах компактності його одиничної кулі відносно слабкої* Блох-топології.

Ключові слова і фрази: векторнозначне Блох-відображення, тензорний добуток, p-сумовний оператор, двоїстість.