ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (2), 616–630
doi:10.15330/cmp.17.2.616-630

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2025, Т.17, №2, С.616–630

Chebyshev polynomials involved in the Householder's method for square roots

Dijoux Y.

The Householder's method is a root-find algorithm which is a natural extension of both the Newton's method and the Halley's method. The current paper focuses on approximating the square root of a positive real number based on these methods. The resulting algorithms can be expressed using Chebyshev polynomials. An extension to the nth root is also proposed.

Key words and phrases: Chebyshev polynomial, Householder's method, Newton's method.

University of Technology of Troyes, 12 Rue Marie Curie, 10300, Troyes, France
E-mail: yann.dijoux@utt.fr

1 The Householder's method for square roots

The Householder's method [20] is an extension of both the Newton's method [28] and the Halley's method [18]. Considering a positive integer d and the function $g(t) = 1/(t^2 - x)$, where x is a positive real number, the Householder's method of order d for square roots is provided by

$$\mathcal{H}_0 = r, \quad \mathcal{H}_{n+1} = \mathcal{H}_n + d\frac{g^{(d-1)}(\mathcal{H}_n)}{g^{(d)}(\mathcal{H}_n)} \quad \forall n \geq 0, \tag{1}$$

with initial guess r, positive constant satisfying the inequality $r^2 \neq x$. This iterative algorithm is such that the sequence $\{\mathcal{H}_n\}_{n\geq 0}$ converges to $\sqrt{x}$ with a rate of convergence of $d+1$.

An explicit expression of the sequence is presented in Theorem 1 below. After deriving a few elementary expressions in Lemma 1, a proof of Theorem 1 is presented. The sequence $\mathcal{H}_n$ naturally depends on the order d, but d is omitted in the notation for the sake of simplicity. The Newton's method can be obtained with $d = 1$ while the Halley's method can be obtained with $d = 2$.

Theorem 1. *Considering the Householder's method of order d for $\sqrt{x}$ with a starting point r, the corresponding sequence $\{\mathcal{H}_n\}_{n\geq 0}$ can be expressed as follows:*

$$\mathcal{H}_0 = r, \quad \mathcal{H}_{n+1} = \left[\sum_{k=0}^{\lceil d/2 \rceil} \binom{d+1}{2k} \mathcal{H}_n^{d+1-2k} x^k\right] \Big/ \left[\sum_{k=0}^{\lfloor d/2 \rfloor} \binom{d+1}{1+2k} \mathcal{H}_n^{d-2k} x^k\right] \quad \forall n \geq 0. \tag{2}$$

Lemma 1. *Considering the function $G(t) = 1/(t^2 - 1)$, its pth derivative is as follows:*

$$G^{(p)}(t) = \frac{(-1)^p p!}{2(t^2-1)^{p+1}}\left((t+1)^{p+1} - (t-1)^{p+1}\right). \tag{3}$$

УДК 519.65
2020 *Mathematics Subject Classification:* 65H05, 11B37.

In addition, the following equalities are verified for any real t and any integer p:

$$\begin{aligned}\frac{1}{2}\left((t+1)^{p+1}-(t-1)^{p+1}\right) &= \sum_{k=0}^{\lfloor p/2\rfloor}\binom{p+1}{1+2k}t^{p-2k},\\ \frac{1}{2}\left((t+1)^{p+1}+(t-1)^{p+1}\right) &= \sum_{k=0}^{\lceil p/2\rceil}\binom{p+1}{2k}t^{p+1-2k}.\end{aligned} \tag{4}$$

Proof. First, the expression of $G^{(p)}$ in Lemma 1 is derived. As $G(t) = (t+1)^{-1}(t-1)^{-1}$, applying the Leibniz rule to G allows us to obtain

$$G^{(p)}(t) = \frac{(-1)^p p!}{(t^2-1)^{p+1}}\sum_{k=0}^{p}(t-1)^{p-k}(t+1)^k.$$

The previous expression is a geometric series with common ratio $(t+1)/(t-1)$ an its closed-formed formula is (3). Next, both equalities in (4) can be obtained using the binomial expansion and elementary parity arguments. Finally, the expression (2) from Theorem 1 is derived. By noticing that $g(t) = x^{-1}G(tx^{-1/2})$, expression (1) can be written as follows

$$\mathcal{H}_{n+1} = \mathcal{H}_n + d\sqrt{x}\frac{G^{(d-1)}(\mathcal{H}_n x^{-1/2})}{G^{(d)}(\mathcal{H}_n x^{-1/2})}.$$

Using successively (3) and (4) from Lemma 1, the previous equality becomes:

$$\begin{aligned}\mathcal{H}_{n+1} &= t\sqrt{x} - \sqrt{x}(t^2-1)\frac{(t+1)^d-(t-1)^d}{(t+1)^{d+1}-(t-1)^{d+1}}\bigg|_{t=\mathcal{H}_n x^{-1/2}}\\ &= \sqrt{x}\frac{(t+1)^{d+1}+(t-1)^{d+1}}{(t+1)^{d+1}-(t-1)^{d+1}}\bigg|_{t=\mathcal{H}_n x^{-1/2}}\\ &= \sqrt{x}\left[\left(\sum_{k=0}^{\lceil d/2\rceil}\binom{d+1}{2k}t^{d+1-2k}\right)\bigg/\left(\sum_{k=0}^{\lfloor d/2\rfloor}\binom{d+1}{1+2k}t^{d-2k}\right)\right]\bigg|_{t=\mathcal{H}_n x^{-1/2}}\\ &= \left(\sum_{k=0}^{\lceil d/2\rceil}\binom{d+1}{2k}\mathcal{H}_n^{d+1-2k}x^k\right)\bigg/\left(\sum_{k=0}^{\lfloor d/2\rfloor}\binom{d+1}{1+2k}\mathcal{H}_n^{d-2k}x^k\right).\end{aligned} \tag{5}$$

□

Before providing an explicit expression of $\mathcal{H}_n$, the Chebyshev polynomials of the first kind $\{T_n\}_{n\geq 0}$, second kind $\{U_n\}_{n\geq 0}$, third kind $\{V_n\}_{n\geq 0}$ and fourth kind $\{W_n\}_{n\geq 0}$ are introduced in (6), (7), (8) and (9), respectively, [1,26], namely

$$T_0(X) = 1, \quad T_1(X) = X, \quad T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X) \quad \forall n \geq 0, \tag{6}$$

$$U_0(X) = 1, \quad U_1(X) = 2X, \quad U_{n+2}(X) = 2XU_{n+1}(X) - U_n(X) \quad \forall n \geq 0, \tag{7}$$

$$V_0(X) = 1, \quad V_1(X) = 2X-1, \quad V_{n+2}(X) = 2XV_{n+1}(X) - V_n(X) \quad \forall n \geq 0, \tag{8}$$

$$W_0(X) = 1, \quad W_1(X) = 2X+1, \quad W_{n+2}(X) = 2XW_{n+1}(X) - W_n(X) \quad \forall n \geq 0. \tag{9}$$

The expression of $\mathcal{H}_n$ is presented in Theorem 2 either from Chebyshev polynomials or from monomials while distinguishing whether the order is odd or even. A proof of Theorem 2 follows.

Theorem 2. *Considering the Householder's method of order* d *for* $\sqrt{x}$ *with a starting point* r, *the corresponding sequence* $\{\mathcal{H}_n\}_{n\geq 0}$ *can be expressed in terms of Chebyshev polynomials as follows:*

1) if d *is even and* n *greater or equal than 0, we obtain*

$$\mathcal{H}_n = r\big(W_{((d+1)^n-1)/2}(X)\big)/\big(V_{((d+1)^n-1)/2}(X)\big)|_{X=(x+r^2)/(x-r^2)}; \tag{10}$$

2) if d *is odd and* n *greater or equal than 1, we obtain*

$$\mathcal{H}_n = rT_{(d+1)^n/2}(X)/\big((X-1)U_{(d+1)^n/2-1}(X)\big)|_{X=(x+r^2)/(x-r^2)}. \tag{11}$$

Furthermore, the sequence $\{\mathcal{H}_n\}_{n\geq 0}$ *can be expressed in function of monomials of* $X = (x+r^2)/(x-r^2)$ *as follows:*

1) if d *is even, then*

$$\mathcal{H}_n = r\prod_{k=1}^{((d+1)^n-1)/2}\Big(X-\cos\Big(\frac{2k\pi}{(d+1)^n}\Big)\Big)\Big/\Big(X-\cos\Big(\frac{(2k-1)\pi}{(d+1)^n}\Big)\Big)\Big|_{X=\frac{x+r^2}{x-r^2}}; \tag{12}$$

2) if d *is odd, then*

$$\mathcal{H}_n = r\prod_{k=0}^{(d+1)^n/2-1}\Big(X-\cos\Big(\frac{(2k+1)\pi}{(d+1)^n}\Big)\Big)\Big/\Big(X-\cos\Big(\frac{2k\pi}{(d+1)^n}\Big)\Big)\Big|_{X=\frac{x+r^2}{x-r^2}}. \tag{13}$$

Proof. The case for even $d = 2p$ is first considered. Let us introduce the two following rational functions

$$K_e(X) = \frac{W_{\frac{(d+1)^n-1}{2}}(X)}{V_{\frac{(d+1)^n-1}{2}}(X)} \quad \text{and} \quad L_e(X) = \prod_{k=1}^{((d+1)^n-1)/2}\frac{X-\cos\Big(\dfrac{2k\pi}{(d+1)^n}\Big)}{X-\cos\Big(\dfrac{(2k-1)\pi}{(d+1)^n}\Big)}.$$

Given an integer k, the degree of V_k and W_k is k, their leading coefficient is 2^k, and their roots are $\{\cos\big(\frac{2j\pi}{2k+1}\big)\}_{1\leq j\leq k}$ and $\{\cos\big(\frac{(2j-1)\pi}{2k+1}\big)\}_{1\leq j\leq k}$, respectively [26]. Therefore K_e and L_e are identical and it implies that the expressions (10) and (12) are the same.

Next, we evaluate $W_{\frac{(d+1)^n-1}{2}}$ and $V_{\frac{(d+1)^n-1}{2}}$ at $\cos\big(\frac{2k\pi}{(d+1)^{n+1}}\big)$ and $\cos\big(\frac{(2k-1)\pi}{(d+1)^{n+1}}\big)$, using the identities $V_k(\cos(\theta)) = \frac{\cos((2k+1)\theta/2)}{\cos(\theta/2)}$ and $W_k(\cos(\theta)) = \frac{\sin((2k+1)\theta/2)}{\sin(\theta/2)}$ [26], namely

$$\begin{aligned}
W_{\frac{(d+1)^n-1}{2}}\Big(\cos\Big(\frac{2k\pi}{(d+1)^{n+1}}\Big)\Big) &= \sin\Big(\frac{k\pi}{d+1}\Big)\Big/\sin\Big(\frac{k\pi}{(d+1)^{n+1}}\Big),\\
V_{\frac{(d+1)^n-1}{2}}\Big(\cos\Big(\frac{2k\pi}{(d+1)^{n+1}}\Big)\Big) &= \cos\Big(\frac{k\pi}{(d+1)}\Big)\Big/\cos\Big(\frac{k\pi}{(d+1)^{n+1}}\Big),\\
W_{\frac{(d+1)^n-1}{2}}\Big(\cos\Big(\frac{(2k-1)\pi}{(d+1)^{n+1}}\Big)\Big) &= \sin\Big(\frac{(2k-1)\pi}{2(d+1)}\Big)\Big/\sin\Big(\frac{(2k-1)\pi}{2(d+1)^{n+1}}\Big),\\
V_{\frac{(d+1)^n-1}{2}}\Big(\cos\Big(\frac{(2k-1)\pi}{(d+1)^{n+1}}\Big)\Big) &= \cos\Big(\frac{(2k-1)\pi}{2(d+1)}\Big)\Big/\cos\Big(\frac{(2k-1)\pi}{2(d+1)^{n+1}}\Big).
\end{aligned} \tag{14}$$

We prove now by induction that the expressions (2) and (10) are identical. The base case for $n = 0$ is immediate from the empty-product rule. Let us assume that both expressions are identical for $n > 0$. Let us introduce a rational function M_e as follows

$$M_e(X) = K_e(X)\frac{\sum_{j=0}^{p}\binom{2p+1}{2j}(X-1)^{p-j}K_e(X)^{2p-2j}(X+1)^j}{\sum_{j=0}^{p}\binom{2p+1}{1+2j}(X-1)^{p-j}K_e(X)^{2p-2j}(X+1)^j}. \tag{15}$$

As for K_e it is straightforward to see that M_e is the ratio of two polynomials of degree $\frac{(d+1)^{n+1}-1}{2}$ and that the overall leading coefficient of M_e is 1. The function M_e can be expressed directly in terms of a ratio of polynomials of degree $\frac{(d+1)^{n+1}-1}{2}$ as follows

$$M_e(X) = \frac{\sum_{j=0}^{p}\binom{2p+1}{2j}(X-1)^{p-j}(X+1)^j\left[W_{\frac{(d+1)^n-1}{2}}(X)\right]^{2p+1-2j}\left[V_{\frac{(d+1)^n-1}{2}}(X)\right]^{2j}}{\sum_{j=0}^{p}\binom{2p+1}{1+2j}(X-1)^{p-j}(X+1)^j\left[W_{\frac{(d+1)^n-1}{2}}(X)\right]^{2p-2j}\left[V_{\frac{(d+1)^n-1}{2}}(X)\right]^{2j+1}}.$$

We will prove now that the roots of $M_e(X)$ are $\left\{\cos\left(\frac{2k\pi}{(d+1)^{n+1}}\right)\right\}$ while the poles of $M_e(X)$ are $\left\{\cos\left(\frac{(2k-1)\pi}{(d+1)^{n+1}}\right)\right\}$ for any k in $\left\{1, 2, \ldots, \frac{(d+1)^{n+1}-1}{2}\right\}$. The trigonometric identities of $\cos(2\theta) \pm 1$ and the results from (14) are used and we obtain

$$M_e\left(\cos\frac{2k\pi}{(d+1)^{n+1}}\right) = \frac{\cos\left(\frac{k\pi}{(d+1)^{n+1}}\right)\sum_{j=0}^{p}\binom{2p+1}{2j}(-1)^{p-j}\sin\left(\frac{k\pi}{d+1}\right)^{2p+1-2j}\cos\left(\frac{k\pi}{d+1}\right)^{2j}}{\sin\left(\frac{k\pi}{(d+1)^{n+1}}\right)\sum_{j=0}^{p}\binom{2p+1}{1+2j}(-1)^{p-j}\sin\left(\frac{k\pi}{d+1}\right)^{2p-2j}\cos\left(\frac{k\pi}{d+1}\right)^{2j+1}}$$

and

$$\left(M_e\left(\cos\frac{(2k-1)\pi}{(d+1)^{n+1}}\right)\right)^{-1} = \frac{\sin\frac{(2k-1)\pi}{2(d+1)^{n+1}}\sum_{j=0}^{p}\binom{2p+1}{1+2j}(-1)^{p-j}\sin\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{2p-2j}\cos\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{2j+1}}{\cos\frac{(2k-1)\pi}{2(d+1)^{n+1}}\sum_{j=0}^{p}\binom{2p+1}{2j}(-1)^{p-j}\sin\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{2p+1-2j}\cos\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{2j}}. \tag{16}$$

Next, we apply the binomial expansion to $\left[\exp\left(\frac{ik\pi}{d+1}\right)\right]^{d+1}$ and $\left[\exp\left(\frac{i(2k-1)\pi}{2(d+1)}\right)\right]^{d+1}$:

$$\left[\exp\left(\frac{ik\pi}{d+1}\right)\right]^{d+1} = \sum_{j=0}^{2p+1}\binom{2p+1}{j}\cos\left(\frac{k\pi}{d+1}\right)^{j}\sin\left(\frac{k\pi}{d+1}\right)^{2p+1-j}i^{2p+1-j} = (-1)^k$$

and

$$\left[\exp\left(\frac{i(2k-1)\pi}{2(d+1)}\right)\right]^{d+1} = \sum_{j=0}^{2p+1}\binom{2p+1}{j}\cos\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{j}\sin\left(\frac{(2k-1)\pi}{2(d+1)}\right)^{2p+1-j}i^{2p+1-j} = (-1)^{k+1}i.$$

Identifying the real and imaginary parts of the previous identities in (16) , it comes automatically that $M_e\big(\cos\big(\frac{2k\pi}{(d+1)^{n+1}}\big)\big) = 0$ and $\big[M_e\big(\cos\big(\frac{(2k-1)\pi}{(d+1)^{n+1}}\big)\big]^{-1} = 0$. It implies that M_e has the following monomial factorization

$$M_e(X) = \prod_{k=1}^{((d+1)^{n+1}-1)/2} \Big(X - \cos\Big(\frac{2k\pi}{(d+1)^{n+1}}\Big)\Big) \Big/ \Big(X - \cos\Big(\frac{(2k-1)\pi}{(d+1)^{n+1}}\Big)\Big). \tag{17}$$

Let us evaluate $rM_e(X)$ for $X = \frac{x+r^2}{x-r^2}$. From (17), $rM_e(X)$ corresponds to the expression developed in (12) at step $n+1$ or equivalently to the expression developed in (10) at step $n+1$. Also $rK_e\big(\frac{x+r^2}{x-r^2}\big)$ is equal to $\mathcal{H}_n$. Noting that $X-1 = \frac{2r^2}{x-r^2}$ and $X+1 = \frac{2x}{x-r^2}$, $rM_e(X)$ from (15) can be assessed as follows

$$rM_e\Big(\frac{x+r^2}{x-r^2}\Big) = rK_e(X)\frac{\sum_{j=0}^{p}\binom{2p+1}{2j}\Big(\frac{2r^2}{x-r^2}\Big)^{p-j}K_e(X)^{2p-2j}\Big(\frac{2x}{x-r^2}\Big)^{j}}{\sum_{j=0}^{p}\binom{2p+1}{1+2j}\Big(\frac{2r^2}{x-r^2}\Big)^{p-j}K_e(X)^{2p-2j}\Big(\frac{2x}{x-r^2}\Big)^{j}}$$

$$= \frac{\sum_{j=0}^{p}\binom{2p+1}{2j}(rK_e(X))^{2p+1-2j}x^j}{\sum_{j=0}^{p}\binom{2p+1}{1+2j}(rK_e(X))^{2p-2j}x^j} = \frac{\sum_{j=0}^{\lceil d/2\rceil}\binom{d+1}{2j}\mathcal{H}_n^{d+1-2j}x^j}{\sum_{j=0}^{\lfloor d/2\rfloor}\binom{d+1}{1+2j}\mathcal{H}_n^{d-2j}x^j}$$

and $rM_e\big(\frac{x+r^2}{x-r^2}\big)$ has the expression of $\mathcal{H}_{n+1}$ provided in (2). We have proven that the expressions in (2) and (10) are identical at the step $n+1$, which ends the proof by induction when d is even.

The case for odd d can be carried out similarly, first proving that equations (11) and (13) are identical for $n \geq 1$ and then establishing the equality between (13) and (2). □

Additional expressions of $\{\mathcal{H}_n\}$ or of its residual sequence $\{\mathcal{H}_{n+1} - \mathcal{H}_n\}$ are presented in Corollary 1, followed by a proof.

Corollary 1. *When d is even ($d = 2p$), the sequence can be expressed as a product as follows*

$$\mathcal{H}_n = r\Big(\prod_{j=1}^{n} W_p\Big(T_{(2p+1)^{j-1}}\Big(\frac{x+r^2}{x-r^2}\Big)\Big)\Big) \Big/ \Big(\prod_{j=1}^{n} V_p\Big(T_{(2p+1)^{j-1}}\Big(\frac{x+r^2}{x-r^2}\Big)\Big)\Big). \tag{18}$$

Furthermore, the residual sequence can be expressed as follows

$$\mathcal{H}_{n+1} - \mathcal{H}_n = 2\mathcal{H}_n\Big[U_{p-1}\Big(T_{(2p+1)^n}\Big(\frac{x+r^2}{x-r^2}\Big)\Big)\Big] \Big/ \Big[V_p\Big(T_{(2p+1)^n}\Big(\frac{x+r^2}{x-r^2}\Big)\Big)\Big]. \tag{19}$$

If d is odd, $\mathcal{H}_n$ can be expressed as follows

$$\mathcal{H}_{n+1} = \mathcal{H}_n \frac{T_{d+1}(X_n)}{X_n U_d(X_n)}\Big|_{X_n = T_{(d+1)^n/2}\left(\frac{x+r^2}{x-r^2}\right)} \tag{20}$$

for n greater or equal than 1.

Finally, the residual sequence can be expressed as follows

$$\mathcal{H}_{n+1} - \mathcal{H}_n = -\mathcal{H}_n \frac{U_{d-1}(X_n)}{X_n U_d(X_n)}\Big|_{X_n = T_{(d+1)^n/2}\left(\frac{x+r^2}{x-r^2}\right)}. \tag{21}$$

Proof. Using their trigonometric definitions, it is easy to verify that

$$W_{\frac{(d+1)^n-1}{2}}(X) = \prod_{j=1}^{n} W_p(T_{(2p+1)^{j-1}}(X))$$

and

$$V_{\frac{(d+1)^n-1}{2}}(X) = \prod_{j=1}^{n} V_p(T_{(2p+1)^{j-1}}(X).$$

Therefore, equations (10) and (18) are identical. Equation (19) can be directly obtained from (18) using the identity $V_p - W_p = -2U_{p-1}$ [26]. Equation (20) can be obtained from (11) using the two classical identities $T_n \circ T_m = T_{n+m}$ and $U_{nm-1} = U_{m-1}(T_n)U_{n-1}$ [26]. Equation (21) is a direct consequence of (20) using the identity $T_{d+1}(X) = XU_d(X) - U_{d-1}(X)$ [26]. □

An algorithm to compute $\mathcal{H}_n$ can be derived using the properties of Corollary 1. It is implemented in Algorithm 1 with input x, the initialization r, the order d and the positive index n and with output S. For an efficient numerical implementation of Algorithm 1, it can be recommended to have a precise asymptotic expansion of either $\frac{2U_{d/2-1}(X)}{V_{d/2}(X)}$ or $\frac{-U_{d-1}(X)}{XU_d(X)}$, further discussed in Section 3.

Algorithm 1 $S = H(x, r, d, n)$

$X \leftarrow \dfrac{x + r^2}{x - r^2}$
if $(mod(d, 2) == 0)$ **then**
    $S \leftarrow r(1 + \text{PHI}(X))$
    $T \leftarrow T_{d+1}(X)$
else
    $T \leftarrow T_{(d+1)/2}(X)$
    $S \leftarrow \dfrac{rT}{(X-1) \times U_{(d+1)/2-1}(X)}$
end if
for i=2:n **do**
    $S \leftarrow S(1 + \text{PHI}(T))$ /*H_i*/
    $T \leftarrow T_{d+1}(T)$
end for
return S
function PHI(X)
    if $(mod(d, 2) == 0)$ **then**
        return $\dfrac{2U_{d/2-1}(X)}{V_{d/2}(X)}$
    else
        return $\frac{-U_{d-1}(X)}{XU_d(X)}$
    end if
end function

The Babylonian method [21], also called Heron's method [16], is a particular case of the Newton's method [28] and is derived in Corollary 2.

Corollary 2. *Let x be a real positive number and $\{u_n\}_{n\geq 0}$ be the sequence*

$$u_0 = r, \quad u_{n+1} = \frac{u_n}{2} + \frac{x}{2u_n} \quad \forall n \geq 0, \tag{22}$$

associated to the Babylonian method reminded in (22)*, where r is a real positive number. An explicit expression of u_n is presented as follows*

$$u_0 = r, \quad u_n = r\frac{T_{2^{n-1}}(X)}{(X-1)U_{2^{n-1}-1}(X)}\Big|_{X=(x+r^2)/(x-r^2)} \quad \forall\, n \in \mathbb{N}^*.$$

An explicit expression solely based on the Chebyshev polynomials of the first kind is proposed for positive n as follows

$$u_n = \frac{x+r^2}{2r} - \frac{x-r^2}{2r}\sum_{k=1}^{n-1}\left[2^k\prod_{j=0}^{k-1}T_{2^j}\left(\frac{x+r^2}{x-r^2}\right)\right]^{-1}.$$

Alternately, the sequence $\{u_n\}_{n\geq 0}$ can be expressed as a product of monomials, namely

$$u_n = r\prod_{k=0}^{2^{n-1}-1}\left(X - \cos\left(\frac{(2k+1)\pi}{2^n}\right)\right)\Big/\left(X - \cos\left(\frac{2k\pi}{2^n}\right)\right)\Big|_{X=\frac{x+r^2}{x-r^2}}.$$

Corollary 3 presents the Halley's method for square roots.

Corollary 3. *Let x and r be real positive numbers and $\{u_n\}_{n\geq 0}$ be the sequence*

$$u_0 = r, \quad u_{n+1} = u_n\frac{{u_n}^2 + 3x}{3{u_n}^2 + x} \quad \forall n \geq 0, \tag{23}$$

associated to the Halley's method for square roots reminded in (23) *The sequence $\{u_n\}_{n\geq 0}$ can be expressed using Chebyshev polynomials of the first kind as follows*

$$u_n = r\prod_{i=1}^{n}\left(2T_{3^{i-1}}\left(\frac{x+r^2}{x-r^2}\right)+1\right)\Big/\left(2T_{3^{i-1}}\left(\frac{x+r^2}{x-r^2}\right)-1\right) \quad \forall n \geq 0.$$

Finally, it is possible to express the general expression of $\mathcal{H}_n$ without the use of trigonometric-related functions as expressed in Theorem 3. A proof follows.

Theorem 3. *Considering the Householder's method of order d for $\sqrt{x}$ with a starting point r, the corresponding sequence $\{\mathcal{H}_n\}_{n\geq 0}$ can be expressed as follows*

$$\mathcal{H}_n = \sqrt{x}\frac{(r+\sqrt{x})^{(d+1)^n} + (r-\sqrt{x})^{(d+1)^n}}{(r+\sqrt{x})^{(d+1)^n} - (r-\sqrt{x})^{(d+1)^n}}. \tag{24}$$

The previous expression can be re-written as a rational function of (x, r) as follows

$$\mathcal{H}_n = r\frac{\displaystyle\sum_{k=0}^{\lceil((d+1)^n-1)/2\rceil}\binom{(d+1)^n}{2k}x^k r^{(d+1)^n-2k}}{\displaystyle\sum_{k=0}^{\lfloor((d+1)^n-1)/2\rfloor}\binom{(d+1)^n}{1+2k}x^k r^{(d+1)^n-2k}}. \tag{25}$$

Proof. We prove by induction that equations (2) and (24) are identical. For the initialization, both equations are equal to r. Let us assume the equality holds at step $n \geq 0$. Let us denote $\phi_n = (r+\sqrt{x})^{(d+1)^n}$ and $\psi_n = (r-\sqrt{x})^{(d+1)^n}$. Therefore $\mathcal{H}_n = \sqrt{x}\frac{\phi_n+\psi_n}{\phi_n-\psi_n}$. The induction step can be obtained using the same method as in (5), namely

$$\mathcal{H}_{n+1} = \sqrt{x}\frac{(t+1)^{d+1}+(t-1)^{d+1}}{(t+1)^{d+1}-(t-1)^{d+1}}\Big|_{t=\mathcal{H}_n x^{-1/2}} = \sqrt{x}\frac{\left(\frac{\phi_n+\psi_n}{\phi_n-\psi_n}+1\right)^{d+1}+\left(\frac{\phi_n+\psi_n}{\phi_n-\psi_n}-1\right)^{d+1}}{\left(\frac{\phi_n+\psi_n}{\phi_n-\psi_n}+1\right)^{d+1}-\left(\frac{\phi_n+\psi_n}{\phi_n-\psi_n}-1\right)^{d+1}}$$

$$= \sqrt{x}\frac{(\phi_n+\psi_n+\phi_n-\psi_n)^{d+1}+(\phi_n+\psi_n-(\phi_n-\psi_n))^{d+1}}{(\phi_n+\psi_n+\phi_n-\psi_n)^{d+1}-(\phi_n+\psi_n-(\phi_n-\psi_n))^{d+1}} = \sqrt{x}\frac{\phi_n^{d+1}+\psi_n^{d+1}}{\phi_n^{d+1}-\psi_n^{d+1}}$$

$$= \sqrt{x}\frac{\phi_n^{d+1}+\psi_n^{d+1}}{\phi_n^{d+1}-\psi_n^{d+1}} = \sqrt{x}\frac{(r+\sqrt{x})^{(d+1)^{n+1}}+(r-\sqrt{x})^{(d+1)^{n+1}}}{(r+\sqrt{x})^{(d+1)^{n+1}}-(r-\sqrt{x})^{(d+1)^{n+1}}}.$$

Equation (24) highlights that the $(n+1)$th term of the Householder's sequence of order d is equal to the second term of the Householder's sequence of order $(d+1)^n - 1$. Equation (25) is simply obtained from (2) for $n = 0$ and the order $(d+1)^n - 1$. Similar functions to the ones in equation (25) have been recently studied in [24] and are related to the tangent analog of the Chebyshev polynomials. □

We can notice that the sequence $\{\mathcal{A}_n\}_{n\geq 1}$ defined by

$$\mathcal{A}_n = \sqrt{x}\frac{(r+\sqrt{x})^n+(r-\sqrt{x})^n}{(r+\sqrt{x})^n-(r-\sqrt{x})^n}$$

is among the slowest sequence to converge to $\sqrt{x}$ while at the same time it is featuring in its subsequences all the sequences of the Householder's method for $\sqrt{x}$ at every order. This sequence has already been obtained by A.K. Yeyios [31] from continued fraction expansions. The Newton's method and more generally the Householder's method for square roots of integer numbers is intimately related to Pell's equations [4, 27].

2 A note the Householder's method for *n*th roots

Algorithms for the nth root computation have already been developed [14]. An introduction to the Householder's method to obtain $\sqrt[p]{x}$ is now discussed. Considering an integer p, an order d and the function $g_p(t) = \frac{1}{t^p - x}$, where x is a positive real number, the Householder's method of order d for pth root is provided by

$$\mathbf{H}_0 = r, \quad \mathbf{H}_{n+1} = \mathbf{H}_n + d\frac{g_p^{(d-1)}(\mathbf{H}_n)}{g_p^{(d)}(\mathbf{H}_n)} \quad \forall n \geq 0, \tag{26}$$

with initial guess r. The sequence $\{\mathbf{H}_n\}_{n\geq 0}$ converges to $\sqrt[p]{x}$ with a rate of convergence of $d+1$.

Numerous expressions in the Householder's method for square roots are based on binomial coefficients. Considering the nth root extraction, we need to introduce the generalized binomial coefficients $\binom{n}{m}_p$ of order p [7, 8], which naturally appear in the development of $B_p(x) = \sum_{k=0}^{p-1} x^k$ at the power n as follows

$$B_p(x)^n = (1+x+\cdots+x^{p-1})^n = \sum_{m=0}^{(p-1)n}\binom{n}{m}_p x^m.$$

The generalized binomial coefficients of order p can be obtained from the binomial coefficients as follows [8]

$$\binom{n}{m}_p = \sum_{k=0}^{\lfloor m/p \rfloor} (-1)^k \binom{n}{k} \binom{n+m-pk-1}{n-1}.$$

In addition, the parity arguments are extended to their modular counterparts involving series multisection [3,5]. Given a function f and a radical p, let us denote the primitive root of unity $\xi = \exp\left(\frac{2i\pi}{p}\right)$ and we introduce p functions $\{[f]_\ell(t)\}_{0\leq \ell \leq p-1}$ as follows

$$[f]_\ell(t) = \frac{1}{p}\sum_{k=0}^{p-1} \xi^{-\ell k} f(\xi^k t). \tag{27}$$

It is straightforward to verify that if f has a power expansion of the type $\sum_{n\geq 0} a_n x^n$, $[f]_\ell$ is expressed as $\sum_{n\geq 0} a_{pn+\ell} x^{pn+\ell}$. The functions are commonly referred as Roots of Unity Filters and f can be reconstructed from their sums.

An explicit expression of the sequence $\{\mathbf{H}_n\}_{n\geq 0}$ is presented in Theorem 4. A proof follows.

Theorem 4. *Considering the Householder's method of order d for $\sqrt[p]{x}$ with a starting point r, the corresponding sequence $\{\mathbf{H}_n\}_{n\geq 0}$ can be expressed as follows*

$$\mathbf{H}_0 = r, \quad \mathbf{H}_{n+1} = \sqrt[p]{x}\frac{[B_p^{d+1}]_{1-d[p]}(\mathbf{H}_n x^{-1/p})}{[B_p^{d+1}]_{-d[p]}(\mathbf{H}_n x^{-1/p})} \quad \forall n \geq 0. \tag{28}$$

The previous expression can be formulated as follows

$$\mathbf{H}_0 = r, \quad \mathbf{H}_{n+1} = \frac{\sum\limits_{k=0}^{\lfloor ((p-1)d+1)/p \rfloor} \binom{d+1}{p(k+1)-2}_p \mathbf{H}_n^{d(p-1)+1-pk} x^k}{\sum\limits_{k=0}^{\lfloor (p-1)d/p \rfloor} \binom{d+1}{p(k+1)-1}_p \mathbf{H}_n^{d(p-1)-pk} x^k} \quad \forall n \geq 0. \tag{29}$$

Proof. The partial fraction decomposition of $\frac{1}{t^p-1}$ is the following elementary result

$$\frac{1}{t^p-1} = \frac{1}{p}\sum_{k=0}^{p-1} \frac{\xi^k}{t-\xi^k}.$$

Therefore, its rth derivative has the next expression

$$\frac{\mathrm{d}^r \frac{1}{t^p-1}}{\mathrm{d}t^r} = \frac{(-1)^r r!}{p}\sum_{k=0}^{p-1} \frac{\xi^k}{(t-\xi^k)^{r+1}}.$$

By noticing that $g_p(t) = \frac{1}{x}\frac{1}{(tx^{-1/p})^p-1}$, expression (26) can be written as follows

$$\mathbf{H}_{n+1} = \sqrt[p]{x}\left[t - \left(\sum_{k=0}^{p-1}\frac{\xi^k}{(t-\xi^k)^d}\right) \Big/ \left(\sum_{k=0}^{p-1}\frac{\xi^k}{(t-\xi^k)^{d+1}}\right)\right]\Bigg|_{t=\mathbf{H}_n x^{-1/p}}.$$

We can now notice that $\frac{t\xi^k}{(t-\xi^k)^{d+1}} = \frac{\xi^k}{(t-\xi^k)^d} + \frac{\xi^{2k}}{(t-\xi^k)^{d+1}}$, which implies

$$\mathbf{H}_{n+1} = \sqrt[p]{x}\left[\left(\sum_{k=0}^{p-1}\frac{\xi^{2k}}{(t-\xi^k)^{d+1}}\right) \Big/ \left(\sum_{k=0}^{p-1}\frac{\xi^k}{(t-\xi^k)^{d+1}}\right)\right]\Bigg|_{t=\mathbf{H}_n x^{-1/p}}. \tag{30}$$

Let us now derive the function $\frac{[B_p^{d+1}]_l(t)}{(t^p-1)^{d+1}}$ using the identity $t^p - 1 = (t-1)B(t)$ and the series multisection defined in (27) as follows

$$\begin{aligned}\frac{[B_p^{d+1}]_\ell(t)}{(t^p-1)^{d+1}} &= \frac{1}{p}\frac{\sum_{k=0}^{p-1}\xi^{-\ell k}B(\xi^k t)^{d+1}}{(t^p-1)^{d+1}} = \frac{1}{p}\sum_{k=0}^{p-1}\frac{\xi^{-\ell k}}{(\xi^k t-1)^{d+1}}\\ &= \frac{1}{p}\sum_{k=0}^{p-1}\frac{\xi^{-\ell k}}{\xi^{k(d+1)}(t-\xi^{p-k})^{d+1}} = \frac{1}{p}\sum_{k=0}^{p-1}\frac{\xi^{k(\ell+d+1)}}{(t-\xi^k)^{d+1}}.\end{aligned}$$

The previous function can match both the numerator and the denominator of (30). For the numerator, we need to identify ℓ such that $\ell + d + 1 = 2[p]$ and for the denominator, to identify ℓ such that $\ell + d + 1 = 1[p]$. Equation (28) ensues. To obtain (29), rather than using the explicit expression of $[B_p^{d+1}]_\ell(t)$ using (27), we use the fact that it filters all powers except the ones congruent to ℓ modulo p. The highest degree of the numerator is $d(p-1)+1$ while the highest degree of the denominator is $d(p-1)$. Therefore, we obtain

$$\begin{aligned}\mathbf{H}_{n+1} &= \sqrt[p]{x}\frac{[B_p^{d+1}]_{1-d[p]}(\mathbf{H}_n x^{-1/p})}{[B_p^{d+1}]_{-d[p]}(\mathbf{H}_n x^{-1/p})} = \sqrt[p]{x}\left.\frac{\sum_{\substack{m=0\\ m+d-1=0[p]}}^{(p-1)(d+1)}\binom{d+1}{m}_p t^m}{\sum_{\substack{m=0\\ m+d=0[p]}}^{(p-1)(d+1)}\binom{d+1}{m}_p t^m}\right|_{t=\mathbf{H}_n x^{-1/p}}\\ &= \sqrt[p]{x}\left.\frac{\sum_{k=0}^{\lfloor((p-1)d+1)/p\rfloor}\binom{d+1}{d(p-1)+1-pk}_p t^{d(p-1)+1-pk}}{\sum_{k=0}^{\lfloor(p-1)d/p\rfloor}\binom{d+1}{d(p-1)-pk}_p t^{d(p-1)-pk}}\right|_{t=\mathbf{H}_n x^{-1/p}}\\ &= \sqrt[p]{x}\frac{\sum_{k=0}^{\lfloor((p-1)d+1)/p\rfloor}\binom{d+1}{d(p-1)+1-pk}_p \mathbf{H}_n^{d(p-1)+1-pk}x^{k-1/p}}{\sum_{k=0}^{\lfloor(p-1)d/p\rfloor}\binom{d+1}{d(p-1)-pk}_p \mathbf{H}_n^{d(p-1)-pk}x^{k}}.\end{aligned}$$

The radical values collapses and we obtain (29) using the identity $\binom{n}{m}_s = \binom{n}{(s-1)n-m}_s$ [8], which ends the proof. ☐

3 A note on the asymptotic expansion of Chebyshev functions

When considering the Householder's method of order d for $\sqrt{x}$, Algorithm 1 identified the need to have a precise asymptotic evaluation of $\frac{U_{d-1}(X)}{XU_d(X)}$ and $\frac{2U_{d/2-1}(X)}{V_{d/2}(X)}$, when d is odd and even, respectively. The goal of this section is to provide an asymptotic expansion of these functions and to discuss the presence of these families of functions in the study of lattice paths.

Let us first denote $f_d(x) = \frac{U_{d-1}(x)}{xU_d(x)}$ for $d \geq 1$. It is straightforward from (7) to obtain the following recurrence relation:

$$f_1(x) = \frac{1}{2x^2}, \quad f_d(x) = \frac{1}{x^2(2 - f_{d-1}(x))}. \tag{31}$$

From (31), we can in particular explicit the pointwise limit of $f_d(x)$ for $x > 1$ which is $f(x) = 1 - \sqrt{1 - 1/x^2}$. Before expressing f_d as a power series, we define one of the most common lattice path called Dyck path [22]. A Dyck path of semilength n and of maximum height h is a lattice walk from $(0,0)$ to $(2n,0)$ with steps of the form $(1,1)$ and $(1,-1)$ with a height bounded in the interval $[0,h]$. The number of Dyck paths of semilength n and of maximum height h is denoted $\Delta_{n,h}$. We can observe that $\Delta_{4,1} = 1$, $\Delta_{4,2} = 8$, $\Delta_{4,3} = 13$ and $\Delta_{4,k} = 14$ for $k \geq 4$. The natural initialization of the sequence is $\{\Delta_{0,h} = 1\}_{h\geq 0}$ and $\{\Delta_{n,0} = 0\}_{n\geq 1}$. Enumerating Dyck paths can be also found in ballot counting problems [6], plane trees [11] or permutations [22]. It is well established that $\Delta_{n,h}$ is equal to the Catalan number $C_n = \binom{2n}{n}/(n+1)$, when $h \geq n$ [2].

An early reference to the sequence $\{\Delta_{n,h}\}_{\{n,h\geq 0\}}$ can be traced back in the work of G. Kreweras [23] using Fibonacci polynomials, which are closely related to the Chebyshev polynomials of the second kind. The relationship between Dyck paths and the ratio of Chebyshev polynomials has been further established in [9,17,22], usually involving generating functions. Theorem 5 presents an asymptotic expansion of f_d expressed as a power series, followed by a proof.

Theorem 5. *We consider the family of functions* $\{f_d(x) = \frac{U_{d-1}(x)}{xU_d(x)}\}_{d\geq 1}$ *over the interval* $[1, \infty[$ *and* $\Delta_{n,h}$ *as the number of Dyck paths of semilength* n *and of maximum height* h*. Then* f_d *can be expressed using power series as follows*

$$f_d(x) = \sum_{i=0}^{\infty} \frac{\Delta_{i,d-1}}{2^{2i+1}x^{2i+2}}. \tag{32}$$

Proof. Equation (32) is correct for $d = 1$, based on (31). Considering Dyck paths of semilength n and maximum height h, a conditioning with respect to the last return to the x-axis leads to [11]

$$\Delta_{n,h} = \sum_{k=0}^{n-1} \Delta_{k,h}\Delta_{n-1-k,h-1}. \tag{33}$$

For $d \geq 2$, given the parity of f_d highlighted in (31), f_d can be expressed as $\sum_{i=0}^{\infty} \frac{\alpha_{i,d}}{x^{2i+2}}$. Rewritting (31) as $2x^2 f_d(x) - x^2 f_d(x) f_{d-1}(x) = 1$ and using the Cauchy product, it leads to

$$\begin{cases} 2\alpha_{0,d} = 1, \\ 2\alpha_{i,d} = \sum_{k=0}^{i} \alpha_{k,d}\alpha_{i-k,d-1}, \quad i \geq 1. \end{cases} \tag{34}$$

Finally, $\alpha_{i,d} = \frac{\Delta_{i,d-1}}{2^{2i+1}}$ is the appropriate candidate for both the initialization and the general case in (34), which ends the proof. □

The approximation of f_d through the sequence $\{\Delta_{n,d-1}\}_{\{n\geq 0\}}$ can be obtained in multiple ways. The sequence $\{\Delta_{n,h}\}_{\{n,h\geq 0\}}$ corresponds to [30, sequence A080934] and the recurrence relation presented in (33) [11] is a common identity to compute the sequence. Closed form expressions are presented in [19] and the sequence has also been studied through its differences in [23].

Let us now study the family of functions $g_d = \frac{2U_{d/2-1}(X)}{V_{d/2}(X)}$. As for f_d, it is interesting to study g_d for odd and even values of d. When d is odd, it is necessary to define Chebyshev functions of half-integer order, which has been comonly studied over the interval $[-1, 1]$ [29]. As we evaluate g_d over the interval $[1, \infty[$ in Algorithm 1, the Chebyshev functions of half integer order are proposed in Lemma 2 over the interval $[1, \infty[$. The proof is simply obtained from the hyperbolic definition of the Chebyshev polynomial over the interval $[1, \infty[$ (see, for example, [26]).

Lemma 2. *The Chebyshev polynomials of the four kinds can be defined at half-integer orders using the following identities*

$$U_{p-1/2} = \sqrt{1/(2(1+z))}W_p(z), \quad T_{p+1/2} = \sqrt{(1+z)/2}V_p(z)$$

for $p \geq 0$.

Based on the identities of Lemma 2, the family of functions g_d is defined as follows

$$g_d(x) = \begin{cases} 2U_{d/2-1}(x)/V_{d/2}(x), & \text{if } d \text{ is even,} \\ W_{(d-1)/2}(x)/T_{(d+1)/2}(x), & \text{if } d \text{ is odd.} \end{cases}$$

In order to express g_d as a power series, Lemma 3 presents an alternate expression of g_d along with a recurrence relation involving both f_d and g_d. A proof follows.

Lemma 3. *The family of functions* $\{g_d(x)\}_{d\geq 1}$ *can be expressed using only Chebyshev polynomials of the second kind as follows*

$$g_d(x) = 2\sum_{k=0}^{d-1} \frac{U_k(x)}{U_d(x)}. \tag{35}$$

In addition, f_d *and* g_d *obey to the following recurrence relation*

$$g_1(x) = 1/x, \quad g_d(x) = xf_d(x)(2 + g_{d-1}(x)), \quad d \geq 1. \tag{36}$$

Proof. The proof of establishing equation (35) is mainly based on Lagrange's trigonometric identity

$$\sum_{k=0}^{n} \sin(k\theta) = \sin\left(\frac{(n+1)\theta}{2}\right) \sin\left(\frac{n\theta}{2}\right) \left(\sin\left(\frac{\theta}{2}\right)\right)^{-1}.$$

Therefore, we obtain

$$2\sum_{k=0}^{d-1} \frac{U_k(\cos(\theta))}{U_d(\cos(\theta))} = \frac{2\sin\left(\frac{(d+1)\theta}{2}\right)\sin\left(\frac{d\theta}{2}\right)}{\sin((d+1)\theta)\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\frac{d\theta}{2}\right)}{\cos\left(\frac{(d+1)\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}.$$

Using the trigonometric definitions of the Chebyshev polynomials of the four kinds [26] and based on the parity of d, equation (35) can be established over the interval $[-1, 1]$, except for the isolated singularities. The equality can be extended to $[1, \infty[$ by analytic continuation. Equation (36) is a direct consequence of equation (35). □

From (36), it is possible to derive the pointwise limit of $g_d(x)$ for $x > 1$, which is $g(x) = \sqrt{(x+1)/(x-1)} - 1$. Before expressing g_d as a power series, we introduce the concept of Symmetric Dyck path [12, 13, 25]. A Symmetric Dyck path of semilength n and of maximum height h is a Dyck path of semilength n and of maximum height h, which is symmetrical from the line $x = n$. The number of Symmetric Dyck paths of semilength n and of maximum height h is denoted $\Delta^S_{n,h}$. For example, $\Delta^S_{4,1} = 1$, $\Delta^S_{4,2} = 4$, $\Delta^S_{4,3} = 5$ and $\Delta^S_{4,k} = 6$ for $k \geq 4$. The initialization of the sequence is similar to the previous sequence with $\{\Delta^S_{0,h} = 1\}_{h\geq 0}$ and $\{\Delta^S_{n,0} = 0\}_{n\geq 1}$. It is well known [13], that $\Delta^S_{n,h}$ is equal to the central binomial coefficient $D_n = \binom{n}{\lfloor n/2\rfloor}$, when $h \geq n$ [15]. A Symmetric Dyck path of semilength n and of maximum height h can be commonly decomposed in the following way: given $k \leq \lfloor n/2\rfloor$, a path is composed of a Dyck path of semilength k and of maximum height h, a step $(1,1)$, a (shifted) Symmetric Dyck path of semilength $n-1-2k$ and of maximum height $h-1$, a step $(1,-1)$ and finally the symmetric of the initial Dyck path of semilength k. The notable exception is the existence, when n is even, of Symmetric Dyck paths of semilength n and of maximum height h composed of two symmetric Dyck paths of semilength $\lfloor n/2\rfloor$. Given the convention $\{\Delta^S_{-1,h} = 1\}_{h\geq 0}$ and conditioning with respect to the last return to the x-axis before n, we finally obtain the following identity $\Delta^S_{n,h} = \sum_{k=0}^{\lfloor n/2\rfloor} \Delta_{k,h}\Delta^S_{n-1-2k,h-1}$ [13, 17].

The relationship between Symmetric Dyck paths and Chebyshev polynomials has been discussed in [10, 17]. Theorem 6 presents an asymptotic expansion of g_d expressed as a power series, followed by a proof.

Theorem 6. *We consider the family of functions*

$$g_d(x) = \begin{cases} 2U_{d/2-1}(x)/V_{d/2}(x), & \text{if } d \text{ is even,} \\ W_{(d-1)/2}(x)/T_{(d+1)/2}(x), & \text{if } d \text{ is odd.} \end{cases}$$

Let $\Delta^S_{n,h}$ be the number of Symmetric Dyck paths of semilength n and of maximum height h. Over the interval $[1,\infty[$, g_d can be expressed using power series as follows

$$g_d(x) = \sum_{i=0}^{\infty} \frac{\Delta^S_{i,d-1}}{2^i x^{i+1}}.$$

Proof. A similar approach using involutions, generating functions and equation (35) has been presented in [17]. Let us denote $\widehat{g_d}(x) = \sum_{i=0}^{\infty} \frac{\Delta^S_{i,d-1}}{2^i x^{i+1}}$. We will prove by induction that $\widehat{g_d} = g_d$. The case $d = 1$ is immediate. Let us assume $\widehat{g_d} = g_d$ for d greater or equal than 1 and we develop equation (36) using the Cauchy product, namely

$$\begin{aligned} g_{d+1} &= xf_{d+1}(x)(2+g_d(x)) = x\Big(\sum_{i=0}^{\infty}\frac{\Delta_{i,d}}{2^{2i+1}x^{2i+2}}\Big)\Big(2+\sum_{i=0}^{\infty}\frac{\Delta^S_{i,d-1}}{2^i x^{i+1}}\Big) \\ &= x\Big(\sum_{i=0}^{\infty}\frac{\Delta_{i,d}}{2^{2i+1}x^{2i+2}}\Big)\Big(\sum_{i=-1}^{\infty}\frac{\Delta^S_{i,d-1}}{2^i x^{i+1}}\Big) = \frac{1}{x}\Big(\sum_{i=0}^{\infty}\frac{\Delta_{i,d}}{2^{2i}x^{2i}}\Big)\Big(\sum_{i=0}^{\infty}\frac{\Delta^S_{i-1,d-1}}{2^i x^i}\Big) \\ &= \frac{1}{x}\Big(\sum_{i=0}^{\infty}\sum_{k=0}^{\lfloor i/2\rfloor}\frac{\Delta_{k,d}}{2^{2k}}\frac{\Delta^S_{i-1-2k,d-1}}{2^{i-2k}}x^{-i}\Big) = \sum_{i=0}^{\infty}\frac{\Delta^S_{i,d}}{2^i x^{i+1}} = \widehat{g_{d+1}}. \end{aligned}$$

□

The sequence $\{\Delta^S_{n,h}\}_{\{n,h\geq 1\}}$ corresponds to [30, sequence A94718]. Closed form expressions are presented in [10, 13].

References

[1] Aghigh K., Masjed-Jamei M., Dehghan M. *A survey on third and fourth kind of Chebyshev polynomials and their applications*. Appl. Math. Comput. 2008, **199** (1), 2–12. doi:10.1016/j.amc.2007.09.018

[2] Aigner M. Catalan and other numbers: a recurrent theme. Springer, Milan, 2001.

[3] Balaich M., Ondrus M. *A generalization of even and odd functions.* Involve J. Math. 2011, **4** (1), 91–102. doi: 10.2140/involve.2011.4.91

[4] Barbeau E. Pell's equation. Problem books in mathematics. Springer, New York, 2003.

[5] Beauregard R.A., Dobrushkin V.A. *Multisection of series.* Math. Gaz. 2016, **100** (549), 460–470. doi: 10.1017/mag.2016.111

[6] Bertrand J. *Solution d'un problème.* C. R. Acad. Sci. Sér. I (Math.) 1887, **105**, 369.

[7] Bollinger R.C. *Extended Pascal triangles.* Math. Mag. 1993, **66** (2), 87–94. doi:10.2307/2691114

[8] Bondarenko B.A. Generalized Pascal triangles and pyramids, their fractals, graphs and applications. The Fibonacci Association, Santa Clara, 1993.

[9] Chow T., West J. *Forbidden subsequences and Chebyshev polynomials.* Discrete Math. 1999, **204** (1), 119–128. doi: 10.1016/S0012-365X(98)00384-7

[10] Cigler J. *Some remarks and conjectures related to lattice paths in strips along the x-axis*. arXiv:1501.04750 [math.CO] doi:10.48550/arxiv.1501.04750

[11] de Bruijn N.G., Knuth D.E., Rice S.O. *The average height of planted plane trees.* Graph Theory & Comput. 1972, **1972**, 15–22. doi:10.1016/B978-1-4832-3187-7.50007-6

[12] Deng L.H., Deng Y.P., Shapiro L.W. *The Riordan group and symmetric lattice paths.* J. Shandong Univ. Nat. Sci. 2015, **50** (4), 82–89. doi:10.6040/j.issn.1671-9352.0.2014.196

[13] Dershowitz N. *Between Broadway and the Hudson: A bijection of corridor paths.* J. Integer Seq. 2021, **24** (2), article 21.2.8.

[14] Dubeau F. *Newton's method and high-order algorithms for the nth root computation*. J. Comput. Appl. Math. 2009, **224** (1), 66–76. doi:10.1016/j.cam.2008.04.014

[15] Ferrari L. *Some combinatorics related to central binomial coefficients: Grand-Dyck paths, coloured noncrossing partitions and signed pattern avoiding permutations.* Graphs Combin. 2010, **26** (1), 51–70. doi:10.1007/s00373-010-0895-z

[16] Fowler D., Robson E. *Square Root Approximations in Old Babylonian Mathematics: YBC 7289 in Context.* Hist. Math. 1998, **25** (4), 366–378. doi:10.1006/hmat.1998.2209

[17] Guibert O., Mansour T. *Restricted 132-involutions.* Sémin. Lothar. Comb. 2002, **48**, B48a.

[18] Hasan M.A. New families of higher order iterative methods for solving equations. In: Proc. of the 45th IEEE Conference on Decision and Control, San Diego, CA, USA, December 13–15, 2006. IEEE, 2007, 6379–6384.

[19] Hein N., Huang J. *Variations of the Catalan numbers from some nonassociative binary operations.* Discrete Math. 2022, **345** (3), 112711. doi:10.1016/j.disc.2021.112711

[20] Householder A. The numerical treatment of a single nonlinear equation. McGraw-Hill, New-York, 1970.

[21] Kosheleva O. Babylonian method of computing the square root: justifications based on fuzzy techniques and on computational complexity. In: NAFIPS 2009 – 2009 Annual Meeting of the North American Fuzzy Information Processing Society, Cincinnati, OH, USA , June 14–17, 2009. IEEE, 2009, 1–6.

[22] Krattenthaler C. *Permutations with restricted patterns and Dyck paths.* Adv. Appl. Math. 2001, **27** (2–3), 510–530. doi:10.1006/aama.2001.0747

[23] Kreweras G. *Sur les éventails de segments.* Cah. Bur. Univ. Rech. Opérationnelle 1970, **15**, 3–41.

[24] Lima J.B., Campello de Souza R.M. *Tangent function and Chebyshev-like rational maps over finite fields.* IEEE Trans. Circuits Syst. II, Exp. Briefs 2020, **67** (4), 775–779. doi:10.1109/TCSII.2019.2923879

[25] Lipton R.J., Zalcstein Y. *Word problems solvable in logspace.* J. ACM 1977, **24** (3), 522–526. doi: 10.1145/322017.322031

[26] Mason J. Handscomb D. Chebyshev polynomials. CRC Press LLC, Boca Raton, 2002.

[27] McBride A. *Remarks on Pell's equation and square root algorithms.* Math. Gaz. 1999, **83** (496), 47–52. doi: 10.2307/3618682

[28] Polyak B. *Newton's method and its use in optimization.* Eur. J. Oper. Res. 2007, **181** (3), 1086–1096. doi: 10.1016/j.ejor.2005.06.076

[29] Ricci P.E. *A survey on pseudo-Chebyshev functions.* 4open 2020, **3**, 2. doi:10.1051/fopen/2020001

[30] Sloane N.J.A. The on-line encyclopedia of integer sequences (OEIS). `https://oeis.org/`

[31] Yeyios A.K. *On two sequences of algorithms for approximating square roots*. J. Comput. Appl. Math. 1992, **40** (1), 63–72. doi:10.1016/0377-0427(92)90042-V

Received 11.12.2024
Revised 21.01.2025

Діжу Й. *Застосування поліномів Чебишева до методу Хаусхолдера для квадратних коренів* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 616–630.

Метод Хаусхолдера – це алгоритм знаходження коренів, який є природним розширенням як методу Ньютона, так і методу Галлея. У цій статті основна увага приділяється наближенню квадратного кореня з додатного дійсного числа на основі цих методів. Отримані алгоритми можна виразити за допомогою поліномів Чебишева. Також пропонується розширення до n-го кореня.

Ключові слова і фрази: поліном Чебишева, метод Хаусхолдера, метод Ньютона.