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Function calculus on rings of multisets associated with symmetric and supersymmetric polynomials

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We investigate symmetric and supersymmetric mappings on Banach spaces with symmetric bases, and rings of multisets related to these spaces. We establish some results about the extension of (super)symmetric mappings and describe continuous complex homomorphisms of the rings of multisets. Also, we study analytic mappings and function calculi on the rings.

Key words and phrases: symmetric function on Banach space, supersymmetric function, ring of multisets, function calculus.

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Introduction

The theory of symmetric functions on infinite dimensional Banach spaces has intensively developed in recent years. Let X be an infinite dimensional complex Banach space with a symmetric basis $(e_n)_{n\in\mathbb{N}}$. For any permutation σ on the set of natural numbers $\mathbb{N}$ (i.e. injective map $\sigma\colon \mathbb{N}\to\mathbb{N}$) the linear map

$$A_\sigma\colon \sum_{i=1}^{\infty} x_i e_i \mapsto \sum_{i=1}^{\infty} x_i e_{\sigma(i)}$$

is an injective isometric operator. A mapping F on X is symmetric if $F(x)=F\big(A_\sigma(x)\big)$ for every permutation σ. The semigroup of all operators A_σ will be denoted by $\mathcal{S}(X)$.

Symmetric polynomials and analytic mappings with respect to actions of operators in $\mathcal{S}(X)$ were considered by many authors (see, e.g., [2, 5, 16] and references therein). In [17, 23], it was described algebraic bases of power symmetric polynomials on a space X and it was proved that polynomials

$$F_k(x)=\sum_{n=1}^{\infty} x_n^k$$

form an algebraic basis in ℓ_p, where $1\le p<\infty$ and $k\ge\lceil p\rceil$. Here $\lceil p\rceil$ is the ceiling of p. Also, in [17, 23] it was observed that the Banach space c_0 does not admit nonconstant symmetric

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polynomials. Supersymmetric analytic functions on Cartesian products of ℓ_1 were considered in [4,10,11,18,19], for the general facts on supersymmetric polynomials of finite many variables see [24]. Investigating spectra of algebras of symmetric and supersymmetric analytic functions (see [5,6,19]) it was indicated that it makes sense to consider the quotient set of X with respect to actions of operators in the semigroup of symmetry. Algebraic and topological structures of this quotient set for more general situation were studied in [4,8–10] . In particular, it was introduced (semi)ring structures on this quotient set and identified it with some set of multisets of nonzero complex numbers. In this paper, we continue investigations of (semi)rings of multisets based on X and related properties of symmetric and supersymmetric mappings.

The paper is organized as follows. Section 1 contains a brief overview of semirings and rings of multisets based on Banach spaces with a symmetric basis. In Section 2, we consider a problem of extension of a symmetric mapping to a larger Banach space with preserving the symmetry. In Section 3, we consider continuous complex-valued homomorphisms on rings of multisets. Also, we discuss how to introduce analytic mappings on these rings. In Section 4, we consider function calculi on rings of multisets with values in rings of analytic functions and rational functions.

The general theory of analytic functions on Banach spaces can be found in [13,22]. Theory of classical symmetric functions is in [21].

1 Preliminary results

Throughout this paper, let X be an infinite dimensional complex Banach space with a symmetric basis $(e_n)_{n\in\mathbb{N}}$. Without loss of generality we may suppose that the basis (e_n) is normalized, monotone, and the norm on X is symmetric [20]. In other words, $\|e_n\| = 1$ for all $n \in \mathbb{N}$, and for every $x \in X$ presented as

$$x = (x_1, x_2, \ldots) = \sum_{n=1}^{\infty} x_n e_n$$

we have $\|x\| \le \max_k |x_k|$ and the function $x \mapsto \|x\|$ is symmetric (invariant) with respect to permutations of basis vectors.

Let us introduce the following relation of equivalence on X: $x \sim z$ if there are σ and μ in $\mathcal{S}(X)$ such that $A_\sigma(x) = A_\mu(z)$. We denote by $X/\sim$ the quotient set with respect to the equivalence and by $[x]$ the class containing x. Note that $X/\sim$ can be considered as the set of potentially infinite multisets such that any element of a given multiset has a finite multiplicity. Of course, if $x \sim z$, then $\|x\| = \|z\|$ and so we can define $\|[x]\| = \|x\|$. As in [6,9], we introduce the following operation of "symmetric addition" on X. Let $x = \sum_{n=1}^{\infty} x_n e_n = (x_1, x_2, \ldots)$, and $z = \sum_{n=1}^{\infty} z_n e_n = (z_1, z_2, \ldots)$ be vectors in X. We set

$$x \bullet z = (x_1, z_1, x_2, z_2, \ldots) = \sum_{n=1}^{\infty} x_n e_{2n-1} + \sum_{n=1}^{\infty} z_n e_{2n}.$$

Since $\|x \bullet z\| \le \|x\| + \|z\|$, vector $x \bullet z$ is in X. Also, the operation "$\bullet$" can be lifted to the quotient set $X/\sim$ by $[x] + [z] = [x \bullet z]$, $x, z \in X$. Clearly, if $x' \sim x$ and $z' \sim z$, then $[x] + [z] = [x'] + [z']$ and so the addition does not depend on representatives. In [7,9], it was introduced a "symmetric multiplication" $x \diamond z$ on X as the vector in X with coordinates $x_i z_j$, $i, j \in \mathbb{N}$, ordered in some fixed order. This operation also can be lifted to $X/\sim$ by

$[x][z] = [x \diamond z]$, and the definition does not depend on representatives (see [9, 19]). Moreover, the quotient set $X/\sim$ with these operations is a commutative semiring with unity $[(1,0,0,\ldots)]$. Using a standard procedure, it is possible to embed this semiring into a commutative ring. Let us describe this in more detail. We represent any element in the Cartesian product $X \times X$ as $(y|x)$, $x, y \in X$. Let us introduce the following relation of equivalence on $X \times X$: $(y|x) \sim (y'|x')$ if there are $a, b \in X$ and operators A_σ and A_μ in $\mathcal{S}$ such that

$$(A_\mu(y \bullet a)|A_\sigma(x \bullet a)) = (y' \bullet b|x' \bullet b).$$

We denote by $\mathcal{M}(X)$ the quotient set $X \times X/\sim$ and by $[(y|x)]$ the class of equivalence containing $(y|x)$.

We will use notation $[(y_m,\ldots,y_1|x_1,\ldots,x_n)]$ instead of $[(\ldots,0,y_m,\ldots,y_1|x_1,\ldots,x_n,0,\ldots)]$. Let $\mathcal{M}^+(X) = \{[(0|x)]\colon x \in X\}$. Then

$$(X/\sim) \ni [x] \longleftrightarrow [(0|x)] \in \mathcal{M}^+(X) \subset \mathcal{M}(X)$$

is an embedding of $X/\sim$ to $\mathcal{M}(X)$. The semiring operations on $X/\sim = \mathcal{M}^+(X)$ can be extended to ring operations on $\mathcal{M}(X)$ by

$$[(y|x)] + [(y'|x')] = [(y \bullet y'|x \bullet x')] \quad \text{and} \quad [(y|x)][(y'|x')] = [((y \diamond x') \bullet (y' \diamond x)|(x \diamond x') \bullet (y \diamond y'))].$$

It is easy to check that $[(y|x)] + [(x|y)] = 0$ and so we can denote $-[(y|x)] = [(x|y)]$.

Let $\mathbf{u} \in \mathcal{M}(X)$ and $u = (y|x)$ be a representative of $\mathbf{u}$. We say that u is *irreducible* if for any other representative $u' = (y'|x')$ of $\mathbf{u}$ there is a vector $a \in X$, $a \neq 0$, and operators A_σ and A_μ such that

$$(A_\mu(y \bullet a)|A_\sigma(x \bullet a)) = (y'|x').$$

In other words, every nonzero coordinate of y for a given irreducible representative $(y|x)$ is not equal to any coordinate of x. In [9], it was observed that every $\mathbf{u} \in \mathcal{M}(X)$ admits an irreducible representative.

Let us define a norm on $\mathcal{M}(X)$ by $\|\mathbf{u}\| = \|x\| + \|y\|$, where $u = (y|x)$ is an irreducible representative of $\mathbf{u}$. Also, we define a metric

$$d(\mathbf{u},\mathbf{u}') = d\big([(y|x)],[(y'|x')]\big) = \big\|[((y|x))] - [(y'|x')]\big\| = \big\|[(y|x)] + [(x'|y')]\big\|.$$

The following theorem collects results from [9] that are important for our work.

Theorem 1. *The set $\mathcal{M}(X)$ with the introduced operation of addition and multiplication is a commutative ring with unity $\mathbf{1} = [(0|1)]$ and zero $0 = [(0|0)]$. $\mathcal{M}^+(X)$ is a subsemiring of $\mathcal{M}(X)$. The norm $\|\cdot\|$ is well-defined on $\mathcal{M}(X)$ and satisfies the following properties:*

(i) $\|\mathbf{u}\| \geq 0$ *and* $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = 0$,

(ii) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$,

(iii) $\|-\mathbf{u}\| = \|\mathbf{u}\|$,

(iv) $\|\mathbf{u}\mathbf{v}\| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ *for every* $\mathbf{u}, \mathbf{v} \in \mathcal{M}(X)$.

The function $d(\mathbf{u},\mathbf{u}')$ defined above is a metric and $\big(\mathcal{M}(X), d\big)$ is a complete metric space.

Let us denote by $n\mathbf{u}$, $n \in \mathbb{Z}$, the element

$$\underbrace{\mathbf{u} + \cdots + \mathbf{u}}_{n} \quad \text{if} \quad n \geq 0, \quad \text{and} \quad \underbrace{-\mathbf{u} - \cdots - \mathbf{u}}_{-n} \quad \text{if} \quad n < 0.$$

If we consider a multiplication by a constant $\lambda \in \mathbb{C}$ using a representative $(\lambda, [u]) \mapsto [\lambda u]$, then it is well defined but disagree with the addition. For example $2[u] = [u] + [u] \neq [2u]$. In the general case there is no good multiplication by rational numbers in $\mathcal{M}(X)$ (see [19]) that would agree with addition. Thus $\mathcal{M}(X)$ is not algebra because it is not a linear space. Moreover, the operation $(\lambda, [u]) \mapsto [\lambda u]$ is discontinuous (with respect to λ) [19]. For example, $d\big([(0|1)], [(0|1+\varepsilon)]\big) = 2 + \varepsilon$ for every $\varepsilon > 0$. Note that by this reason, $\big(\mathcal{M}(X), d\big)$ is nonseparable even if X is separable [19].

In [10], it was observed that if $X = \ell_1$, the polynomials T_k on $\ell_1 \times \ell_1$ defined as

$$T_k((y|x)) = \sum_{n=1}^{\infty} x_n^k - \sum_{n=1}^{\infty} y_n^k$$

can be lifted to $\mathcal{M}(\ell_1)$ by $T_k([(y|x)]) = T_k((y|x))$ and the value does not depend on representatives. Moreover, the mapping $[u] \mapsto T_k(u)$ is a ring homomorphism from $\mathcal{M}(X)$ to $\mathbb{C}$ for every $k \in \mathbb{N}$. Note that polynomials T_k also defined on $\ell_p \times \ell_p$, $1 \leq p < \infty$, but only for $k \geq \lceil p \rceil$, where $\lceil p \rceil$ is the minimal integer grater than or equal to p.

2 Extension of symmetric mappings

A subset $U \subset X$ is said to be *symmetric* if $x \in U$ implies $z \in U$ for every $x \sim z$. In other words, if $x \in U$, then every representative $z \in [x]$ belongs to U. A subset $U \subset X \times X$ is said to be *supersymmetric* if for every $u \in U$ each irreducible representation of $[u]$ belongs to U. From the definition it follows that polynomials T_k on $\ell_p \times \ell_p$ defined above are supersymmetric, and their restrictions to ℓ_p, namely $F_k(x) = T_k(0|x)$, are symmetric. Clearly that any algebraic combination of (super)symmetric functions is a (super)symmetric function. Another example of symmetric function on X is the norm. Note that the norm on $X \times X$ is not supersymmetric but the map $X \times X \ni u \mapsto \|[u]\|$ is supersymmetric.

We denote by $\Phi\colon u \to [u]$ the quotient map from $X \times X$ to $\mathcal{M}(X)$. In [15], it is proved that Φ is open and discontinuous.

Proposition 1. *The following assertions hold.*

(i) Let $\mathcal{U}$ be a subset in $\mathcal{M}(X)$. Then $U = \Phi^{-1}(\mathcal{U})$ is supersymmetric.

(ii) If U and V are supersymmetric and $U \cap V = \varnothing$, then $[U] \cap [V] = \varnothing$, where $[U] = \Phi(U)$.

(iii) If f is a supersymmetric function defined on a supersymmetric domain U with values in a topological space Y, then $\widehat{f}([u]) := f(u)$ is a well-defined function on $[U]$. If f is continuous, then $\widehat{f}$ is continuous as well.

Proof. (i) Let $u \in \Phi^{-1}(\mathbf{u}) \in \mathcal{U}$ and $(y|x)$ be an irreducible representation of $\mathbf{u} = [u]$. Then $\Phi((y|x)) = \mathbf{u}$ and so $(y|x) \in U$.

(ii) Let $\mathbf{u} \in [U] \cap [V]$. Then an irreducible representation u of $\mathbf{u}$ belongs to U and an irreducible representation v of $\mathbf{u}$ belongs to V. Thus $u \sim v$ and so both u and v belong to $U \cap V$.

(iii) Let V be an open set in Y. Since f is continuous, $f^{-1}(V)$ is open and so is $[f^{-1}(V)]$, because the quotient map is open. □

Note that Proposition 1 is evidently true if we replace everywhere word "supersymmetric" by "symmetric" and $\mathcal{M}(X)$ by $\mathcal{M}(X)^+$.

We write $X_0 \Subset X$ if there is an infinite subsequence (n_k) of $\mathbb{N}$ such that X_0 is the closed linear subspace of X, spanned on e_{n_k}.

Theorem 2. *Let $X_0 \Subset X$ and U be a supersymmetric subset of $X_0 \times X_0$. Then there is a linear isomorphism $\pi\colon X \times X \to X_0 \times X_0$ such that*

(i) $\widetilde{\pi}[u] := [\pi(u)]$, $[u] \in \mathcal{M}(X)$ does not depend on the representative, and $\widetilde{\pi}\colon \mathcal{M}(X) \to \mathcal{M}(X_0)$ is a topological ring isomorphism,

(ii) there is a supersymmetric subset $\widetilde{U}$ of $X \times X$, $\widetilde{U} \supset U$, such that every supersymmetric map f from U to a topological space Y can be extended to a supersymmetric map $\widetilde{f}\colon \widetilde{U} \to Y$,

(iii) if f in (ii) is continuous, then $\widetilde{f}$ is continuous. If Y is a normed space, then

$$\sup_{u\in U} \|f(u)\| = \sup_{w\in \widetilde{U}} \|\widetilde{f}(w)\|,$$

(iv) the extension is unique.

Proof. (i) Let (n_k) be a subsequence of $\mathbb{N}$ such that X_0 is the closed linear subspace of X, spanned on e_{n_k}. For a given $u = (y|x)$ we define $\pi(u) = (\pi(y)|\pi(x))$, where

$$\pi\colon x = \sum_{k=1}^{\infty} x_k e_k \longmapsto \pi(x) = \sum_{k=1}^{\infty} x_k e_{n_k},$$

and $\pi(y)$ is defined by the same way. Since basis (e_{n_k}) of X_0 is equivalent to the basis (e_n) of X, the restriction of π to X is an isomorphism from X to X_0. Thus, π is an isomorphism. If $u \sim u'$ in X_0, $u = (y|x)$, $u' = (y'|x')$, then there are vectors a, b in X_0 and permutations σ and μ such that

$$\big(A_\sigma(y \bullet a)|A_\mu(x \bullet a)\big) = (y' \bullet b|x' \bullet b).$$

Let $\sigma'\colon (n_k) \to (n_{\sigma(k)})$ be the permutation on the subsequence (n_k). Then $A_\sigma(x) = A_{\sigma'}(\pi(x))$ for every $x \in X$. Also, it is easy to see that $\pi(x \bullet a) = \pi(x) \bullet \pi(a)$. Thus,

$$\big(A_{\sigma'}(\pi(y) \bullet \pi(a))|A_{\mu'}(\pi(x) \bullet \pi(a))\big) = \big(\pi(y') \bullet b|\pi(x') \bullet b\big),$$

that is, $\pi(y|x) \sim \pi(y'|x')$. Hence, $\widetilde{\pi}$ does not depend on the representative.

The operator $\widetilde{\pi}$ acts as the identical map in the sense that

$$\widetilde{\pi}\colon \mathcal{M}(X) \ni (\ldots, y_2, y_1|x_1, x_2, \ldots) \longmapsto (\ldots, y_2, y_1|x_1, x_2, \ldots) \in \mathcal{M}(X_0)$$

and it is, obviously, surjective.

Thus, if U is supersymmetric, then $\widetilde{U} = \pi^{-1}(U)$ is supersymmetric because if $(y|x)$ is an irreducible representation of $\mathbf{u} \in \mathcal{M}_{X_0}$, then $(y|x)$ is an irreducible representation of $\pi(\mathbf{u}) \in \mathcal{M}(X)$. If f is continuous on U, then $\widetilde{f} = f \circ \pi$ well-defined on $\widetilde{U}$ and continuous as the composition of continuous mappings. If Y is a normed space, then

$$\sup_{u\in U} \|f(u)\| = \sup_{[u]\in[U]} \|\widehat{f}([u])\| = \sup_{w\in \widetilde{U}} \|\widetilde{f}(w)\|.$$

Therefore *(ii)* and *(iii)* are proved.

(iv) The uniqueness of the extension easily follows from the fact that if $f \equiv 0$ on U, then its symmetric extension is only zero-function. □

3 Complex homomorphisms and analyticity

As we have observed, supersymmetric polynomials T_k generate complex-valued homomorphisms on $\mathcal{M}(\ell_1)$ by $T_k([u]) = T_k(u)$. Let us consider how a complex ring homomorphism looks in the general case of $\mathcal{M}(X)$. To be a ring homomorphism a mapping must be additive and multiplicative.

Theorem 3. *A continuous function $h\colon \mathcal{M}(X) \to \mathbb{C}$ is additive if and only if it is of the form*

$$h\big([(y|x)]\big) = \sum_{n=1}^{\infty} \xi(x_n) - \sum_{n=1}^{\infty} \xi(y_n) \tag{1}$$

for some function $\xi\colon \mathbb{C} \to \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} \xi(x_n) < \infty \tag{2}$$

for every $x \in X$, and ξ is continuous at zero. The additive function h is multiplicative if and only if ξ is multiplicative.

Proof. Let h be an additive and continuous function. Since $[(0|t)] + [(t|0)] = 0$, we have that $h\big([(t|0)]\big) = -h\big([(0|t)]\big)$. We set $\xi(t) = h\big([(0|t)]\big)$, $t \in \mathbb{C}$. For any element $[(y|x)] \in \mathcal{M}_0(X)$ of the form $(y|x) = (y_m, \ldots, y_1|x_1, \ldots x_k)$ by additivity of h we have

$$h\big([(y|x)]\big) = h\Big(\sum_{n=1}^{k} [(0|x_n)] + \sum_{n=1}^{m} [(y_n|0)]\Big) = \sum_{n=1}^{k} \xi(x_n) - \sum_{n=1}^{\infty} \xi(y_n).$$

The general case $[(y|x)] \in \mathcal{M}(X)$ follows from the continuity of h, the density of $\mathcal{M}_0(X)$ in $\mathcal{M}(X)$ and (2).

Conversely, if h satisfies (1), then, obviously, $h\big([(y|x)]\big)$ does not depend on the representative and

$$\begin{aligned} h\big([(y|x)]\big) + h\big([(y'|x')]\big) &= \sum_{n=1}^{\infty} \xi(x_n) - \sum_{n=1}^{\infty} \xi(y_n) + \sum_{n=1}^{\infty} \xi(x'_n) - \sum_{n=1}^{\infty} \xi(y'_n) \\ &= h\big([(y|x)] + [(y'|x')]\big). \end{aligned}$$

Thus, h is additive. It is enough to check the continuity of h at zero. Let $[(y^{(m)}|x^{(m)})]$ be a sequence in $\mathcal{M}(X)$ approaching zero. Without loss of generality, we may assume that $(y^{(m)}|x^{(m)})$ are irreducible representatives of $[(y^{(m)}|x^{(m)})]$, $m \in \mathbb{N}$. Then

$$\big\|[(y^{(m)}|x^{(m)})]\big\| = \|y^{(m)}\| + \|x^{(m)}\| \to 0 \qquad \text{as} \quad m \to \infty.$$

In particular, $x^{(m)} \to 0$ and $y^{(m)} \to 0$ as $m \to \infty$. From (2) it follows that vectors $\big(\xi(x_1^{(m)}), \xi(x_2^{(m)}), \ldots\big)$ and $\big(\xi(y_1^{(m)}), \xi(y_2^{(m)}), \ldots\big)$ are in ℓ_1 for every m. These sequences are bounded and coordinate-wise convergent to zero. So, they are weakly zeros, in particular,

$$\sum_{n=1}^{k} \xi(x_n^{(m)}) - \sum_{n=1}^{\infty} \xi(y_n^{(m)}) \to 0 \qquad \text{as} \quad m \to \infty.$$

That is, h is continuous.

Let now h be additive. We already know that it satisfies (1). If h is multiplicative, then

$$\xi(t_1)\xi(t_2) = h\big([(0|t_1)]\big)h\big([(0|t_2)]\big) = h\big([(0|t_1t_2)]\big) = \xi(t_1t_2).$$

Conversely, if ξ is multiplicative, then

$$\begin{aligned} h\big([(y|x)]\big)h\big([(y'|x')]\big) &= \Big(\sum_{n=1}^{\infty}\xi(x_n) - \sum_{n=1}^{\infty}\xi(y_n)\Big)\Big(\sum_{n=1}^{\infty}\xi(x'_n) - \sum_{n=1}^{\infty}\xi(y'_n)\Big) \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\xi(x_n)\xi(x'_j) + \xi(y_n)\xi(y'_j)) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(\xi(x_n)\xi(y'_j) + \xi(y_n)\xi(x'_j)) \\ &= h\big([(y|x)][(y'|x')]\big). \end{aligned}$$

□

Corollary 1. *There is no nonzero continuous complex homomorphism of the ring $\mathcal{M}(c_0)$.*

Proof. Let h be a continuous complex homomorphism of $\mathcal{M}(c_0)$ and $h \neq 0$. Then h is of the form (1) for some multiplicative function ξ satisfying (2). From the multiplicativity of ξ it follows that $\xi(1) = 1$ and $\xi(t) \neq 0$ if $t \neq 0$. Let $x \in c_0$ such that all $x_n \neq 0$, $n \in \mathbb{N}$. Then

$$h(x) = \sum_{n=1}^{\infty}\xi(x_n) < \infty,$$

and $\xi(x_n) \neq 0$ for every $n \in \mathbb{N}$. Let m_n be a positive integer such that $m_n|\xi(x_n)| > 1$. For the following vector

$$z = \big(\underbrace{x_1,\ldots,x_1}_{m_1},\underbrace{x_2,\ldots,x_2}_{m_2},\ldots,\underbrace{x_n,\ldots,x_n}_{m_n},\ldots\big)$$

we have that $z \in c_0$, but $h(z)$ is undefined, because the series

$$\sum_{n=1}^{\infty} m_n\xi(x_n)$$

diverges. □

Example 1. *If $X = \ell_p$, $1 \leq p < \infty$, then supersymmetric polynomials T_k generate continuous complex homomorphisms on $\mathcal{M}(\ell_p)$ for $k \geq \lceil p \rceil$. For this case, the function ξ is of the form $\xi(t) = t^k$. There are other continuous complex homomorphisms on $\mathcal{M}(\ell_p)$. For example, we take h of the form (1) for $\xi(t) = |t|^q$ for any $q \geq p$.*

Proposition 2. *Let K be a subset of $\mathbb{C}$ such that $t_1t_2 \in K$ for all t_1, $t_2 \in K$. We denote by*

$$\mathcal{M}_K(X) = \big\{[(y|x)] \in \mathcal{M}(X)\colon x_n, y_n \in K\big\}$$

a subring of $\mathcal{M}(X)$. If $1 \in K$, then $\mathcal{M}_K(X)$ is a ring with unity.

Proof. Clearly, $\mathcal{M}_K(X)$ is a subset of $\mathcal{M}(X)$. Also, $\mathcal{M}_K(X)$ is closed with respect to the ring operations. So, it is a subring. If $1 \in K$, then $\mathbf{1} = [(0|1)] \in \mathcal{M}_K(X)$. □

Example 2. *Let*

$$\overline{\mathbb{D}} = \{t \in \mathbb{C} \colon |t| \leq 1\}$$

be the closed unit disc in $\mathbb{C}$ and $\mathbb{S}$ be the unit circle in $\mathbb{C}$, centered at zero. The ring $\mathcal{M}_{\overline{\mathbb{D}}}(c_0)$ has the following nontrivial continuous complex homomorphism

$$h_0\big([(y|x)]\big) = \sum_{n=1}^{\infty} \chi_{\mathbb{S}}(x_n) - \sum_{n=1}^{\infty} \chi_{\mathbb{S}}(y_n),$$

where $\chi_{\mathbb{S}}$ is the characteristic function of $\mathbb{S}$. In other words, $h_0\big([(y|x)]\big)$ is the number of coordinates x_i of x such that $|x_i| = 1$ minus the number of coordinates y_j of y such that $|y_j| = 1$. Homomorphism h_0 is well defined and additive. Moreover, $\chi_{\mathbb{S}}$ is multiplicative on $\overline{\mathbb{D}}$ and so, like in the proof of Theorem 3, we can show that h_0 is multiplicative. The continuity of h_0 follows from the continuity of $\chi_{\mathbb{S}}$ at zero.

It is well-known that any complex homomorphism of a Banach algebra is necessary continuous (see, e.g., [12, p. 21]). Unfortunately we do not know if it is true for the ring $\mathcal{M}(X)$.

As we already observed, the ring $\mathcal{M}(X)$ is not a linear space, and so we can not consider analytic mapping on $\mathcal{M}(X)$ in the classical sense. On the other hand, if f is a supersymmetric analytic function on $X \times X$, then $\widehat{f}([u]) = f(u)$ can be considered as an "analytic" function on $\mathcal{M}(X)$. Moreover, if $\mathcal{F} \colon \mathcal{M}(X) \to \mathcal{M}(X)$ is a mapping such that for every supersymmetric analytic function f on $X \times X$ the function

$$X \times X \ni u \mapsto \widehat{f} \circ \mathcal{F}([u])$$

is analytic, we can say that intuitively $\mathcal{F}$ looks like analytic. The problem is that many spaces X have poor amount of supersymmetric analytic functions. For example, there is no symmetric analytic functions (excepting constants) on c_0 and so there is no nonconstant supersymmetric analytic functions on $c_0 \times c_0$. The opposite situation is in the case $X = \ell_1$. It is known [10] that $[u] = [v]$ in $\mathcal{M}(\ell_1)$ if and only if $T_k(u) = T_k(v)$ for every $k \in \mathbb{N}$. In other words, supersymmetric polynomials on $\ell_1 \times \ell_1$ separate points of $\mathcal{M}(\ell_1)$. On the other hand, ℓ_1 is densely embedded in any Banach space X with a symmetric basis, since

$$\|x\|_X \leq \sum_{n=1}^{\infty} |x_n| = \|x\|_{\ell_1}$$

for every $x \in \ell_1$. Thus

$$\mathcal{M}_{\ell_1}(X) = \big\{[(x|y)] \in \mathcal{M}(X) \colon x, y \in \ell_1\big\}$$

is a dense subring of $\mathcal{M}(X)$ which pointwise coincides with $\mathcal{M}(\ell_1)$. Thus, we can introduce the concept of analyticity for mappings on $\mathcal{M}(X)$ using the dense subring $\mathcal{M}_{\ell_1}(X)$. Let us denote by τ_k the densely defined function on $\mathcal{M}(X)$ such that $\tau_k([u]) = T_k(u)$ for $[u] \in \mathcal{M}_{\ell_1}(X)$. Clearly that τ_k is a (discontinuous in the general case) complex homomorphism with domain $\mathcal{M}_{\ell_1}(X)$ for every $k \in \mathbb{N}$.

Definition 1. *Let $\mathcal{U}$ be an open set of $\mathcal{M}(X)$. A mapping $\mathcal{F} \colon \mathcal{U} \to \mathcal{M}(X)$ is analytic at $[u_0] \in \mathcal{U}$ if it is continuous and for every $k \in \mathbb{N}$ the function*

$$\ell_1 \times \ell_1 \ni u \longmapsto [u] \in \mathcal{U} \longmapsto \tau_k \circ \mathcal{F}([u])$$

is analytic on an open set $U_{u_0} \subset \Phi^{-1}(\mathcal{U})$.

Example 3. *It is easy to check that the following mappings are analytic on $\mathcal{M}(X)$.*

1. *For every $k \in \mathbb{N}$ the mapping $[u] \mapsto [u]^k$ is analytic on $\mathcal{M}(X)$.*
2. *For a given $\mathbf{u} = [(y|x)]$, we set*

$$\mathbf{u}^{[k]} = [(\ldots, y_2^k, y_1^k | x_1^k, x_2^k, \ldots)].$$

Then $\mathbf{u}^{[k]}$ is well-defined on $\mathcal{M}(X)$ and analytic.

4 Function calculus and rational symmetric functions

Let

$$\gamma(t) = \sum_{n=0}^{\infty} m_n t^n$$

be an analytic function with integer coefficients m_n on an open disc centered at origin

$$\mathbb{D}_r = \{t \in \mathbb{C} \colon |t| < r\},$$

where $0 < r \le \infty$. We define

$$\gamma_\diamond(\mathbf{u}) = \sum_{n=0}^{\infty} m_n \mathbf{u}^n = \sum_{n=0}^{\infty} [(0|\underbrace{1, 1, \ldots, 1}_{m_n})]\mathbf{u}^n.$$

Clearly, if $r = \infty$, then γ is a polynomial and so $\gamma_\diamond$ is well-defined on $\mathcal{M}(X)$. In the general case, let $\check{\gamma}_\diamond(u) := \gamma_\diamond(\mathbf{u})$, where $\mathbf{u} = [u]$. Then

$$\|\gamma_\diamond(\mathbf{u})\| = \|\check{\gamma}_\diamond(u)\| = \Big\| \sum_{n=0}^{\infty} m_n u^{\diamond n} \Big\| \le \sum_{n=0}^{\infty} m_n \|u\|^n < \infty, \tag{3}$$

if $\|u\| < r$. Since $\|\mathbf{u}\| = \|u\|$ for an irreducible representation u, mapping $\gamma_\diamond$ is well-defined on the set

$$\mathbf{B}_r := \{\mathbf{u} \in \mathcal{M}(X) \colon \|\mathbf{u}\| < r\}.$$

Note, that $\gamma_\diamond$ takes values in $\mathcal{M}(X)$, and the operator $\gamma \mapsto \gamma_\diamond$ is additive and multiplicative.

Let us denote by $\mathcal{O}_0^{\mathbb{Z}}$ the algebra of germs of analytic functions with integer coefficients at zero. That is, every element of $\mathcal{O}_0^{\mathbb{Z}}$ is of the form $(\gamma, \mathbb{D}_r)$, where γ is an analytic function

$$\gamma(t) = \sum_{n=0}^{\infty} t^n m_n, \quad t \in \mathbb{D}_r,$$

on $\mathbb{D}_r$ with integer coefficients and $\mathbb{D}_r$ is the open disk of radius $r > 0$ centered at zero.

Theorem 4. *There is a homomorphism $\gamma \mapsto \gamma_\diamond$ from $\mathcal{O}_0^{\mathbb{Z}}$ to an algebra of mappings with domains in $\mathcal{M}(X)$ to $\mathcal{M}(X)$ such that for every $(\gamma, \mathbb{D}_r) \in \mathcal{O}_0^{\mathbb{Z}}$, mapping $\gamma_\diamond$ is defined by* (3) *for $\mathbf{u} \in \mathbf{B}_r$ and is analytic at* $[0]$.

Proof. If $\|\mathbf{u}\| < r$, then

$$\|\gamma_\diamond(\mathbf{u})\| \le \sum_{n=1}^{\infty} \|\mathbf{u}\|^n |m_n| < \infty,$$

because

$$\varrho_0(\gamma) = \Big(\limsup_{n\to+\infty} |m_n|^{1/n}\Big)^{-1} \leq r.$$

Thus $\gamma_\diamond(\mathbf{u})$ is well-defined on $\mathbf{B}_r$. The continuity of $\gamma_\diamond(\mathbf{u})$ follows from the continuity of the ring operations and the convergence of the series. For every complex homomorphism τ_k and $\mathbf{u} \in \mathcal{M}_{\ell_1}(X) \cap \mathbf{B}_r$, we have

$$\tau_k(\gamma_\diamond(\mathbf{u})) = \sum_{n=1}^{\infty} m_n(\tau_k(\mathbf{u}))^n = \sum_{n=1}^{\infty} m_n(T_k(u))^n.$$

Vector u is in $\Phi^{-1}(\mathbf{B}_r) \cap \ell_1 \times \ell_1$. The ball $B_r^{\ell_1} \subset \ell_1 \times \ell_1$ of radius r centered at 0 belongs to $\Phi^{-1}(\mathbf{B}_r) \cap \ell_1 \times \ell_1$ and the series above converges on this ball. Thus, $\gamma_\diamond$ is an analytic map at 0. □

The following theorem can be proved similarly as in the case of Banach algebras.

Theorem 5. *Let* $\mathbf{u} \in \mathcal{M}(X)$ *and* $\|\mathbf{u}\| < 1$. *Then* $\mathbf{1} - \mathbf{u}$ *is invertible in* $\mathcal{M}(X)$ *and*

$$(\mathbf{1} - \mathbf{u})^{-1} = \sum_{n=0}^{\infty} \mathbf{u}^n,$$

where $\mathbf{u}^0 = \mathbf{1}$.

Proof. Since

$$\Big\| \sum_{n=0}^{\infty} \mathbf{u}^n \Big\| \leq \sum_{n=0}^{\infty} \|u\|^n \leq \frac{1}{1 - \|u\|},$$

the element

$$\sum_{n=0}^{\infty} \mathbf{u}^n$$

is in $\mathcal{M}(X)$, because $\mathcal{M}(X)$ is complete. The direct computation shows that it is inverse to $\mathbf{1} - \mathbf{u}$. □

Example 4. *Let*

$$\gamma(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1.$$

Then

$$\gamma_\diamond(\mathbf{u}) = \sum_{n=0}^{\infty} \mathbf{u}^n = (\mathbf{1} - \mathbf{u})^{-1},$$

and so we can write

$$\gamma_\diamond(\mathbf{u}) = \frac{1}{\mathbf{1} - \mathbf{u}},$$

if $\|\mathbf{u}\| < 1$.

We can iterate the process of construction of fractions and get a formally continued fraction of the form

$$\cfrac{m_1\mathbf{u}}{\mathbf{1} + \cfrac{m_2\mathbf{u}}{\mathbf{1} + \cfrac{m_3\mathbf{u}}{\mathbf{1} + \ddots}}} \tag{4}$$

for some integers m_n and ask under which conditions the fraction converges to an element in $\mathcal{M}(X)$. If $X = \ell_1$, then we can apply homomorphisms τ_k associated with polynomials T_k for this formula and get functional continued fraction

$$\cfrac{m_1\tau_k(\mathbf{u})}{\mathbf{1}+\cfrac{m_2\tau_k(\mathbf{u})}{\mathbf{1}+\cfrac{m_3\tau_k(\mathbf{u})}{\mathbf{1}+\ddots}}} = \cfrac{m_1T_k(u)}{\mathbf{1}+\cfrac{m_2T_k(u)}{\mathbf{1}+\cfrac{m_3T_k(u)}{\mathbf{1}+\ddots}}}, \tag{5}$$

where $\mathbf{u} = [u]$. So, continued fraction (4) converges if (5) converges for every k. It is known that continued fraction (5) converges to an analytic function of $z = T_k(u)$ in the domain $|\arg z| < \pi$ if and only if there is a sequence of positive numbers d_n such that $d_0 = 1$, $\sum_{n=1}^{\infty} d_n$ diverges and $m_n = 1/(d_{n-1}d_n)$ (see, e.g., [3]). To make sure that $|\arg T_k(u)| < \pi$ we may assume that $\mathbf{u} \in \mathcal{M}_{\mathbb{R}_+}(\ell_1)$, that is, all coordinates x_k, y_k are positive numbers. So we have the next theorem.

Theorem 6. *The continued fraction* (4) *converges if* $\mathbf{u} \in \mathcal{M}_{\mathbb{R}_+}(\ell_1)$ *and the sequence of positive integers* m_n *satisfies the condition above. In particular, it is so if* $m_n = 1$ *for every* n.

If m_k, $k \geq 2$, are positive integers such that $m_k r \leq 1$, $k \geq 2$, where r is a positive constant, then the continued fraction (4) converges to a function analytic of $z = T_k(u)$ in the domain $|\arg(z + r/4)| < \pi$ (see [3]).

Note that some special functions can be represented by brunched continued fractions. For example, the hypergeometric function $H_4(a, b; c, d; z_1, z_2)$ of two complex variables z_1 and z_2 can be represented as a brunched continued fraction with certain conditions on integer coefficients a, b, c, and d (see [14]). For this case it can be defined for $\mathbf{u} \in \mathcal{M}_{\mathbb{R}_+}(\ell_1)$.

One more example, the hypergeometric function $F_K(a_1, a_2, b_1, b_2; a_1, b_2, c_3; z_1, z_2, z_3)$ of three complex variables z_1, z_2, and z_3 can be represented as a brunched continued fraction with certain conditions on parameters of this function (see [1]).

Let f be a symmetric function on ℓ_p (respectively, supersymmetric function on $\ell_p \times \ell_p$). We say that f factorizes through $\mathcal{M}^+(\ell_p)$ (respectively, $\mathcal{M}(\ell_p)$) if there is a mapping $\Delta_f\colon \mathcal{M}^+(\ell_p) \to \mathcal{M}^+(\ell_p)$ (respectively, $\Delta_f\colon \mathcal{M}(\ell_p) \to \mathcal{M}(\ell_p)$) such that $\widehat{f}(\mathbf{u}^{[k]}) = \widehat{F}_k(\Delta_f(\mathbf{u}))$, (respectively, $\widehat{f}(\mathbf{u}^{[k]}) = \widehat{T}_k(\Delta_f(\mathbf{u}))$) for every positive integer $k \geq \lceil p \rceil$.

Proposition 3. *Let*

$$Q = \sum_{k_1+k_2+\cdots+k_n=n} a_{k_1k_2\ldots k_n} P_{k_1} \cdots P_{k_n}$$

be an n*-homogeneous symmetric (respectively, supersymmetric) polynomial on* ℓ_p*. Then* Q *factorizes through* $\mathcal{M}^+(\ell_p)$ *(respectively,* $\mathcal{M}(\ell_p)$*).*

Proof. It is enough to set

$$\Delta_Q(\mathbf{u}) = \sum_{n=0}^{\infty} \sum_{k_1+k_2+\cdots+k_n=n} a_{k_1k_2\ldots k_n} \mathbf{u}^{[k_1]} \cdots \mathbf{u}^{[k_n]}.$$

Then, for the supersymmetric case ($P_j = T_j$), we get

$$\begin{aligned}\widehat{Q}(\mathbf{u}^{[k]}) &= \sum_{k_1+k_2+\cdots+k_n=n} a_{k_1k_2\ldots k_n} \widehat{P}_{k_1}(\mathbf{u}^{[k]}) \cdots \widehat{P}_{k_n}(\mathbf{u}^{[k]}) \\ &= \sum_{k_1+k_2+\cdots+k_n=n} a_{k_1k_2\ldots k_n} \widehat{T}_k\big(\mathbf{u}^{[k_1]} \cdots \mathbf{u}^{[k_n]}\big) = \widehat{T}_k(\Delta_Q(\mathbf{u})).\end{aligned}$$

The same works for the symmetric case. ∎

Example 5. *It is known [19] that if $[u] \neq 0$, $[u] \in \mathcal{M}(\ell_1)$, then there exists $[v] \in \mathcal{M}(\ell_1)$ such that $T_k(u) = \lambda T_k(v)$ for all $k \in \mathbb{N}$ if and only if $\lambda \in \mathbb{Z}$. By this reason the supersymmetric analytic function $f(u) = \frac{1}{2}T_1(u)$ on $\ell_1 \times \ell_1$ does not factorize through $\mathcal{M}^+(\ell_1)$. Indeed, if $v = \Delta_f(u)$, then*

$$T_k(v) = \frac{1}{2}\widehat{T}_1(\mathbf{u}^{[k]}) = \frac{1}{2}T_k(u), \quad k \in \mathbb{N},$$

but it is impossible as mentioned above.

The question when a function factorizes through $\mathcal{M}(\ell_p)$ can be interesting from the point of view of applications of supersymmetric polynomials in quantum physics (see [8]). The proof of following proposition is evident.

Proposition 4. *For every polynomial q on $\mathbb{C}$ and $k \geq \lceil p \rceil$ the function $T_k \circ q_\diamond$ factorizes through $\mathcal{M}(\ell_p)$.*

The class of supersymmetric functions admitting the factorization through $\mathcal{M}(\ell_p)$ will be considered in further investigations.

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Досліджено симетричні та суперсиметричні відображення на банахових просторах із симетричними базами та кільця мультимножин, пов'язані з цими просторами. Встановлено деякі результати про розширення (супер)симетричних відображень та описано неперервні комплексні гомоморфізми кілець мультимножин. Також розглянуто аналітичні відображення та функційні числення на цих кільцях.

Ключові слова і фрази: симетрична функція на банаховому просторі, суперсиметрична функція, кільце мультимножин, функційне числення.