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Superextensions of doppelsemigroups

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A family $\mathcal{U}$ of non-empty subsets of a set D is called an *upfamily* if for each set $U \in \mathcal{U}$ any set $F \supset U$ belongs to $\mathcal{U}$. An upfamily $\mathcal{L}$ of subsets of D is said to be *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. A linked upfamily $\mathcal{M}$ of subsets of D is *maximal linked* if $\mathcal{M}$ coincides with each linked upfamily $\mathcal{L}$ on D that contains $\mathcal{M}$. The *superextension* $\lambda(D)$ of D consists of all maximal linked upfamilies on D. Any associative binary operation $* : D \times D \to D$ can be extended to an associative binary operation

$$* : \lambda(D) \times \lambda(D) \to \lambda(D), \quad \mathcal{M} * \mathcal{L} = \Big\langle \bigcup_{a \in M} a * L_a : M \in \mathcal{M},\ \{L_a\}_{a \in M} \subset \mathcal{L} \Big\rangle.$$

In the paper, we investigate the structure of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$ of maximal linked upfamilies on a doppelsemigroup $(D, \dashv, \vdash)$. In particular, we study right and left zeros and identities, commutativity, the center, ideals of the superextension $(\lambda(D), \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$. We introduce the superextension functor λ in the category **DSG**, whose objects are doppelsemigroups and morphisms are doppelsemigroup homomorphisms, and show that this functor preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups. Also we prove that the automorphism group of the superextension of a doppelsemigroup $(D, \dashv, \vdash)$ contain a subgroup, isomorphic to the automorphism group of $(D, \dashv, \vdash)$.

Key words and phrases: semigroup, superextension, maximal linked upfamily, doppelsemigroup.

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Introduction

Given a semigroup $(S, \dashv)$, consider a semigroup $(S, \vdash)$ defined on the same set. We say that the semigroups $(S, \vdash)$ and $(S, \dashv)$ are *interassociative* provided $(x \dashv y) \vdash z = x \dashv (y \vdash z)$ and $(x \vdash y) \dashv z = x \vdash (y \dashv z)$ for all $x, y, z \in S$. When this occurs, $(S, \vdash)$ is said to be an *interassociate* of $(S, \dashv)$, or that the semigroups are interassociates of each other. The present concept of interassociative semigroups originated in 1986 in M. Drouzy [11], where it is noted that every group is isomorphic to each of its interassociates. In 1983, M. Gould and R.E. Richardson [23] introduced *strong interassociativity*, defined by the above equations along with $x \dashv (y \vdash z) = x \vdash (y \dashv z)$. J.B. Hickey in 1983 [24] dealt with the special case of interassociativity in which the operation $\vdash$ is defined by specifying $a \in S$ and stipulating that $x \vdash y = x \dashv a \dashv y$ for all $x, y \in S$. Clearly $(S, \vdash)$, which Hickey calls a *variant* of $(S, \dashv)$, is a semigroup that is an interassociate of $(S, \dashv)$. Methods of constructing interassociates were developed, for semigroups in general and for specific classes of semigroups, in 1997 by S.J. Boyd, M. Gould and A. Nelson [9].

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This paper is devoted to study of doppelsemigroups which are sets with two associative binary operations satisfying axioms of interassociativity. More accurately, a *doppelsemigroup* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a non-empty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(D_1)\ (x \dashv y) \vdash z = x \dashv (y \vdash z), \quad (D_2)\ (x \vdash y) \dashv z = x \vdash (y \dashv z).$$

Thus, we can see that for any doppelsemigroup $(D, \dashv, \vdash)$, the semigroups $(D, \vdash)$ and $(D, \dashv)$ are interassociative, and conversely, if a semigroup $(D, \vdash)$ is an interassociate of a semigroup $(D, \dashv)$, then $(D, \dashv, \vdash)$ and $(D, \vdash, \dashv)$ are doppelsemigroups. If $(D, \dashv, \vdash)$ is a doppelsemigroup, then rearranging the parentheses in an expression that contains only operations $\vdash$, $\dashv$ and elements of D do not change the result. A doppelsemigroup $(D, \dashv, \vdash)$ is called *commutative* [34] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. A doppelsemigroup $(D, \dashv, \vdash)$ is said to be *strong* [36] if it satisfies the axiom $x \dashv (y \vdash z) = x \vdash (y \dashv z)$.

The study of doppelsemigroups was initiated by A. Zhuchok [34]. The idea of doppelsemigroups bases on the study of dimonoids in the sense of J.L. Loday [26]. Doppelalgebras introduced by B. Richter [29] in the context of algebraic K-theory are linear analogs of doppelsemigroups and commutative dimonoids are examples of doppelsemigroups. Consequently, doppelsemigroup theory has connections to doppelalgebra theory and dimonoid theory. A doppelsemigroup can also be determined by using the notion of a duplex [28]. Doppelsemigroups are closely related to bisemigroups considered in the work of B.M. Schein [30]. The latter algebras have applications in the theory of binary relations [31]. If operations of a doppelsemigroup coincide, we obtain the notion of a semigroup.

Many classes of doppelsemigroups were studied by A. Zhuchok and his coauthors. The free product of doppelsemigroups, the free (strong) doppelsemigroup, the free commutative (strong) doppelsemigroup, the free n-nilpotent (strong) doppelsemigroup, the free rectangular doppelsemigroup and the free abelian doppelsemigroup were constructed in [34, 36, 37, 39]. Relatively free doppelsemigroups were studied in [38]. The free n-dinilpotent (strong) doppelsemigroup was constructed in [33, 36]. In [35], A. Zhuchok described the free left n-dinilpotent doppelsemigroup. Representations of ordered doppelsemigroups by binary relations were studied by Yu. Zhuchok and J. Koppitz, see [40].

In [18,21], all pairwise non-isomorphic doppelsemigroups of order at most three were classified. There exist 8 two-element doppelsemigroups, 6 of which are commutative, and all are strong. Up to isomorphism, there exist 77 three-element doppelsemigroups, 41 of which are commutative; the noncommutative ones form 18 pairs of dual doppelsemigroups. Moreover, 65 of these are strong. In [20], we studied cyclic doppelsemigroups. A doppelsemigroup $(G, \dashv, \vdash)$ is called *a group doppelsemigroup* if $(G, \dashv)$ is a group. In this case $(G, \vdash)$ is a group isomorpic to $(G, \dashv)$. A group doppelsemigroup $(G, \dashv, \vdash)$ is said to be *cyclic* if $(G, \dashv)$ is a cyclic group. It was proved that up to isomorphism there exist $\tau(n)$ finite cyclic (strong) doppelsemigroups of order n, where τ is the number of divisors function. There exist infinite many pairwise non-isomorphic infinite cyclic (strong) doppelsemigroups.

In this paper, we investigate the superextension $(\lambda(D), \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$. The thorough study of various extensions of semigroups was started in [13] and continued in [1–8, 14–17]. The largest among these extensions is the semigroup $v(S)$ of all upfamilies on a semigroup S. The extension $v(S)$ is called *the upfamily extension* of S. A family $\mathcal{U}$ of non-empty subsets of a set X is called an *upfamily*[1] if for each set $U \in \mathcal{U}$ any subset $F \supset U$ belongs to $\mathcal{U}$. Each family $\mathcal{A}$ of non-empty subsets of X generates the upfamily $\langle A \subset X : A \in \mathcal{A}\rangle = \{U \subset X : \exists A \in \mathcal{A}\ (A \subset U)\}$. An upfamily $\mathcal{F}$ that is closed under taking finite intersections is called a *filter*. A filter $\mathcal{B}$ is called an *ultrafilter* if $\mathcal{B} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{B}$. The family $\beta(X)$ of all ultrafilters on a set X is called the Stone-Čech compactification of X, see [25]. An ultrafilter, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. Each point $x \in X$ is identified with the principal ultrafilter $\langle\{x\}\rangle$ generated by the singleton $\{x\}$, and hence we consider $X \subset \beta(X) \subset v(X)$.

It was shown in [13] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $* : v(S) \times v(S) \to v(S)$ by the formula

$$\mathcal{U} * \mathcal{V} = \Big\langle \bigcup_{a \in U} a * V_a : U \in \mathcal{U},\ \ \{V_a\}_{a \in U} \subset \mathcal{V} \Big\rangle \tag{1}$$

for upfamilies $\mathcal{U}, \mathcal{V} \in v(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the semigroup $v(S)$. The semigroup $v(S)$ contains many other important extensions of S. In particular, it contains the semigroup $\lambda(S)$ of maximal linked upfamilies. The space $\lambda(S)$ is well-known in General and Categorial Topology as the *superextension* of S, see [27,32]. An upfamily $\mathcal{L}$ of subsets of S is *linked* if $A \cap B \neq \varnothing$ for all $A, B \in \mathcal{L}$. The family of all linked upfamilies on S is denoted by $N_2(S)$. It is a subsemigroup of $v(S)$. The superextension $\lambda(S)$ consists of all maximal elements of $N_2(S)$, see [12,13].

For a finite set X, the cardinality of the set $\lambda(X)$ grows very quickly as $|X|$ tends to infinity. The calculation of the cardinality of $\lambda(X)$ seems to be a difficult combinatorial problem, which can be reformulated as the problem of counting the number $\lambda(n)$ of self-dual monotone Boolean functions of n variables, which is well-known in Discrete Mathematics. According to [10, Proposition 1.1], we have

$$\log_2 \lambda(n) = \frac{2^n}{\sqrt{2\pi n}} + o(1),$$

which means that the sequence $(\lambda(n))_{n=1}^{\infty}$ has double exponential growth. The sequence of numbers $\lambda(n)$ (known in Discrete Mathematics as Hoşten-Morris numbers) is included in the `On-line Encyclopedia of Integer Sequences` as the sequence A001206. All known precise values of this sequence (taken from [10]) are presented in the Table 1.

n	1	2	3	4	5	6	7	8	9
$\lambda(n)$	1	2	4	12	81	2646	1422564	229809982112	423295099074735261880

Table 1. The values of the function $\lambda(n)$ for $1 \le n \le 9$.

Each map $f : X \to Y$ induces the map (see [12])

$$\lambda(f) : \lambda(X) \to \lambda(Y), \quad \lambda(f) : \mathcal{M} \mapsto \big\langle f(M) \subset Y : M \in \mathcal{M}\big\rangle.$$

If $\varphi : S \to S'$ is a semigroup homomorphism, then $\lambda(\varphi) : \lambda(S) \to \lambda(S')$ is a semigroup homomorphism as well, see [16].

[1] In [13], instead of the notion "upfamily" it was used the notion "inclusion hyperspace".

In [22], it was proved that the upfamily extension $(\upsilon(D), \dashv, \vdash)$ of a (strong) doppelsemigroup $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup as well. Doppelsemigroups of k-linked upfamilies were investigated in [19]. This paper is devoted to investigate the subdoppelsemigroup $(\lambda(D), \dashv, \vdash)$ of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$. We study right and left zeros and identities, commutativity, the center, ideals of the superextension $(\lambda(D), \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$. Also we introduce the superextension functor λ in the category **DSG**, whose objects are doppelsemigroups and morphisms are doppelsemigroup homomorphisms, and show that this functor preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups. On the other hand, the functor λ does not preserve commutative doppelsemigroups and group doppelsemigroups. Using functoriality of λ we prove that the automorphism group of the superextension of a doppelsemigroup $(D, \dashv, \vdash)$ contain a subgroup, isomorphic to the automorphism group of $(D, \dashv, \vdash)$.

1 Zeros and identities of the superextesion $(\lambda(D), \dashv, \vdash)$

In [22], it was shown that the upfamily extension $(\upsilon(D), \dashv, \vdash)$ of a (strong) doppelsemigroup $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup as well, where for $* \in \{\dashv, \vdash\}$ and $\mathcal{U}, \mathcal{V} \in \upsilon(D)$, we have

$$\mathcal{U} * \mathcal{V} = \Big\langle \bigcup_{a \in U} a * V_a : U \in \mathcal{U}, \ \ \{V_a\}_{a \in U} \subset \mathcal{V} \Big\rangle.$$

Taking into account that for any semigroup S, the superextension $\lambda(S)$ is a subsemigroup of the semigroup $\upsilon(S)$ (see [13]), we conclude the following proposition.

Proposition 1. *If $(D, \dashv, \vdash)$ is a (strong) doppelsemigroup, then $(\lambda(D), \dashv, \vdash)$ is a (strong) subdoppelsemigroup of the doppelsemigroup $(\upsilon(D), \dashv, \vdash)$.*

An element z of a doppelsemigroup $(D, \dashv, \vdash)$ is called a *zero* (resp. a *left zero*, a *right zero*) if $a \dashv z = z \dashv a = a \vdash z = z \vdash a = z$ (resp. $z \dashv a = z \vdash a = z$, $a \dashv z = a \vdash z = z$) for any $a \in D$.

Let $(D, \dashv, \vdash)$ be a doppelsemigroup and $z \notin D$. The binary operations defined on D can be extended to $D \cup \{z\}$ putting $z \dashv d = d \dashv z = z = z \vdash d = d \vdash z$ for all $d \in D \cup \{z\}$. The notation $(D, \dashv, \vdash)^{+0}$ denotes a doppelsemigroup $(D \cup \{z\}, \dashv, \vdash)$ obtained from $(D, \dashv, \vdash)$ by adjoining the extra zero z. If $(D, \dashv, \vdash)$ is a strong doppelsemigroup, then $(D, \dashv, \vdash)^{+0}$ is a strong doppelsemigroup as well.

Proposition 2. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. For each element $z \in D \subset \lambda(D)$ the following conditions are equivalent:*

(1) *z is a left (right) zero of a semigroup $(D, \dashv)$;*

(2) *z is a left (right) zero of a doppelsemigroup $(D, \dashv, \vdash)$;*

(3) *z is a left (right) zero of a semigroup $(D, \vdash)$;*

(4) *z is a left (right) zero of the semigroup $(\lambda(D), \vdash)$;*

(5) *z is a left (right) zero of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$;*

(6) *z is a left (right) zero of the semigroup $(\lambda(D), \dashv)$.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) were proved in [22] while the implications (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) are trivial.

(3) $\Rightarrow$ (4) Let z be a left zero of a semigroup $(D, \vdash)$. Then for every maximal linked upfamily $\mathcal{M} \in \lambda(D)$ we get $z \vdash \mathcal{M} = \langle z \vdash M : M \in \mathcal{M} \rangle = \langle \{z\} : M \in \mathcal{M} \rangle = z$, which means that (the principal ultrafilter generated by) z is a left zero of $(\lambda(D), \vdash)$.

(4) $\Rightarrow$ (5) Assume that z is a left zero of the semigroup $(\lambda(D), \vdash)$. Taking into account that for any $\mathcal{M} \in \lambda(D)$, $z \dashv \mathcal{M} = (z \vdash \mathcal{M}) \dashv \mathcal{M} = z \vdash (\mathcal{M} \dashv \mathcal{M}) = z$, we conclude that z is also a left zero of the semigroup $(\lambda(D), \dashv)$, and hence z is a left zero of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.

In the same way one can check the right zero case. $\square$

A doppelsemigroup $(D, \dashv, \vdash)$ is said to be a *left (right) zero doppelsemigroup* if each of its elements is a left (right) zero. By LO_X (RO_X) we denote the left (right) zero doppelsemigroup on a set X. If X is finite of cardinality $|X| = n$, then instead of LO_X and RO_X we use LO_n and RO_n, respectively. Note that the operations of LO_n and RO_n coincide and they can be considered as a left zero semigroup and a right zero semigroup, respectively.

Let us note that for a subdoppelsemigroup $(T, \dashv, \vdash)$ of a doppelsemigroup $(D, \dashv, \vdash)$ the homomorphism $i : \lambda(T) \to \lambda(D)$, $i : \mathcal{M} \to \langle \mathcal{M} \rangle_D$ is injective, and thus we can identify the doppelsemigroup $(\lambda(T), \dashv, \vdash)$ with the doppelsubsemigroup $i((\lambda(T), \dashv, \vdash)) \subset (\lambda(D), \dashv, \vdash)$.

Proposition 3. *If T is a left (right) zero subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$, then $\lambda(T)$ is a left (right) zero subdoppelsemigroup of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$ as well.*

Proof. Let T be a left zero subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$. Then for any $\mathcal{M}, \mathcal{L} \in \lambda(T)$, we have

$$\begin{aligned} \mathcal{M} \dashv \mathcal{L} &= \Big\langle \bigcup_{a \in M} a \dashv L_a : M \in \mathcal{M},\ M \subset T,\ \ L_a \in \mathcal{L},\ L_a \subset T \text{ for all } a \in M \Big\rangle \\ &= \Big\langle \bigcup_{a \in M} \{a\} : M \in \mathcal{M} \Big\rangle = \mathcal{M}. \end{aligned}$$

It follows from Proposition 2 that $\mathcal{M} \vdash \mathcal{L} = \mathcal{M}$, and hence $(\lambda(T), \dashv, \vdash)$ is a left zero subdoppelsemigroup of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.

For a right zero subdoppelsemigroup the proof is similar. $\square$

Propositions 2 and 3 imply the following corollary.

Corollary 1. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. Then the following conditions are equivalent:*

1) $(D, \dashv)$ is a left (right) zero semigroup;

2) $(D, \dashv, \vdash)$ is a left (right) zero doppelsemigroup;

3) $(D, \vdash)$ is a left (right) zero semigroup;

4) $(\lambda(D), \vdash)$ is a left (right) zero semigroup;

5) $(\lambda(D), \dashv, \vdash)$ is a left (right) zero doppelsemigroup;

6) $(\lambda(D), \dashv)$ is a left (right) zero semigroup.

Unlike the upfamily extension $(v(G), \dashv, \vdash)$ of a group doppelsemigroup $(G, \dashv, \vdash)$ which always contains right zeros, the superextension $(\lambda(G), \dashv, \vdash)$ contains right zeros only for so-called odd group doppelsemigroups. We define a group doppelsemigroup $(G, \dashv, \vdash)$ to be *odd* if each element g of a group $(G, \dashv)$ has odd order. Since $(G, \vdash)$ is isomorphic to $(G, \dashv)$, each element g of $(G, \vdash)$ has odd order as well. If $(G, \dashv, \vdash)$ is a finite odd group doppelsemigroup, then the maximal linked upfamily $\mathcal{Z} = \{A \subset G : |A| > |G|/2\}$ is a right zero of the semigroup $(\lambda(G), \dashv)$ (see [8]), and hence $\mathcal{Z}$ is a right zero of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ by Proposition 2. The proposition below follows from [8, Theorem 3.2].

Proposition 4. *Let $(G, \dashv, \vdash)$ be a group doppelsemigroup. The doppelsemigroup $(\lambda(G), \dashv, \vdash)$ has a right zero if and only if $(G, \dashv, \vdash)$ is odd.*

Proof. If a group doppelsemigroup $(G, \dashv, \vdash)$ is odd, then according to [8, Theorem 3.2] the semigroup $(\lambda(G), \dashv)$ contains a right zero $\mathcal{Z}$. By Proposition 2, $\mathcal{Z}$ is a right zero of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$. If $(G, \dashv, \vdash)$ is not odd, then by [8, Theorem 3.2] the semigroup $(\lambda(G), \dashv)$ contains no a right zero, and hence the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ has no a right zero. $\square$

An element e of a doppelsemigroup $(D, \dashv, \vdash)$ is called an *identity* (respectively, a *left identity*, a *right identity*) if $a \dashv e = e \dashv a = a \vdash e = e \vdash a = a$ (respectively, $e \dashv a = e \vdash a = e$, $a \dashv e = a \vdash e = a$) for any $a \in D$.

Proposition 5. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup. For each element $e \in D \subset \lambda(D)$ the following conditions are equivalent:*

(1) *e is a left (right) identity of a doppelsemigroup $(D, \dashv, \vdash)$;*

(2) *e is a left (right) identity of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.*

Proof. The implication $(2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (2)$ Let e be a left identity of a doppelsemigroup $(D, \dashv, \vdash)$. Then for every maximal linked upfamily $\mathcal{M} \in \lambda(D)$ we get $e \vdash \mathcal{M} = \{e \vdash M : M \in \mathcal{M}\} = \{M : M \in \mathcal{M}\} = \mathcal{M}$ and $e \dashv \mathcal{M} = \{e \dashv M : M \in \mathcal{M}\} = \{M : M \in \mathcal{M}\} = \mathcal{M}$, which means that (the principal ultrafilter generated by) e is a left identity of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.

In the same way one can check the right identity case. $\square$

Remark 1. *In general case, a left (right) identity of a semigroup $(D, \dashv)$ is not a left (right) identity of a doppelsemigroup $(D, \dashv, \vdash)$. For the doppelsemigroup $(\mathbb{R}, \cdot, *)$, where $a * b = 0$ for any $a, b \in \mathbb{R}$, the semigroup $(\mathbb{R}, \cdot)$ is a monoid while the semigroup $(\mathbb{R}, *)$ contains no a left (right) identity.*

2 Commutativity and the center of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$

In the next proposition we use [8, Theorem 5.1] to characterize group doppelsemigroups with commutative superextensions.

Proposition 6. *The superextension $(\lambda(G), \dashv, \vdash)$ of a group doppelsemigroup $(G, \dashv, \vdash)$ is commutative if and only if $|G| \leq 4$.*

Proof. If $|G| \leq 4$, then according to [8, Theorem 5.1] the semigroups $(\lambda(G), \dashv)$ and $(\lambda(G), \vdash)$ are commutative, and hence the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ is commutative as well. If $|G| > 4$, then by [8, Theorem 5.1] the semigroup $(\lambda(G), \dashv)$ is not commutative, and thus the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ is not commutative as well. $\square$

By definition, the *center* of a doppelsemigroup $(D, \dashv, \vdash)$ is the set

$$C(D, \dashv, \vdash) = \{a \in D : a \dashv x = x \dashv a \text{ and } a \vdash x = x \vdash a \text{ for all } x \in D\}.$$

It follows from this definition that $C(D, \dashv, \vdash) = C(D, \dashv) \cap C(D, \vdash)$. If the center $C(D, \dashv, \vdash)$ of the doppelsemigroup $(D, \dashv, \vdash)$ is non-empty, then it is a subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$, see [22, Proposition 3].

It was proved in [22, Proposition 4] that the center of the upfamily extension $(\upsilon(G), \dashv, \vdash)$ of a group doppelsemigroup $(G, \dashv, \vdash)$ coincides with the center of $(G, \dashv, \vdash)$. On the other hand, it follows from Proposition 6 that $C(\lambda(G), \dashv, \vdash) = (\lambda(G), \dashv, \vdash)$ for each group doppelsemigroup $(G, \dashv, \vdash)$ of order $|G| \leq 4$. In the next theorem we show that the center of the doppelsemigroup of maximal linked upfamilies on a countable infinite group doppelsemigroup $(G, \dashv, \vdash)$ coincides with the center of $(G, \dashv, \vdash)$.

Theorem 1. *The center of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$ contains the center of $(D, \dashv, \vdash)$. If $(D, \dashv, \vdash)$ is a countable infinite group doppelsemigroup, then*

$$C(\lambda(D), \dashv, \vdash) = C(D, \dashv) \cap C(D, \vdash).$$

Proof. Let $a \in C(D, \dashv, \vdash)$. Then for every maximal linked upfamily $\mathcal{M} \in \lambda(D)$ we get

$$a \dashv \mathcal{M} = \{a \dashv M : M \in \mathcal{M}\} = \{M \dashv a : M \in \mathcal{M}\} = \mathcal{M} \dashv a$$

and

$$a \vdash \mathcal{M} = \{a \vdash M : M \in \mathcal{M}\} = \{M \vdash a : M \in \mathcal{M}\} = \mathcal{M} \vdash a,$$

which means that (the principal ultrafilter generated by) a belongs to the center of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.

If $(D, \dashv)$ is a countable infinite group, then $(D, \vdash)$ is a countable infinite group as well. Applying [1, Theorem 4.2], we get that $C(\lambda(D), \dashv) = C(D, \dashv)$ and $C(\lambda(D), \vdash) = C(D, \vdash)$, and hence $C(\lambda(D), \dashv, \vdash) = C(\lambda(D), \dashv) \cap C(\lambda(D), \vdash) = C(D, \dashv) \cap C(D, \vdash)$. $\square$

Proposition 7. *Let $(G, \cdot)$ be a group of order $|G| \geq 3$. The superextension $\lambda(G)$ contains at least two idempotents.*

Proof. Let e be the identity of G. Consider the linked upfamily $\mathcal{I} = \langle G \setminus \{e\}, \{e, g\} : g \in G \setminus \{e\} \rangle$. Let us show that $\mathcal{I}$ is a maximal linked upfamily. It suffices to show that each set $A \subset G$ that intersects every element of the upfamily $\mathcal{I}$ belongs to $\mathcal{I}$. Consider two cases.

Case 1) Let $e \in A$. Then $A \supset \{e, a\}$, where $a \in A \cap (G \setminus \{e\})$. Since $\mathcal{I}$ is an upfamily, we get $A \in \mathcal{I}$.

Case 2) Let $e \notin A$. Taking into account that A intersects $\{e, g\}$ for each $g \in G \setminus \{e\}$, we conclude that $A \supset G \setminus \{e\}$, and hence $A \in \mathcal{I}$.

Now let us show that the maximal linked upfamily $\mathcal{I}$ is an idempotent of the semigroup $\lambda(G)$. By maximality, it is sufficient to show that $\mathcal{I} \subset \mathcal{I}^2$. Since $\{e, g\} = \bigcup_{a \in \{e,g\}} aI_a$, where $I_e = \{e, g\}$ and $I_g = \{e, g^{-1}\}$, we conclude that $\{e, g\} \in \mathcal{I}^2$ for any $g \in G \setminus \{e\}$. To prove that $G \setminus \{e\} \in \mathcal{I}^2$, for each $g \in G \setminus \{e\}$ fix any $h_g \notin G \setminus \{e, g^{-1}\}$. Then $G \setminus \{e\} = \bigcup_{g \in G\setminus\{e\}} g\{e, h_g\}$ and we are done.

We conclude that the superextension $\lambda(G)$ contains at least two idempotents: e and $\mathcal{I}$. $\square$

Proposition 8. *Let $(G, \cdot)$ be a group. The superextension $\lambda(G)$ is a group if and only if $|G| \in \{1, 2\}$.*

Proof. If $|G| \in \{1, 2\}$, then the semigroup $\lambda(G)$ contains only principal ultrafilters, see Table 1. It follows that $\lambda(G)$ is isomorphic to G, and hence $\lambda(G)$ is a group. If $|G| \geq 3$, then according to Proposition 7 the semigroup $\lambda(G)$ contains at least two idempotents, and thus $\lambda(G)$ can not be a group. $\square$

Corollary 2. *Let $(G, \dashv, \vdash)$ be a group doppelsemigroup. The superextension $(\lambda(G), \dashv, \vdash)$ is a group doppelsemigroup if and only if $|G| \in \{1, 2\}$.*

3 The functor λ in the category DSG

A map $\varphi : D_1 \to D_2$ is called a *homomorphism of doppelsemigroups* $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ if $\varphi(a \dashv_1 b) = \varphi(a) \dashv_2 \varphi(b)$ and $\varphi(a \vdash_1 b) = \varphi(a) \vdash_2 \varphi(b)$ for all $a, b \in D_1$.

A bijective homomorphism is called an *isomorphism of doppelsemigroups*. If there exists an isomorphism between the doppelsemigroups $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$, then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are said to be *isomorphic*, denoted $(D_1, \dashv_1, \vdash_1) \cong (D_2, \dashv_2, \vdash_2)$. An isomorphism $\psi : D \to D$ is called an *automorphism* of a doppelsemigroup $(D, \dashv, \vdash)$. By $\mathrm{Aut}(D, \dashv, \vdash)$ we denote the automorphism group of a doppelsemigroup $(D, \dashv, \vdash)$.

According to [13, Proposition 8], each homomorphism $\varphi : D_1 \to D_2$ of doppelsemigroups $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ induces the homomorphism

$$\lambda(\varphi) : \lambda(D_1) \to \lambda(D_2), \quad \lambda(\varphi) : \mathcal{M} \mapsto \langle \varphi(M) \subset D_2 : M \in \mathcal{M} \rangle$$

of doppelsemigroups $(\lambda(D_1), \dashv_1, \vdash_1)$ and $(\lambda(D_2), \dashv_2, \vdash_2)$.

By **DSG** we denote the category of doppelsemigroups whose objects are doppelsemigroups and morphisms are doppelsemigroup homomorphisms.

A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of an object map $F : v\mathcal{C} \to v\mathcal{D}$, which assigns to each $a \in v\mathcal{C}$ an object $F(a) \in v\mathcal{D}$ and a morphism map F which assigns to each morphism $f : a \to b$ in $\mathcal{C}$ a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{D}$ such that

1) $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for each $a \in v\mathcal{C}$;

2) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms $f, g \in \mathcal{C}$ for which the composition $f \circ g$ exists.

Let us consider $\lambda : \mathbf{DSG} \to \mathbf{DSG}$ which assigns to each doppelsemigroup $(D, \dashv, \vdash)$ the doppelsemgroup $(\lambda(D), \dashv, \vdash)$ of maximal linked upfamilies on D, its morphism map assigns to each doppelsemigroup homomorphism $\varphi : D_1 \to D_2$, the doppelsemigroup homomorphism $\lambda(\varphi) : v(D_1) \to v(D_2)$. Taking into account that for any $\mathcal{M} \in \lambda(D)$ we have

$$\lambda(\mathrm{id}_D)(\mathcal{M}) = \langle \mathrm{id}_D(M) : M \in \mathcal{M} \rangle = \langle M : M \in \mathcal{M} \rangle = \mathcal{M} = \mathrm{id}_{\lambda(D)}(\mathcal{M})$$

and

$$\lambda(\varphi)\circ\lambda(\psi)(\mathcal{M})=\lambda(\varphi)(\langle\psi(M):M\in\mathcal{M}\rangle)=\langle\varphi\circ\psi(M):M\in\mathcal{M}\rangle=\lambda(\varphi\circ\psi)(\mathcal{M}),$$

we conclude that $\lambda(\mathrm{id}_D)=\mathrm{id}_{\lambda(D)}$ and $\lambda(\varphi\circ\psi)=\lambda(\varphi)\circ\lambda(\psi)$, and hence this construction defines the covariant functor $\lambda:\mathbf{DSG}\to\mathbf{DSG}$. This functor are said to be the *superextension functor* in the category of doppelsemigroups.

Combining Propositions 1, 2 and 5 with Corollary 1, we get the following theorem.

Theorem 2. *The superextension functor λ in* **DSG** *preserves strong doppelsemigroups, doppelsemigroups with left (right) zero, doppelsemigroups with left (right) identity, left (right) zeros doppelsemigroups.*

On the other hand, Proposition 6 and Corollary 2 imply that the functor λ in **DSG** does not preserve commutative doppelsemigroups and group doppelsemigroups.

4 Extending automorphisms from a doppelsemigroup to its superextension

Taking into account that the construction λ defines a covariant functor in the category of doppelsemigroups, we conclude the following proposition.

Proposition 9. *If $\psi:D_1\to D_2$ is an isomorphism from a doppelsemigroup $(D_1,\dashv_1,\vdash_1)$ to a doppelsemigroup $(D_2,\dashv_2,\vdash_2)$, then $\lambda(\psi):\lambda(D_1)\to\lambda(D_2)$ is an isomorphism as well.*

Proof. Since $\psi:D_1\to D_2$ is an isomorphism from a doppelsemigroup $(D_1,\dashv_1,\vdash_1)$ to a doppelsemigroup $(D_2,\dashv_2,\vdash_2)$, there exists a doppelsemigroup homomorphism $\phi:D_2\to D_1$ such that $\psi\circ\phi=\mathrm{id}_{D_2}$ and $\phi\circ\psi=\mathrm{id}_{D_1}$. Taking into account that the construction λ is a covariant functor in the category of doppelsemigroups, we conclude that

$$\lambda(\psi)\circ\lambda(\phi)=\lambda(\psi\circ\phi)=\lambda(\mathrm{id}_{D_2})=\mathrm{id}_{\lambda(D_2)},$$
$$\lambda(\phi)\circ\lambda(\psi)=\lambda(\phi\circ\psi)=\lambda(\mathrm{id}_{D_1})=\mathrm{id}_{\lambda(D_1)},$$

and so $\lambda(\psi):\lambda(D_1)\to\lambda(D_2)$ is an isomorphism from $(\lambda(D_1),\dashv_1,\vdash_1)$ to $(\lambda(D_2),\dashv_2,\vdash_2)$. □

Corollary 3. *If $\psi:D\to D$ is an automorphism of a doppelsemigroup $(D,\dashv,\vdash)$, then $\lambda(\psi):\lambda(D)\to\lambda(D)$ is an automorphism of the superextension $(\lambda(D),\dashv,\vdash)$.*

In the following proposition we show that the automorphism group of the superextension $(\lambda(D),\dashv,\vdash)$ of a doppelsemigroup $(D,\dashv,\vdash)$ contain a subgroup, isomorphic to the group $\mathrm{Aut}(D,\dashv,\vdash)$.

Proposition 10. *The automorphism group of the superextension $(\lambda(D),\dashv,\vdash)$ of a doppelsemigroup $(D,\dashv,\vdash)$ contain a subgroup, isomorphic to $\mathrm{Aut}(D,\dashv,\vdash)$.*

Proof. The functoriality of λ in the category of doppelsemigroups implies that $\lambda(\psi_1\circ\psi_2)=\lambda(\psi_1)\circ\lambda(\psi_2)$ for any $\psi_1,\psi_2\in\mathrm{Aut}(D)$.

Let $\psi_1\neq\psi_2$, then $\psi_1(a)\neq\psi_2(a)$ for some $a\in D$, and hence

$$\lambda(\psi_1)(\langle\{a\}\rangle)=\langle\{\psi_1(a)\}\rangle\neq\langle\{\psi_2(a)\}\rangle=\lambda(\psi_2)(\langle\{a\}\rangle).$$

It follows that the map $\varphi:\mathrm{Aut}(D)\to\mathrm{Aut}(\lambda(D))$, $\varphi:\psi\mapsto\lambda(\psi)$, is an injective homomorphism. Therefore, $\mathrm{Aut}(D)$ is isomorphic to the subgroup $\{\lambda(\psi):\psi\in\mathrm{Aut}(D)\}$ of the group $\mathrm{Aut}(\lambda(D))$. □

5 Ideals of the superextesion $(\lambda(D), \dashv, \vdash)$

A non-empty subset I of a doppelsemigroup $(D, \dashv, \vdash)$ is called an *ideal* (respectively, a *left ideal*, a *right ideal*) if $(S \dashv I) \cup (S \vdash I) \cup (I \dashv S) \cup (I \vdash S) \subset I$ (respectively, $(S \dashv I) \cup (S \vdash I) \subset I$, $(I \dashv S) \cup (I \vdash S) \subset I$).

Proposition 11. *If I is a left (right) ideal of a doppelsemigroup $(D, \dashv, \vdash)$, then $\lambda(I)$ is a left (right) ideal of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$ as well.*

Proof. Indeed, let $\mathcal{M} \in \lambda(D)$, $\mathcal{L} \in \lambda(I)$. Then

$$\begin{aligned}\mathcal{M} \dashv \mathcal{L} &= \Big\langle \bigcup_{a \in M} a \dashv L_a : M \in \mathcal{M},\ L_a \in \mathcal{L},\ L_a \subset I \text{ for all } a \in M \Big\rangle \\ &= \Big\langle \bigcup_{a \in M} a \dashv L_a : M \in \mathcal{M},\ \{L_a\}_{a \in M} \subset \mathcal{L},\ \bigcup_{a \in M} a \dashv L_a \subset I \Big\rangle \in \lambda(I).\end{aligned}$$

By analogy $\mathcal{M} \vdash \mathcal{L} \in \lambda(I)$, and therefore $\lambda(I)$ is a left ideal of the doppelsemigroup $(\lambda(D), \dashv, \vdash)$.

In the same way one can check the right ideal case. □

Formula (1) implies that the product $\mathcal{M} * \mathcal{L}$ of any two maximal linked upfamilies $\mathcal{M}$ and $\mathcal{L}$ on a group $(G, *)$ is principal ultrafilter if and only if both $\mathcal{M}$ and $\mathcal{L}$ are principal ultrafilters. So we get the following proposition.

Proposition 12. *For any group doppelsemigroup $(G, \dashv, \vdash)$ the set $\lambda(G) \setminus G$ is an ideal in $(\lambda(G), \dashv, \vdash)$.*

By definition, the *minimal ideal* of a doppelsemigroup $(D, \dashv, \vdash)$ is an ideal containing no other ideal of $(D, \dashv, \vdash)$. It is also called the *kernel* of a doppelsemigroup $(D, \dashv, \vdash)$, denoted $K(D)$.

Proposition 13. *If $(G, \dashv, \vdash)$ is an odd group doppelsemigroup, then the minimal ideal $K(\lambda(G))$ of $(\lambda(G), \dashv, \vdash)$ coincides with the set of all right zeros of $(\lambda(G), \dashv, \vdash)$.*

Proof. According to Proposition 4, the superextension $(\lambda(G), \dashv, \vdash)$ of an odd group doppelsemigroup $(G, \dashv, \vdash)$ contains a right zero. Since the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ contains a right zero, then by [22, Proposition 9] the minimal ideal $K(\lambda(G))$ of $(\lambda(G), \dashv, \vdash)$ coincides with the set of all right zeros of $(\lambda(G), \dashv, \vdash)$. □

Following the algebraic tradition, we take for a model of the class of cyclic groups of order n the multiplicative group $C_n = \{z \in \mathbb{C} : z^n = 1\}$ of nth roots of 1. In fact, the group $(C_n, \cdot)$ can be consider as a (strong) group doppelsemigroup $(C_n, \cdot, \cdot)$, and we denote this *trivial* group doppelsemigroup by C_n. Note that if a and b are generators of the cyclic group C_n, then the cyclic doppelsemigroups $(C_n, \cdot, \cdot_a)$ and $(C_n, \cdot, \cdot_b)$ are isomorphic (see [20]), where $x \cdot_h y = x \cdot h \cdot y$. Let $g = e^{2\pi i/n}$ be a generator of the cyclic group C_n. Following [18], by $C_n \between C_n^{gen}$ we denote the cyclic doppesemigroup $(C_n, \cdot, \cdot_g)$. So $C_n \between C_n^{gen}$ is a model of the class of cyclic doppelsemigroups of order n isomorphic to $(C_n, \cdot, \cdot_g)$.

Now we describe group doppelsemigroups $(G, \dashv, \vdash)$ such that the minimal ideals $K(\lambda(G))$ of the superextensions $(\lambda(G), \dashv, \vdash)$ are singletons.

Theorem 3. *Let $(G, \dashv, \vdash)$ be a group doppelsemigroup. The following conditions are equivalent:*

1) *the minimal ideal $K(\lambda(G))$ of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ is a singleton;*

2) *the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ contains a zero;*

3) *the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ contains a left zero;*

4) *$(G, \dashv, \vdash)$ is isomorpic to C_1, C_3, $C_3 \between C_3^{gen}$, C_5 or $C_5 \between C_5^{gen}$.*

Proof. 1) $\Rightarrow$ 2) Let $K(\lambda(G)) = \{\mathcal{Z}\}$ be the minimal ideal of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$. Then

$$(\lambda(G) \dashv \{\mathcal{Z}\}) \cup (\{\mathcal{Z}\} \dashv \lambda(G)) \cup (\lambda(G) \vdash \{\mathcal{Z}\}) \cup (\{\mathcal{Z}\} \vdash \lambda(G)) \subset \{\mathcal{Z}\}.$$

It follows that $\mathcal{M} \dashv \mathcal{Z} = \mathcal{Z} \dashv \mathcal{M} = \mathcal{M} \vdash \mathcal{Z} = \mathcal{Z} \vdash \mathcal{M} = \mathcal{Z}$ for any $\mathcal{M} \in \lambda(G)$, and hence $\mathcal{Z}$ is a zero of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$.

2) $\Rightarrow$ 1) Let $\mathcal{Z}$ be a zero of the doppelsemigroup $(\lambda(G), \dashv, \vdash)$. Then $\mathcal{M} \dashv \mathcal{Z} = \mathcal{Z} \dashv \mathcal{M} = \mathcal{M} \vdash \mathcal{Z} = \mathcal{Z} \vdash \mathcal{M} = \mathcal{Z}$ for any $\mathcal{M} \in \lambda(G)$. It follows that $\lambda(G) \dashv \{\mathcal{Z}\} = \{\mathcal{Z}\} \dashv \lambda(G) = \lambda(G) \vdash \{\mathcal{Z}\} = \{\mathcal{Z}\} \vdash \lambda(G) = \{\mathcal{Z}\}$, and thus $\{\mathcal{Z}\}$ is an ideal of $(\lambda(G), \dashv, \vdash)$. It is clear that $\{\mathcal{Z}\}$ is minimal.

The implication 2) $\Rightarrow$ 3) is trivial.

3) $\Rightarrow$ 4) Since the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ contains a left zero, the semigroup $(\lambda(G), \dashv)$ contains a left zero as well. According to [8, Proposition 4.1, Theorem 4.2], $(G, \dashv)$ is isomorphic to one of the cyclic groups: C_1, C_3 or C_5. By [20, Theorem 1], for a prime number p, there exist two pairwise non-isomorphic cyclic doppelsemigroups of order p: C_p and $C_p \between C_p^{gen}$. It follows that $(G, \dashv, \vdash)$ must be isomorphic to one of the cyclic doppelsemigroups: C_1, C_3, $C_3 \between C_3^{gen}$, C_5 or $C_5 \between C_5^{gen}$.

4) $\Rightarrow$ 2) If $(G, \dashv, \vdash)$ is isomorphic to C_1, C_3, $C_3 \between C_3^{gen}$, C_5 or $C_5 \between C_5^{gen}$, then $(G, \dashv)$ is isomorphic to one of the cyclic groups: C_1, C_3 or C_5. According to [8, Theorem 4.2] the semigroup $(\lambda(G), \dashv)$ has a zero. By Proposition 2 the doppelsemigroup $(\lambda(G), \dashv, \vdash)$ has a zero as well. $\square$

Example 1. *Let us consider group doppelsemigroups of order 3. According to [20, Theorem 1], there exist two pairwise non-isomorphic group doppelsemigroups of order 3: C_3 and $C_3 \between C_3^{gen} = (C_3, \cdot, \cdot_g)$, where $g = e^{2\pi i/3}$ is a generator of the cyclic group C_3. The set $\lambda(C_3)$ contains three principal ultrafilters $1, g, g^{-1}$ and the maximal linked upfamily $\triangle = \langle\{1, g\}, \{1, g^{-1}\}, \{g, g^{-1}\}\rangle$. By Proposition 12 the set $\{\triangle\} = \lambda(C_3) \setminus C_3$ is an ideal of the doppelsemigroups $\lambda(C_3)$ and $\lambda(C_3 \between C_3^{gen})$, and hence $\triangle$ is the zero of $\lambda(C_3)$ and $\lambda(C_3 \between C_3^{gen})$. It follows that $\lambda(C_3) \cong (C_3)^{+0}$ and $\lambda(C_3 \between C_3^{gen}) \cong (C_3 \between C_3^{gen})^{+0}$.*

Taking into account that zeros are preserved by automorphisms and $\mathrm{Aut}(C_3 \between C_3^{gen}) \cong C_1$ (see [18, Table 4]), we conclude that $\mathrm{Aut}(\lambda(C_3)) \cong \mathrm{Aut}((C_3)^{+0}) \cong \mathrm{Aut}(C_3) \cong C_2$ and

$$\mathrm{Aut}(\lambda(C_3 \between C_3^{gen})) \cong \mathrm{Aut}((C_3 \between C_3^{gen})^{+0}) \cong \mathrm{Aut}(C_3 \between C_3^{gen}) \cong C_1.$$

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Гаврилків В.М. *Суперрозширення допельнапівгруп* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 591–603.

Сім'я $\mathcal{U}$ непорожніх підмножин множини D називається *монотонною*, якщо для кожної множини $U \in \mathcal{U}$ довільна множина $F \supset U$ також належить $\mathcal{U}$. Монотонна сім'я $\mathcal{L}$ підмножин множини D називається *зчепленою*, якщо $A \cap B \neq \varnothing$ для всіх $A, B \in \mathcal{L}$. Зчеплена сім'я $\mathcal{M}$ підмножин множини D називається *максимальною зчепленою*, якщо $\mathcal{M}$ збігається з кожною зчепленою сім'єю $\mathcal{L}$ на D, що містить $\mathcal{M}$. *Суперрозширення* $\lambda(D)$ множини D складається з усіх максимальних зчеплених сімей на D. Кожна асоціативна бінарна операція $* : D \times D \to D$ продовжується до асоціативної бінарної операції

$$*: \lambda(D) \times \lambda(D) \to \lambda(D), \quad \mathcal{M} * \mathcal{L} = \Big\langle \bigcup_{a \in M} a * L_a : M \in \mathcal{M},\ \{L_a\}_{a \in M} \subset \mathcal{L} \Big\rangle.$$

У цій статті ми досліджуємо будову допельнапівгрупи $(\lambda(D), \dashv, \vdash)$ максимальних зчеплених сімей на допельнапівгрупі $(D, \dashv, \vdash)$. Зокрема ми вивчаємо праві і ліві нулі та одиниці, комутативність, центр, ідеали суперрозширення $(\lambda(D), \dashv, \vdash)$ допельнапівгрупи $(D, \dashv, \vdash)$. Ми вводимо функтор суперрозширення λ у категорії **DSG**, об'єктами якої є допельнапівгрупи, а морфізмами — гомоморфізми допельнапівгруп, і показуємо, що функтор λ зберігає сильні допельнапівгрупи, допельнапівгрупи з лівим (правим) нулем, допельнапівгрупи з лівою (правою) одиницею, допельнапівгрупи лівих (правих) нулів. Також ми доводимо, що група автоморфізмів суперрозширення допельнапівгрупи $(D, \dashv, \vdash)$ містить підгрупу, ізоморфну до групи автоморфізмів допельнапівгрупи $(D, \dashv, \vdash)$.

Ключові слова і фрази: напівгрупа, суперозширення, максимальна зчеплена сім'я, допельнапівгрупа.