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Weakly p-nuclear bilinear operators

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The purpose of this article is to investigate the class of weakly p-nuclear bilinear operators between Banach spaces. This notion extends the classical theory of nuclear operators introduced by A. Grothendieck, as well as its multilinear generalizations developed by A. Pietsch and others. In particular, we characterize weakly p-nuclear bilinear operators through appropriate tensor norms, showing that the space of such operators forms a Banach ideal and analyzing some of its structural properties. Moreover, in the context of duality theory for these operator spaces, we introduce the class of quasi-Cohen p-nuclear bilinear operators and establish a Pietsch-type domination theorem.

Key words and phrases: weakly p-nuclear bilinear operator, quasi Cohen p-nuclear bilinear operator, duality.

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Introduction

The concept of a nuclear linear operator was first established by A. Grothendieck in the 1950's. Later on, in 1980, A. Pietsch extended [5] this idea to multilinear operators. Again, the notion of weakly nuclear linear operators, which are an expansion of nuclear linear operators, was introduced by J.M. Kim [4]. In this paper, inspired by Peitsch's contributions, we present the concept of weakly p-nuclear operators in the context of bilinear operators. This class constitutes a special subclass of the σ-nuclear bilinear operators previously introduced by G. Botelho and X. Mujica [2]. In this framework, we show that the space of weakly p-nuclear bilinear operators can be characterized by means of a suitable tensor norm, up to isometric isomorphism. Furthermore, we study the structure of the dual space of weakly p-nuclear bilinear operators, establishing connections with related classes of bilinear operators.

In order to clarify the structure of the paper, we outline its organization as follows. In Section 1, we introduce the notion of weakly p-nuclear bilinear operators and establish their basic properties. Section 2 is devoted to the study of the relationship between weakly p-nuclear operators and tensor products, providing characterizations via associated tensor norms. In Section 3, we investigate the dual space of weakly p-nuclear bilinear operators and present the class of quasi Cohen p-nuclear bilinear operators together with a Pietsch-type domination

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theorem. Finally, the paper ends with some remarks and concluding comments on possible extensions of the results.

In this paper, we clarify the notation used. Let D, D_i, E, E_i, F, F_i represent Banach spaces. We define E^* as the topological dual of E, and B_E as the closed unit ball in E. For $p \geq 1$, we denote its conjugate by p^*, which satisfies the relation $1 = \frac{1}{p} + \frac{1}{p^*}$. Additionally, we define the Banach space $\mathcal{L}(E_1, E_2; F)$, consisting of all continuous bilinear operators from $E_1 \times E_2$ into F, with the norm given by

$$\|T\| = \sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \|T(x_1, x_2)\|.$$

Let $\ell_p(E)$ denote the Banach space of all absolutely p-summable sequences $(x_n)_{n=1}^{\infty}$ in E for $1 \leq p < \infty$, equipped with the norm

$$\|(x_n)_{n=1}^{\infty}\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}.$$

We define $\ell_p^w(E)$ for $1 \leq p < \infty$ as the Banach space of all weakly p-summable sequences $(x_n)_{n=1}^{\infty}$ in X, with the norm given by

$$\|(x_n)_{n=1}^{\infty}\|_p^w = \sup_{x^* \in B_{E^*}} \|(\langle x^*, x_n \rangle)_{n=1}^{\infty}\|_p.$$

We define $x_1^* \otimes x_2^* \otimes y \in \mathcal{L}(E_1, E_2; F)$ by

$$x_1^* \otimes x_2^* \otimes y \colon E_1 \times E_2 \ni (x_1, x_2) \longmapsto (x_1^* \otimes x_2^* \otimes y)(x_1, x_2) = x_1^*(x_1) x_2^*(x_2) y \in F.$$

The collection of all these operators generates the vector space $\mathcal{L}_f\,(E_1, E_2; F)$ of the bilinear operators of finite type.

1 Weakly p-nuclear bilinear operators

Now, we extend the class of weakly p-nuclear linear operators, presented by J.M. Kim [4], to the case of bilinear operators. We begin by recalling the linear case. A linear operator $u : E \to F$ is weakly p-nuclear, $1 \leq p \leq \infty$, if it can be represented as $u = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, where $(x_n^*)_n \in \ell_p^w(E^*)$ and $(y_n)_n \in \ell_{p^*}^w(F)$. We write by $\mathcal{N}_{wp}(E; F)$ the space of all weakly p-nuclear linear operators from E into F, equipped with the norm

$$\|u\|_{\mathcal{N}_{wp}} := \inf \|(x_n^*)_n\|_p^w \|(y_n)_n\|_{p^*}^w,$$

where the infimum is taken over all such representations as above.

Definition 1. *Let $1 \leq p < \infty$. A bilinear operator $T : E_1 \times E_2 \to F$ is called weakly p-nuclear if T can be represented as*

$$T = \sum_{n=1}^{\infty} x_{1n}^* \otimes x_{2n}^* \otimes y_n,$$

where $\left(x_{in}^*\right)_{n=1}^{\infty} \in E_i^*$ for $i = 1, 2$, and $(y_n)_{n=1}^{\infty} \in \ell_{p^*}^w(F)$ satisfy

$$\sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \left(\sum_{n=1}^{\infty} |\langle x_{1n}^*, x_1 \rangle \langle x_{2n}^*, x_2 \rangle|^p \right)^{1/p} < \infty.$$

Moreover,

$$\|T\|_{\mathcal{N}_{wp}} := \inf\left\{ \sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \left(\sum_{n=1}^{\infty} |\langle x_{1n}^{*}, x_1\rangle \, \langle x_{2n}^{*}, x_2\rangle|^p \right)^{1/p} \sup_{y^*\in B_{F^*}} \left(\sum_{n=1}^{\infty} |\langle y^*, y_n\rangle|^{p^*} \right)^{1/p^*} \right\},$$

where the infimum is taken over all such representations as above. We write $\mathcal{N}_{wp}\,(E_1, E_2; F)$ to denote all weakly p-nuclear operators from $E_1 \times E_2$ into F.

Theorem 1. *For $1 \le p < \infty$, $[\mathcal{N}_{wp}\,(E_1, E_2; F)\,, \|\cdot\|_{\mathcal{N}_{wp}}]$ is a Banach ideal of 2-linear operators.*

Proof. First, let us show that $[\mathcal{N}_{wp}(E_1, E_2; F), \|\cdot\|_{\mathcal{N}_{wp}}]$ is a normed space. While this is obvious for $p = 1$, the statement that $T_1 + T_2 \in \mathcal{N}_{wp}(E_1, E_2; F)$ and $\|T_1 + T_2\|_{\mathcal{N}_{wp}} \le \|T_1\|_{\mathcal{N}_{wp}} + \|T_2\|_{\mathcal{N}_{wp}}$ whenever $T_1, T_2 \in \mathcal{N}_{wp}(E_1, E_2; F)$ requires proof if $1 < p < \infty$.

Consequently, fix $\epsilon > 0$ and choose representations $T_1 = \sum_{n=1}^{\infty} x_{2n-1}^{*(1)} \otimes x_{2n-1}^{*(2)} \otimes y_{2n-1}$ and $T_2 = \sum_{n=1}^{\infty} x_{2n}^{*(1)} \otimes x_{2n}^{*(2)} \otimes y_{2n}$ such that

$$\sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \left(\sum_{n=1}^{\infty} |\langle x_{2n-1}^{*(1)}, x_1\rangle \langle x_{2n-1}^{*(2)}, x_2\rangle|^p \right)^{1/p} \le \|T_1\|_{\mathcal{N}_{wp}} + \epsilon,$$

$$\sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \left(\sum_{n=1}^{\infty} |\langle x_{2n}^{*(1)}, x_1\rangle \langle x_{2n}^{*(2)}, x_2\rangle|^p \right)^{1/p} \le \|T_2\|_{\mathcal{N}_{wp}} + \epsilon,$$

and $\|(y_{2n})\|_{p^*}^{w} = \|(y_{2n-1})\|_{p^*}^{w} = 1$. Fix any $r, s > 0$ and define $(\tilde{y}_n) \in \ell_{p^*}^{w}(F)$ by $\tilde{y}_{2n} = s^{-1} y_{2n}$ and $\tilde{y}_{2n-1} = r^{-1} y_{2n-1}$. Let $\tilde{x}_{2n}^{*(1)} \otimes \tilde{x}_{2n}^{*(2)} = s x_{2n}^{*(1)} \otimes x_{2n}^{*(2)}$ and $\tilde{x}_{2n-1}^{*(1)} \otimes \tilde{x}_{2n-1}^{*(2)} = r x_{2n-1}^{*(1)} \otimes x_{2n-1}^{*(2)}$. Then

$$\begin{aligned}
\sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \sum_{n=1}^{\infty} |\langle \tilde{x}_n^{*(1)}, x_1\rangle \langle \tilde{x}_n^{*(2)}, x_2\rangle|^p &= r^p \sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \sum_{n=1}^{\infty} |\langle x_{2n-1}^{*(1)}, x_1\rangle \langle x_{2n-1}^{*(2)}, x_2\rangle|^p \\
&\quad + s^p \sup_{x_1\in B_{E_1},\, x_2\in B_{E_2}} \sum_{n=1}^{\infty} |\langle x_{2n}^{*(1)}, x_1\rangle \langle x_{2n}^{*(2)}, x_2\rangle|^p \\
&\le r^p (\|T_1\|_{\mathcal{N}_{wp}} + \epsilon)^p + s^p (\|T_2\|_{\mathcal{N}_{wp}} + \epsilon)^p
\end{aligned}$$

and

$$\left(\|(\tilde{y}_n)_n\|_{p^*}^{w} \right)^{p^*} \le \left(\frac{1}{r} \|(y_{2n-1})_n\|_{p^*}^{w} \right)^{p^*} + \left(\frac{1}{s} \|(y_{2n})_n\|_{p^*}^{w} \right)^{p^*} = r^{-p^*} + s^{-p^*}.$$

Writing $T_1 + T_2 = \sum_{n=1}^{\infty} \tilde{x}_n^{*(1)} \otimes \tilde{x}_n^{*(2)} \otimes \tilde{y}_n$, we see that $T_1 + T_2$ is weakly p-nuclear with

$$\begin{aligned}
\|T_1 + T_2\|_{\mathcal{N}_{wp}} &\le \left(\sup_{B_{E_1},\, B_{E_2}} \sum_{n=1}^{\infty} |\langle \tilde{x}_n^{*(1)}, x_1\rangle \langle \tilde{x}_n^{*(2)}, x_2\rangle|^p \right)^{1/p} \|(\tilde{y}_n)_n\|_{p^*}^{w} \\
&\le \frac{1}{p} \sup_{B_{E_1},\, B_{E_2}} \sum_{n=1}^{\infty} |\langle \tilde{x}_n^{*(1)}, x_1\rangle \langle \tilde{x}_n^{*(2)}, x_2\rangle|^p + \frac{1}{p^*} \left(\|(\tilde{y}_n)_n\|_{p^*}^{w} \right)^{p^*} \\
&\le \frac{r^p}{p} \left(\|T_1\|_{\mathcal{N}_{wp}} + \epsilon \right)^p + \frac{r^{-p^*}}{p^*} + \frac{s^p}{p} \left(\|T_2\|_{\mathcal{N}_{wp}} + \epsilon \right)^p + \frac{s^{-p^*}}{p^*}.
\end{aligned}$$

With $r = (\|T_1\|_{\mathcal{N}_{wp}} + \epsilon)^{-1/p^*}$ and $s = (\|T_2\|_{\mathcal{N}_{wp}} + \epsilon)^{-1/p^*}$, we obtain

$$\|T_1 + T_2\|_{\mathcal{N}_{wp}} \le \|T_1\|_{\mathcal{N}_{wp}} + \|T_2\|_{\mathcal{N}_{wp}} + 2\epsilon.$$

To show completeness, let (T_k) be a sequence such that $\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{N}_{wp}} < \infty$. Since $\|T_k\| \le \|T_k\|_{\mathcal{N}_{wp}}$, the sum $T = \sum_{k=1}^{\infty} T_k$ exists in $\mathcal{L}(E_1, E_2; F)$. The sequence of partial sums $J_m = \| \sum_{k=1}^{m} T_k \|_{\mathcal{N}_{wp}}$ is bounded and increasing, hence it converges. Letting $m \to \infty$, we conclude $\|T\|_{\mathcal{N}_{wp}} < \infty$. □

Proposition 1. *Let $1 \le p$. Let a bilinear operator $T : E_1 \times E_2 \to F$ be weakly p-nuclear, then T has a factorization $T = R \circ S$, such that $S \in \mathcal{L}(E_1, E_2; \ell_p)$ and $R \in \mathcal{L}(\ell_p; F)$. In addition, $\|T\|_{\mathcal{N}_{wp}} \ge \inf \|R\| \, \|S\|$.*

Proof. Assume that T is weakly p-nuclear and let $T = \sum\limits_{n=1}^{\infty} x_{1n}^* \otimes x_{2n}^* \otimes y_n$. Define

$$S : E_1 \times E_2 \ni (x_1, x_2) \mapsto (x_{1n}^*(x_1) x_{2n}^*(x_2))_n \in \ell_p,$$
$$R : \ell_p \ni (s_n)_n \mapsto \sum_{n=1}^{\infty} s_n y_n \in F.$$

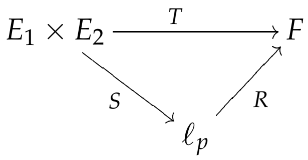
Then we see that $\|S\| = \sup\limits_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \left\| (x_{1n}^*(x_1) x_{2n}^*(x_2))_n \right\|_p$ and we have

$$\|R((s_n)_n)\| = \sup_{y^* \in B_{F^*}} \left| \left\langle y^*, \sum_{n=1}^{\infty} s_n y_n \right\rangle \right| \le \sup_{y^* \in B_{F^*}} \left(\sum_{n=1}^{\infty} |s_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |\langle y^*, y_n \rangle|^{p^*} \right)^{1/p^*}.$$

Thus

$$\|R\| \le \sup_{y^* \in B_{F^*}} \left(\sum_{n=1}^{\infty} |\langle y^*, y_n \rangle|^{p^*} \right)^{1/p^*} \qquad \text{for } 1 \le p < \infty,$$

and the following diagram



commutes.

Thus the weakly p-nuclear representation of T is arbitrary and $\|T\|_{\mathcal{N}_{wp}} \ge \inf \|R\| \|S\|$. □

2 Relationship with tensor products

We consider tensor norm and associate it with an operator ideal. Let us define a cross norm $\omega_p(\cdot)$, $1 \le p < \infty$, on the tensor product $E_1 \otimes E_2 \otimes F$ as follows. If $u \in E_1 \otimes E_2 \otimes F$, then

$$\omega_p(u) = \inf \left\{ \|(\lambda_i)_{i=1}^n\|_\infty \sup_{x_1^* \in B_{E_1^*},\, x_2^* \in B_{E_2^*}} \left(\sum_{i=1}^{n} |\langle x_1^*, x_{1i} \rangle \, \langle x_2^*, x_{2i} \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \right\},$$

where the infimum is taken over all representations of u of the form $u = \sum_{i=1}^{n} \lambda_i x_{1i} \otimes x_{2i} \otimes y_i$, where $(x_{1i})_{i=1}^n \subset E_1$, $(x_{2i})_{i=1}^n \subset E_2$ and $(y_i)_{i=1}^n \subset F$.

Proposition 2. *If E_1 and E_2 are finite dimensional, then*

$$\|T\|_{\mathcal{N}_{wp}} \ge \omega_p(T)$$

for every $T \in \mathcal{L}_f(E_1, E_2; F)$.

Proof. In this situation $\mathcal{L}(E_1, E_2; F) = \mathcal{L}_f(E_1, E_2; F)$ and there exists a constant $C \geq 0$ such that

$$\omega_p(T) \leq C\|T\|_{\mathcal{N}_{wp}}$$

for every $T \in \mathcal{L}_f(E_1, E_2; F)$. For each $\epsilon > 0$ we choose a representation $T = \sum_{i=1}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i$ such that

$$\sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \left(\sum_{i=1}^{m-1} |\langle x_{1i}^*, x_1 \rangle \langle x_{2i}^*, x_2 \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{m-1} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \leq (1 + \frac{\epsilon}{2})\|T\|_{\mathcal{N}_{wp}}.$$

In particular, for each $m \in \mathbb{N}$, we get

$$\begin{aligned} \omega_p\left(\sum_{i=1}^{m-1} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right) & \\ \leq \sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} & \left(\sum_{i=1}^{m-1} |\langle x_{1i}^*, x_1 \rangle \langle x_{2i}^*, x_2 \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{m-1} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \\ \leq & \left(1 + \frac{\epsilon}{2}\right) \|T\|_{\mathcal{N}_{wp}}. \end{aligned}$$

For a sufficiently large $m \in \mathbb{N}$ we can write

$$\begin{aligned} \left\| \sum_{i=m}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right\|_{\mathcal{N}_{wp}} & \\ \leq \sup_{\substack{x_1 \in B_{E_1} \\ x_2 \in B_{E_2}}} & \left(\sum_{i=m}^{\infty} |\langle x_{1i}^*, x_1 \rangle \langle x_{2i}^*, x_2 \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=m}^{\infty} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \\ \leq & \frac{\epsilon}{2C} \|T\|_{\mathcal{N}_{wp}}. \end{aligned}$$

It follows that

$$\begin{aligned} \omega_p(T) &\leq \omega_p\left(\sum_{i=1}^{m-1} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right) + \omega_p\left(\sum_{i=m}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right) \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \|T\|_{\mathcal{N}_{wp}} + C\left\| \sum_{i=m}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right\|_{\mathcal{N}_{wp}} \\ &\leq \left(1 + \frac{\epsilon}{2}\right) \|T\|_{\mathcal{N}_{wp}} + \frac{\epsilon}{2}\|T\|_{\mathcal{N}_{wp}} = (1 + \epsilon)\|T\|_{\mathcal{N}_{wp}}, \end{aligned}$$

and this holds for every $\epsilon > 0$. □

Proposition 3. *If $T \in \mathcal{N}_{wp}(E_1, E_2; F)$ and $S_i \in \mathcal{L}_f(D_i; E_i)$ for $i = 1, 2$, then*

$$\omega_p(T \circ (S_1, S_2)) \leq \|T\|_{\mathcal{N}_{wp}} \|S_1\| \|S_2\|.$$

Proof. If J_i denotes the natural injection of $S_i(D_i)$ into E_i, we can write $S_i = J_i \circ \tilde{S}_i$ with $\|\tilde{S}_k\| = \|S_k\|$. Hence,

$$T \circ (J_1, J_2) \in \mathcal{L}_f(S_1(D_1), S_2(D_2); F).$$

Now, by applying Proposition 2 and the ideal property, we obtain the desired result. □

Proposition 4. *If E_1^* and E_2^* have the bounded approximation property, then*

$$\|T\|_{\mathcal{N}_{wp}} \geq \omega_p(T)$$

for every $T \in \mathcal{L}_f(E_1, E_2; F)$.

Proof. We note that the operators $T_1 \in \mathcal{L}(E_1; \mathcal{L}(E_2; F))$ and $T_2 \in \mathcal{L}(E_2; \mathcal{L}(E_1; F))$, defined by

$$T_1(x_1)(x_2) = T_2(x_2)(x_1) = T(x_1, x_2),$$

is of finite type. Since E_i^* has the λ_i-approximation property for some $\lambda_i > 0$ and for each $\epsilon > 0$, we can find $S_i \in \mathcal{L}_f(E_i; E_i)$, such that $T_i = T_i \circ S_i$ and $\|S_i\| \leq (1+\epsilon)\lambda_i$. Therefore, for all $x_i \in E_i$ with $i = 1, 2$, we have

$$T(x_1, S_2(x_2)) = T(S_1(x_1), x_2) = T(x_1, x_2).$$

Now, we can write $T(x_1, x_2) = T(S_1(x_1), S_2(x_2))$. Thus, by Proposition 4, we have

$$\omega_p(T) \leq \|T\|_{\mathcal{N}_{wp}} \|S_1\| \, \|S_1\| \leq \|T\|_{\mathcal{N}_{wp}} \lambda_1 \lambda_2.$$

Hence

$$\omega_p(T) \leq \lambda_1 \lambda_2 \|T\|_{\mathcal{N}_{wp}}.$$

With the same argument used in the proof of Proposition 2, we obtain $\omega_p(T) \leq \|T\|_{\mathcal{N}_{wp}}$. □

Corollary 1. *If E_1^* and E_2^* have the bounded approximation property, then $\mathcal{N}_{wp}(E_1, E_2; F)$ is isometrically isomorphic to the completion of $(E_1^* \otimes E_2^* \otimes F, \omega_p)$.*

3 The dual space of weakly p-nuclear bilinear operators

The quest for a class of operators that characterize bounded linear functionals on the space of weakly p-nuclear bilinear operators prompted us to present new class of quasi Cohen p-nuclear bilinear operators.

Definition 2. *Let $1 \leq p$. Suppose that $T : E_1 \times E_2 \to F^*$ is a bilinear operator. We say that T is quasi Cohen p-nuclear if there exists a constant $C > 0$ such that for all $x_{j1}, \ldots, x_{jn} \in E_j$, $1 \leq j \leq 2$, and $y_1, \ldots, y_n \in F$, we have*

$$\left| \sum_{i=1}^{n} \langle T(x_{1i}, x_{2i}), y_i \rangle \right| \leq C \sup_{x_1^* \in B_{E_1^*},\, x_2^* \in B_{E_2^*}} \left(\sum_{i=1}^{n} |\langle x_1^*, x_{1i} \rangle \langle x_2^*, x_{2i} \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*}.$$

The smallest constant C for which this inequality always holds is denoted by $\|T\|_{\mathcal{N}_{q(p)}}$. We write $\mathcal{N}_{q(p)}(E_1, E_2; F^*)$ for the space of quasi Cohen p-nuclear bilinear operators.

Remark 1. *Denoting by $\mathcal{N}_p(E_1, E_2; F^*)$ the space of Cohen p-nuclear bilinear operators as defined in [1], it is straightforward that $\mathcal{N}_p(E_1, E_2; F^*) \subseteq \mathcal{N}_{q(p)}(E_1, E_2; F^*)$ with $\|\cdot\|_{\mathcal{N}_{q(p)}} \leq \|\cdot\|_{\mathcal{N}_p}$ for every F. Furthermore, $\mathcal{N}_{q(p)}(E_1, E_2; F^*) = \mathcal{N}_p(E_1, E_2; F^*)$ isometrically for reflexive F.*

The core result of this section is a Pietsch domination theorem, whose proof is obtained through an application of Ky Fan's lemma (see [6, Lemma E.4.2]).

Lemma 1 (Ky Fan's Lemma)**.** *Let E be a Hausdorff topological vector space, and let $\mathcal{C}$ be a compact convex subset of E. Let $\mathcal{F}$ be a set of functions on $\mathcal{C}$ with values in $(-\infty, \infty]$ having the following properties:*

(a) each $f \in \mathcal{F}$ is convex and lower semicontinuous,

(b) if $g \in \operatorname{conv}(\mathcal{F})$, then there is an $f \in \mathcal{F}$ with $g(x) \leq f(x)$ for every $x \in \mathcal{C}$,

(c) there is an $r \in \mathbb{R}$ such that each $f \in \mathcal{F}$ has a value not greater than r.

Then there is an $x_0 \in \mathcal{C}$ such that $f(x_0) \leq r$ for all $f \in \mathcal{F}$.

Theorem 2. *Let $1 \leq p$. For a bilinear operator $T \in \mathcal{L}(E_1, E_2; F^*)$ The following are equivalent.*

(i) *T is quasi Cohen p-nuclear.*

(ii) *There is a constant $C > 0$ and for all $x_{j1}, \dots, x_{jn}$ in E_j, $j = 1, 2$, and $y_1, \dots, y_n$ in F we have*

$$\sum_{i=1}^{n} |\langle T(x_{1i}, x_{2i}), y_i\rangle| \leq C \sup_{x_1^* \in B_{E_1^*},\, x_2^* \in B_{E_2^*}} \left(\sum_{i=1}^{n} |\langle x_1^*, x_{1i}\rangle \langle x_2^*, x_{2i}\rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i\rangle|^{p^*} \right)^{1/p^*}.$$

(iii) *There are regular Borel probability measures $\mu_j \in \mathcal{C}\left(B_{E_j^*}\right)^*$, $j = 1, 2$, and $\lambda \in \mathcal{C}(B_{F^*})^*$ such that*

$$|\langle T(x_1, x_2), y\rangle| \leq C \|x_1\|_{L_p(B_{E_1^*}, \mu_1)} \|x_2\|_{L_p(B_{E_2^*}, \mu_j)} \|y\|_{L_{p^*}(B_{F^*}, \lambda)} \tag{1}$$

for all $(x_1, x_2) \in E_1 \times E_2$ and $y \in F$.

Proof. The first implication (i) $\Rightarrow$ (ii) is clear.

The proof of the implication (ii) $\Rightarrow$ (iii) follows the approach of [1]. We consider the sets $P\left(B_{E_j^*}\right)$, $j = 1, 2$, and $P(B_{F^*})$ of regular Borel probability measures in $\mathcal{C}\left(B_{E_j^*}\right)^*$ and $\mathcal{C}(B_{F^*})^*$, respectively. These sets are convex and compact when $\mathcal{C}\left(B_{E_j^*}\right)^*$ and $\mathcal{C}(B_{F^*})^*$ are endowed with their weak* topologies. Given $x_{j1}, \dots, x_{jn} \in E_j$, $j = 1, 2$, and $y_1, \dots, y_n \in F$, the function $f : P\left(B_{E_1^*}\right) \times P\left(B_{E_2^*}\right) \times P(B_{F^*}) \to \mathbb{R}^+$ is defined by

$$f(\mu_1, \mu_2, \lambda) := \sum_{i=1}^{n} |\langle T(x_{1i}, x_{2i}), y_i\rangle| - \frac{C}{p} \sum_{i=1}^{n} \int_{B_{E_1^*}} |\langle x_1^*, x_{1i}\rangle|^p \, d\mu_1(x_1^*) \int_{B_{E_2^*}} |\langle x_2^*, x_{2i}\rangle|^p \, d\mu_2(x_2^*)$$
$$- \frac{C}{p^*} \sum_{i=1}^{n} \int_{B_{F^*}} |\langle y^*, y_i\rangle|^{p^*} \, d\lambda(y^*).$$

It is obvious that all such f is continuous and convex. Thanks to weak compactness entitles us to consider each $x_0^{*j} \in B_{E_j^*}$ as an element of $B_{E_j^*}$ and $y_0^* \in B_{F^*}$ as an element of B_{F^*}, such that

$$\sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i\rangle|^{p^*} \right) = \sum_{i=1}^{n} |\langle y_0^*, y_i\rangle|^{p^*}.$$

Using the elementary identity

$$ab = \inf_{\epsilon > 0} \left\{ \frac{1}{p} \left(\frac{a}{\epsilon} \right)^p + \frac{1}{p^*} (\epsilon b)^{p^*} \right\}, \quad \forall a, b \in \mathbb{R}_+^*,$$

we obtain

$$a=\left(\sup_{x_1^*\in B_{E_1^*},\, x_2^*\in B_{E_2^*}}\sum_{i=1}^{n}|\langle x_1^*,x_{1i}\rangle\langle x_2^*,x_{2i}\rangle|^p\right)^{1/p},\qquad b=\sup_{y^*\in B_{F^*}}\left(\sum_{i=1}^{n}|\langle y^*,y_i\rangle|^{p^*}\right)^{1/p^*}.$$

Let $\delta_{x_0^{*1}},\delta_{x_0^{*2}},\delta_{y_0}$ be the Dirac measures at x_0^{*1},x_0^{*2},y_0, respectively. Since T is quasi Cohen p-nuclear, we have

$$\begin{aligned}f\left(\delta_{x_0^{*1}},\delta_{x_0^{*2}},\delta_{y_0}\right)&=\sum_{i=1}^{n}|\langle T(x_{1i},x_{2i}),y_i\rangle|-\frac{C}{p}\sup_{x_1^*\in B_{E_1^*},\, x_2^*\in B_{E_2^*}}\left(\sum_{i=1}^{n}|\langle x_1^*,x_{1i}\rangle\langle x_2^*,x_{2i}\rangle|^p\right)\\&\qquad-\frac{C}{p^*}\sup_{y^*\in B_{F^*}}\left(\sum_{i=1}^{n}|\langle y^*,y_i\rangle|^{p^*}\right)\\&\le\sum_{i=1}^{n}|\langle T\left(x_{1i},x_{2i}\right),y_i\rangle|-C\sup_{x_1^*\in B_{E_1^*},\, x_2^*\in B_{E_2^*}}\left(\sum_{i=1}^{n}|\langle x_1^*,x_{1i}\rangle\langle x_2^*,x_{2i}\rangle|^p\right)^{1/p}\\&\qquad\times\sup_{y^*\in B_{F^*}}\left(\sum_{i=1}^{n}|\langle y^*,y_i\rangle|^{p^*}\right)^{1/p^*}\le 0.\end{aligned}$$

Since the set $\mathcal{F}$ of all such functions f is concave, by Ky Fan's Lemma there is an element $(\mu_1,\mu_2,\lambda)\in P\left(B_{E_1^*}\right)\times P\left(B_{E_2^*}\right)\times P\left(B_{F^*}\right)$ such that $f\left(\mu_1,\mu_2,\lambda\right)\le 0$ for all $f\in\mathcal{F}$. Consider the function f, generated by the single elements $(x_1,x_2)\in E_1\times E_2$ and $y\in F$, then

$$|\langle T(x_1,x_2),y\rangle|\le\frac{C}{p}\int_{B_{E_1^*}}|\langle x_1^*,x_1\rangle|^p d\mu_1(x_1^*)\int_{B_{E_2^*}}|\langle x_2^*,x_2\rangle|^p d\mu_2(x_2^*)+\frac{C}{p^*}\int_{B_{F^*}}|\langle y^*,y\rangle|^{p^*}d\lambda(y^*).$$

Fix $\epsilon>0$. Replacing x_j by $\epsilon^{-1/2}x_j$, y by ϵy and taking the infimum over all $\epsilon>0$, we find

$$\begin{aligned}|\langle T(x_1,x_2),y\rangle|&\le C\left[\frac{1}{p}\left(\frac{1}{\epsilon}\left(\int_{B_{E_1^*}}|\langle x_1^*,x_{1i}\rangle|^p d\mu_1(x_1^*)\int_{B_{E_1^*}}|\langle x_2^*,x_{2i}\rangle|^p d\mu_2(x_2^*)\right)^{1/p}\right)^p\right.\\&\qquad\left.+\frac{1}{p^*}\left(\epsilon\left(\int_{B_{F^*}}|\langle y^*,y\rangle|^{p^*}d\lambda(y)\right)^{1/p^*}\right)^{p^*}\right]\\&\le C\left(\int_{B_{E_1^*}}|\langle x_1^*,x_{1i}\rangle|^p d\mu_1(x_1^*)\right)^{1/p}\left(\int_{B_{E_2^*}}|\langle x_2^*,x_{2i}\rangle|^p d\mu_2(x_2^*)\right)^{1/p}\\&\qquad\times\left(\int_{B_F^*}|\langle y^*,y\rangle|^{p^*}d\lambda(y^*)\right)^{1/p^*}.\end{aligned}$$

Now we will show that (iii) $\Rightarrow$ (i). Let $(x_{1i},x_{2i})\in E_1\times E_2$ and $y_i\in F$. According to inequality (1), we get

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle T(x_{1i},x_{2i}),y_i\rangle\right|&\le\sum_{i=1}^{n}|\langle T(x_{1i},x_{2i}),y_i\rangle|\\&\le C\sum_{i=1}^{n}\left(\|x_{1i}\|_{L_p(B_{E_1^*},\mu_1)}\|x_{2i}\|_{L_p(B_{E_2^*},\mu_j)}\|y_i\|_{L_{p^*}(B_{F^*},\lambda)}\right).\end{aligned}$$

Applying Hölder's inequality, yields

$$\begin{aligned}\Big|\sum_{i=1}^{n}\langle T(x_{1i},x_{2i}),y_i\rangle\Big| &\le C\Big(\sum_{i=1}^{n}\|x_{1i}\|^p_{L_p(B_{E_1^*},\mu_1)}\|x_{2i}\|^p_{L_p(B_{E_2^*},\mu_2)}\Big)^{1/p}\Big(\sum_{i=1}^{n}\|y_i^*\|^{p^*}_{L_{p^*}(B_{F^*},\lambda)}\Big)^{1/p^*}\\ &= C\Big(\sum_{i=1}^{n}\int_{B_{E_1^*}\times B_{E_2^*}}|x_1^*(x_{1i})x_2^*(x_{2i})|^p d(\mu_1\otimes\mu_2)(x_1^*,x_2^*)\Big)^{1/p}\\ &\quad\times\Big(\sum_{i=1}^{n}\int_{B_{F^*}}|y^*(y_i)|^{p^*}d\lambda(y^*)\Big)^{1/p^*}\\ &\le C\Big(\sup_{x_1^*\in B_{E_1^*},\,x_2^*\in B_{E_2^*}}\sum_{i=1}^{n}|x_1^*(x_{1i})x_2^*(x_{2i})|^p\Big)^{1/p}\sup_{y^*\in B_{F^*}}\Big(\sum_{i=1}^{n}|y^*(y_i)|^{p^*}\Big)^{1/p^*}.\end{aligned}$$

Thus, T is quasi Cohen p-nuclear, additionally, $\|T\|_{\mathcal{N}_{q(p)}}\le C$, which completes the proof. □

In [2], G. Botelho and X. Mujica introduced the notation $\mathcal{L}_{q\tau(p)}(E_1,E_2;F)$ to represent the space of quasi $\tau(p)$-summing bilinear operators. The Pietsch Domination Theorem for quasi $\tau(p)$-summing bilinear operators is the key to the following result.

Corollary 2. *Let E_1 and E_2 be Banach spaces. For every Banach space F, we have*

$$\begin{cases}\mathcal{L}_{q\tau(p)}(E_1,E_2;F^*)\subset\mathcal{N}_{q(p)}(E_1,E_2;F^*), & \text{if } 1\le p\le 2,\\ \mathcal{N}_{q(p)}(E_1,E_2;F^*)\subset\mathcal{L}_{q\tau(p)}(E_1,E_2;F^*), & \text{if } p\ge 2.\end{cases}$$

Theorem 3. *If E_1^* and E_2^* have the bounded approximation property, then for any Banach space F and $1\le p<\infty$ the space $\left[\mathcal{N}_{wp}(E_1,E_2;F)\right]^*$ is isometrically isomorphic to $\mathcal{N}_{q(p)}(E_1^*,E_2^*;F^*)$.*

Proof. Given $\psi\in\left[\mathcal{N}_{wp}(E_1,E_2;F)\right]^*$, we define

$$\begin{aligned}S_\psi : E_1^*\times E_2^* &\longrightarrow F^*\\ (x_1^*,x_2^*) &\longmapsto S_\psi(x_1^*,x_2^*) : F\longrightarrow\mathbb{K}\\ &\qquad\qquad y\longmapsto S_\psi(x_1^*,x_2^*)(y)=\psi(x_1^*\otimes x_2^*\otimes y).\end{aligned}$$

In order to show that $S_\psi\in\mathcal{N}_p(E_1^*,E_2^*;F^*)$, let $n\in\mathbb{N}$, $x_{j1}^*,\dots,x_{jn}^*\in E_j^*$, $j=1,2$, and $y_1,\dots,y_n\in F$. So,

$$\begin{aligned}\Big|\sum_{i=1}^{n}S_\psi(x_{1i}^*\otimes x_{2i}^*)(y_i)\Big| &= \Big|\sum_{i=1}^{n}\psi(x_{1i}^*\otimes x_{2i}^*\otimes y_i)\Big| = \Big|\psi\Big(\sum_{i=1}^{n}x_{1i}^*\otimes x_{2i}^*\otimes y_i\Big)\Big|\\ &\le\|\psi\|\Big\|\sum_{i=1}^{n}x_{1i}^*\otimes x_{2i}^*\otimes y_i\Big\|_{\mathcal{N}_{wp}}\\ &\le\|\psi\|\sup_{x_1\in B_{E_1},\,x_2\in B_{E_2}}\Big(\sum_{i=1}^{n}|\langle x_{1i}^*,x_1\rangle\langle x_{2i}^*,x_2\rangle|^p\Big)^{1/p}\\ &\quad\times\sup_{y^*\in B_{F^*}}\Big(\sum_{i=1}^{n}|\langle y^*,y_i\rangle|^{p^*}\Big)^{1/p^*},\end{aligned}$$

which proves that S_ψ is quasi Cohen p-nuclear and $\|S_\psi\|_{\mathcal{N}_{q(p)}}\le\|\psi\|$.

Conversely, consider $S \in \mathcal{N}_{q(p)}(E_1^*, E_2^*; F^*)$ defined by

$$\begin{aligned} S : E_1^* \times E_2^* &\longrightarrow F^* \\ (x_1^*, x_2^*) &\longmapsto S(x_1^*, x_2^*) : F \longrightarrow \mathbb{K} \\ &\qquad\qquad\qquad y \longmapsto S(x_1^*, x_2^*)(y). \end{aligned}$$

We define

$$T_S : E_1^* \times E_2^* \times F \ni (x_1^*, x_2^*, y) \longmapsto T_S(x_1^*, x_2^*, y) = S(x_1^*, x_2^*)(y) \in \mathbb{K}.$$

It is evident that T_S is linear in each of the three variables.

Considering $E_1^* \otimes E_2^* \otimes F = \mathcal{L}_f(E_1, E_2; F)$ and utilizing the universal property of the tensor product, we can define a linear operator $\mathcal{T}_{\mathcal{S}}$ as follows

$$\mathcal{T}_{\mathcal{S}} : \mathcal{L}_f(E_1, E_2; F) \ni x_1^* \otimes x_2^* \otimes y \longmapsto \mathcal{T}_{\mathcal{S}}(x_1^* \otimes x_2^* \otimes y) = T_S(x_1^*, x_2^*, y) = S(x_1^*, x_2^*)(y) \in \mathbb{K}.$$

Let us show that $\mathcal{T}_{\mathcal{S}}$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. Given $\epsilon > 0$ and $T \in \mathcal{L}_f(E_1, E_2; F)$, according to the definition of the norm $\omega_p(\cdot)$, we can represent T in the form

$$T = \sum_{i=1}^{n} x_{1i}^* \otimes x_{2i}^* \otimes y_i,$$

such that

$$\sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \left(\sum_{i=1}^{n} |\langle x_{1i}^*, x_1 \rangle \langle x_{2i}^*, x_2 \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \leq (1+\epsilon)\omega_p(T).$$

Therefore,

$$\begin{aligned} |\mathcal{T}_{\mathcal{S}}(T)| &= \left| \mathcal{T}_{\mathcal{S}}\left(\sum_{i=1}^{n} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right) \right| = \left| \sum_{i=1}^{n} S(x_{1i}^*, x_{2i}^*)(y_i) \right| \\ &\leq \|S\|_{\mathcal{N}_{q(p)}} \sup_{x_1 \in B_{E_1},\, x_2 \in B_{E_2}} \left(\sum_{i=1}^{n} |\langle x_{2i}^*, x_1 \rangle \langle x_{2i}^*, x_2 \rangle|^p \right)^{1/p} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |\langle y^*, y_i \rangle|^{p^*} \right)^{1/p^*} \\ &\leq \|S\|_{\mathcal{N}_{q(p)}} (1+\epsilon)\omega_p(T). \end{aligned}$$

Since this holds for any $\varepsilon > 0$ and the spaces E_1^* and E_2^* have the bounded approximation property, we can conclude, based on Proposition 4, that

$$|\mathcal{T}_{\mathcal{S}}(T)| \leq \|S\|_{\mathcal{N}_{q(p)}} \cdot \omega_p(T) = \|S\|_{\mathcal{N}_{q(p)}} \cdot \|T\|_{\mathcal{N}_{wp}}.$$

So, $\mathcal{T}_{\mathcal{S}} \in [\mathcal{L}_f(E_1, E_2; F), \|\cdot\|_{\mathcal{N}_{wp}}]^*$ and $\|\mathcal{T}_{\mathcal{S}}\| \leq \|S\|_{\mathcal{N}_{q(p)}}$.

Given that $\mathcal{L}_f(E_1, E_2; F)$ is $\|\cdot\|_{\mathcal{N}_{wp}}$-dense in $\mathcal{N}_{wp}(E_1, E_2; F)$, then $\mathcal{T}_S \colon \mathcal{L}_f(E_1, E_2; F) \longrightarrow \mathbb{K}$ admits a unique norm preserving linear extension $\psi_S \colon \mathcal{N}_{wp}(E_1, E_2; F) \longrightarrow \mathbb{K}$. Naturally, $\|\psi_S\| \leq \|S\|_{\mathcal{N}_{q(p)}}$ and for $T = \sum_{i=1}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i \in \mathcal{N}_{wp}(E_1, E_2; F)$, we get

$$\begin{aligned} \psi_S(T) = \psi_S\left(\sum_{i=1}^{\infty} x_{1i}^* \otimes x_{2i}^* \otimes y_i \right) &= \sum_{i=1}^{\infty} \psi_S(x_{1i}^* \otimes x_{2i}^* \otimes y_i) \\ &= \sum_{i=1}^{\infty} \mathcal{T}_{\mathcal{S}}(x_{1i}^* \otimes x_{2i}^* \otimes y_i) = \sum_{i=1}^{\infty} S(x_{1i}^*, x_{2i}^*)(y_i). \end{aligned}$$

It follows easily from the above expression that the correspondences $\psi \mapsto S_\psi$ and $S \mapsto \psi_S$ are two one-to-one, meaning that $\psi_{S_\psi} = \psi$ and $S_{\psi_S} = S$ for $\psi \in [\mathcal{N}_{wp}(E_1, E_2; F)]^*$. □

E. Çalışkan and D.M. Pellegrino in [3] noted by $\mathcal{L}_{si,p}\left(E_1, E_2; F\right)$ the space of p-semi-integral bilinear operators.

In the previous theorem, if we take $F = \mathbb{K}$, we obtain the following result.

Corollary 3. *If E_1^* and E_2^* have the bounded approximation property, then*

$$\left[\mathcal{N}_{wp}\left(E_1, E_2\right)\right]^* = \mathcal{L}_{si,p}\left(E_1^*, E_2^*\right)$$

isometrically isomorphic.

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Хамму А., Беласель А., Бугутая А., Тіайба А. *Слабо p-ядерні білінійні оператори* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 160–170.

Метою цієї статті є дослідження класу слабо p-ядерних білінійних операторів між банаховими просторами. Це поняття розширює класичну теорію ядерних операторів, запроваджену А. Гротендіком, а також її багатолінійні узагальнення, розвинуті А. Пічем та іншими авторами. Зокрема, ми характеризуємо слабо p-ядерні білінійні оператори за допомогою відповідних тензорних норм, показуючи, що простір таких операторів утворює банахів ідеал, а також аналізуючи деякі його структурні властивості. Крім того, у контексті теорії двоїстості для цих просторів операторів ми вводимо клас квазі-Коенових p-ядерних білінійних операторів і встановлюємо теорему про домінування типу Піча.

Ключові слова і фрази: слабо p-ядерний білінійний оператор, квазі-Коеновий p-ядерний білінійний оператор, двоїстість.