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On boundary distortion estimates of plane Sobolev mappings

Sevost'yanov E.O.[1,2], Ilkevych N.S.[1], Desyatka V.S.[1]

We study mappings of the Sobolev classes defined in some plane domain. We have obtained estimates of the distortion of the distance under these mappings at the boundary. In particular, we have proved that if the integral averages of the characteristic of mappings are finite, then these mappings are Hölder continuous.

Key words and phrases: quasiconformal mapping, mapping with finite distortion, Sobolev mapping, Hölder continuity, Lipschitz continuity.

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[1] Zhytomyr Ivan Franko State University, 40 Velyka Berdychivs'ka str., 10008, Zhytomyr, Ukraine
[2] Institute of Applied Mathematics and Mechanics of NAS of Ukraine, 19 Henerala Batyuka str., 84116, Slovyans'k, Ukraine
E-mail: `esevostyanov2009@gmail.com` (Sevost'yanov E.O.), `ilkevych1980@gmail.com` (Ilkevych N.S.),
`victoriazehrer@gmail.com` (Desyatka V.S.)

1 Introduction

Recently, we have studied problems related to the boundary behavior of Sobolev and Orlicz-Sobolev mappings in the unit ball (see, e.g., [10, 11, 16]). In particular, we have proved that these mappings are Hölder continuous on the unit sphere under some conditions. The present manuscript is devoted to the distortion estimates for more general plane domains, not only the unit disk. Moreover, we consider not only homeomorphisms but mappings with branching, and not only usual Hölder continuity, but Hölder continuity and more general distortion estimates with respect to prime ends. We should note that the main results of the paper concern not so much the Hölder-type distance distortion as a more general distortion. Hölder-type inequalities are obtained as consequences of the main results in the last section. In this regard, we will point out several of our recent works on the same topic (see [2, 3, 18]). The results of this article are similar to those obtained in the indicated works, at the same time, they are not a consequence of them.

Throughout this manuscript, D denotes a domain in $\mathbb{C}$. For any $z_0 \in \mathbb{C}$, $z_0 \neq \infty$, let

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}, \qquad S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\},$$

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}. \tag{1}$$

We assume that the reader is familiar with the definitions of Sobolev classes $W^{1,1}_{\rm loc}$ and some of their basic properties (see, e.g., [15, 2.I]). For a mapping $f : D \to \mathbb{C}$ having partial derivatives almost everywhere, we set $J(z, f) := \det f'(z)$ for the Jacobian of f at z. Observe that for sense-preserving mappings we have $J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2$ (see [1, relation (9).A.I]).

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We define the *maximal* dilatation of the mapping f at a point z by the relation

$$K_{\mu_f}(z)=\begin{cases}\frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|}, & J(z,f)\neq 0,\\ 1, & f'(z)=0,\\ \infty, & \text{otherwise.}\end{cases}$$

Given a mapping $f:D\rightarrow\mathbb{C}$, a set $E\subset D$ and $y\in\mathbb{C}$, we define the *multiplicity function* $N(y,f,E)$ as a number of preimages of the point y in a set E, i.e.

$$N(y,f,E)\ =\ \text{card}\,\{z\in E: f(z)=y\}\,,\qquad \text{and let}\qquad N(f,E)\ =\ \sup\limits_{y\in\mathbb{C}} N(y,f,E)\,. \tag{2}$$

Let h be a chordal metric in $\overline{\mathbb{C}}$, and

$$h(z,\infty)=\frac{1}{\sqrt{1+{|z|}^2}},\qquad h(z,y)=\frac{|z-y|}{\sqrt{1+{|z|}^2}\sqrt{1+{|y|}^2}},\quad z\neq\infty\neq y, \tag{3}$$

and let

$$h(E):=\sup\limits_{z,y\in E}h(z,y) \tag{4}$$

be a chordal diameter of a set $E\subset\overline{\mathbb{C}}$ (see, e.g., [19, Definition 12.1]). Let X, Y be metric spaces. A map $f:X\rightarrow Y$ is *discrete* if $\{f^{-1}(y)\}$ is discrete for all $y\in Y$ and f is *open* if f maps open sets onto open sets. A mapping $f:X\rightarrow Y$ is called *closed* if $f(A)$ is closed in $f(X)$ whenever A is closed in X. Recall that a mapping f between domains D and D' in $\mathbb{C}$ is of *finite distortion* if $f\in W_{\rm loc}^{1,1}$ and $\|f'(z)\|^2\leqslant K(z)J(z,f)$ for almost all $z\in D$ and some finite function $K(z)<\infty$.

Following [4], a domain G in $\mathbb{C}$ is called a *quasiextremal distance domain* (*QED-domain* for short), if there is a number $A_0\geqslant 1$, such that for any continua $E,F\subset G$ the following inequality

$$M(\Gamma(E,F,\mathbb{C}))\leqslant A_0\cdot M(\Gamma(E,F,G)) \tag{5}$$

holds. For any $A_0>0$, $R_0>0$ and $\delta>0$, a domain $D\subset\mathbb{C}$, a path connected continuum $A\subset D$ and a Lebesgue measurable function $Q:D\rightarrow[0,\infty]$, denote by $\mathfrak{F}_{Q,A,\delta}^{A_0,R_0}(D)$ a family of all homeomorphisms $f:D\rightarrow B(0,R_0)$ with a finite distortion such that $K_{\mu_f}(z)\leqslant Q(z)$ almost everywhere, ${\rm diam}\,(f(A))\geqslant\delta$ and $D_f^{\,\prime}=f(D)$ satisfies the condition (5) with $G=D_f^{\,\prime}$.

We say that a function $\varphi:\mathbb{C}\rightarrow\mathbb{R}$, $\varphi(z)\equiv 0$ for $z\in\mathbb{C}\setminus D$, has a *finite mean oscillation* at a point $z_0\in\overline{D}$, $D\subset\mathbb{C}$, write $\varphi\in FMO(z_0)$, if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\,\varepsilon)}|\varphi(z)-\overline{\varphi}_{\varepsilon}|\,dm(z)<\infty\,,\qquad \text{where}\quad \overline{\varphi}_{\varepsilon}=\frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\,\varepsilon)}\varphi(z)\,dm(z).$$

Theorem 1. *Let $z_0\in\partial D$, $z_0\neq\infty$, and A be a continuum in D. Let $Q:\mathbb{C}\rightarrow[0,\infty]$ be a Lebesgue measurable function in $\mathbb{C}$ vanishing outside D. Assume that the following conditions hold:*

1) there is $\varepsilon_0=\varepsilon_0(z_0)>0$ such that the set $B(z_0,r)\cap D$ is connected for each $0<r<\varepsilon_0$;

2) $Q\in FMO(z_0)$.

Then any $f\in\mathfrak{F}_{Q,A,\delta}^{A_0,R_0}(D)$ has a continuous extension to z_0 and, in addition, there are $\widetilde{C}=\widetilde{C}(\delta,R_0,z_0,Q)>0$ and $\widetilde{\varepsilon_0}=\widetilde{\varepsilon_0}(z_0)>0$ such that the inequality

$$|f(z)-f(z_0)|\leqslant\frac{\widetilde{C}}{\left(\log\frac{1}{|z-z_0|}\right)^{\frac{2\pi}{A_0K_0}}}$$

holds for any $z\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ and any $f\in\mathfrak{F}_{A,\delta}^{A_0,R_0}(D)$, where $K_0>0$ is some constant depending only on the function Q.

Let

$$q_{z_0}(r)=\frac{1}{2\pi}\int\limits_0^{2\pi}Q(z_0+re^{i\varphi})\,d\varphi\,. \tag{6}$$

Theorem 2. *Let us assume that under the conditions of Theorem 1 instead of the condition 2) the following is true: there is* $\delta_0=\delta_0(z_0)>0$ *such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{z_0}(t)}=\infty.$$

Then any $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *has a continuous extension to* z_0 *and, in addition, there is* $\widetilde{\varepsilon_0}=\widetilde{\varepsilon_0}(z_0)>0$ *such that*

$$|f(z)-f(z_0)|\leqslant\frac{32(1+R_0^2)}{\delta}\exp\left\{-\frac{1}{A_0}\int\limits_{|z-z_0|}^{\delta_0}\frac{dt}{tq_{z_0}(t)}\right\}$$

holds for any $z\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ *and any* $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$.

Let us formulate analogs of Theorems 1 and 2 for mappings with branching. For this end, consider the definition of the following class of mappings. Given numbers $A_0>0$, $R_0>0$, $N\in\mathbb{N}$ and $\delta>0$, a domain $D\subset\mathbb{C}$ and a Lebesgue measurable function $Q:D\rightarrow[0,\infty]$, we denote $\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ the family of all open, discrete and closed mappings $f:D\rightarrow B(0,R_0)$ with a finite distortion such that $K_{\mu_f}(z)\leqslant Q(z)$ for almost all $z\in D$, $N(f,D)\leqslant N$, the domain $D_f^{\,\prime}=f(D)$ satisfies the condition (5) with $G=D_f^{\,\prime}$, and, in addition, there exists a path connected continuum $K_f\subset D_f^{\,\prime}$ with ${\rm diam}\,(K_f)\geqslant\delta$ and $h(f^{-1}(K_f),\partial D)\geqslant\delta>0$.

Theorem 3. *Let* $z_0\in\partial D$, $z_0\ne\infty$, *and let* δ *be some positive number. Assume that there is* $\varepsilon_0=\varepsilon_0(z_0)>0, 0<\varepsilon_0<\min\{\delta,1\}$, *such that the following conditions hold:*

1) the set $B(z_0,r)\cap D$ *is connected for any* $0<r<\varepsilon_0$;

2) $Q\in FMO(z_0)$.

Then any $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ *has a continuous extension to* z_0 *and, in addition, there are* $\widetilde{C}=\widetilde{C}(\delta,R_0,z_0,Q,N)>0$ *and* $\widetilde{\varepsilon_0}=\widetilde{\varepsilon_0}(z_0)>0$ *such that*

$$|f(z)-f(z_0)|\leqslant\frac{\widetilde{C}}{\left(\log\frac{1}{|z-z_0|}\right)^{\frac{2\pi}{A_0K_0N}}}$$

holds for any $z\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ *and any* $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$, *where* $K_0>0$ *is some constant depending only on the function* Q.

Theorem 4. *Assume that under the conditions of Theorem 3 instead of the condition 2) the following is true: there is* $\delta_0=\delta_0(z_0)>0$ *such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{z_0}(t)}=\infty.$$

Then f *has a continuous extension to* z_0 *and, in addition, there is* $\widetilde{\varepsilon_0}=\widetilde{\varepsilon_0}(z_0)>0$ *such that the inequality*

$$|f(z)-f(z_0)|\leqslant\frac{32(1+R_0^2)}{\delta}\exp\left\{-\frac{1}{NA_0}\int\limits_{|z-z_0|}^{\delta_0}\frac{dt}{tq_{z_0}(t)}\right\}$$

holds for any $z\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ *and any* $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$.

The results similar to Theorems 1 – 4 hold also for domains with bad boundaries. Let us recall some definitions (see, e.g., [5,6,14]).

Let ω be an open set in $\mathbb{R}^1$. A continuous mapping $\sigma\colon\,\omega\rightarrow\mathbb{C}$ is called a *dashed line* in $\mathbb{C}$. A dashed line σ is called *Jordan*, if $\sigma(x)\neq\sigma(y)$ for $x\neq y$. In the following, we use σ instead of $\sigma(\omega)\subset\mathbb{C}$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$ and $\partial\sigma$ instead of $\overline{\sigma(\omega)}\setminus\sigma(\omega)$. A Jordan dashed line $\sigma\colon\,\omega\rightarrow D$ is called a *cut* of D, if σ separates D, that is $D\setminus\sigma$ has more than one component, $\partial\sigma\cap D=\varnothing$ and $\partial\sigma\cap\partial D\neq\varnothing$.

A sequence of cuts $\sigma_1,\sigma_2,\ldots,\sigma_m,\ldots$ in D is called *a chain*, if

(i) the set σ_{m+1} is contained in exactly one component d_m of the set $D\setminus\sigma_m$, wherein $\sigma_{m-1}\subset D\setminus(\sigma_m\cup d_m)$;

(ii) $\bigcap\limits_{m=1}^{\infty}\,d_m=\varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma_k^{\prime}\}$ are called *equivalent*, if for each $m=1,2,\ldots$ the domain d_m contains all the domains $d_k^{\prime}$, except for a finite number, and for each $k=1,2,\ldots$ the domain $d_k^{\prime}$ also contains all domains d_m, except for a finite number.

The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{C}$, then the set $I(K)=\bigcap\limits_{m=1}^{\infty}\overline{d_m}$ is called *the impression of the end* K. One may to prove that, $I(P)\subset\partial D$ (see, e.g., [6, Proposition 1]).

Following [14], we say that the end K is *a prime end*, if K contains a chain of cuts $\{\sigma_m\}$ such that

$$\lim\limits_{m\rightarrow\infty}M(\Gamma(C,\sigma_m,D))=0 \tag{7}$$

for some continuum C in D. The set of prime ends in D is denoted by E_D, and the completion of the domain D by its prime ends is denoted $\overline{D}_P$. Consider the following definition, which goes back to R. Näkki [14] (see also [6]). We say that the boundary of the domain D in $\mathbb{C}$ is *locally quasiconformal*, if each point $x_0\in\partial D$ has a neighborhood U in $\mathbb{C}$, which can be mapped by a quasiconformal mapping φ onto the unit disk $\mathbb{D}\subset\mathbb{C}$ so that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{D}$ with the coordinate axis.

The sequence of cuts σ_m, $m=1,2,\ldots$, is called *regular*, if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in\mathbb{N}$ and, in addition, $d(\sigma_m)\rightarrow 0$ as $m\rightarrow\infty$, where $d(\sigma_m)=\sup\limits_{x,y\in\sigma_m}|x-y|$. If the end K contains at least one regular chain, then K will be called *regular*. We say that a bounded domain D in $\mathbb{C}$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{C}$, and, besides that, every prime end in D is regular. Note that space $\overline{D}_P=D\cup E_D$ is metrizable, which can be demonstrated as follows. If $g:D_0\rightarrow D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x,y\in\overline{D}_P$ we put

$$\rho(x,y):=|g^{-1}(x)-g^{-1}(y)|\,, \tag{8}$$

where the element $g^{-1}(x)$, $x\in E_D$, is to be understood as some (single) boundary point of the domain D_0. It is easy to verify that ρ in (8) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$ does not depend on the choice of the map g with the indicated property.

We say that a sequence $x_m\in D$, $m=1,2,\ldots$, converges to a prime end of $P\in E_D$ as $m\rightarrow\infty$, if for any $k\in\mathbb{N}$ all elements x_m belong to d_k except for a finite number. Here d_k denotes a sequence of nested domains corresponding to the definition of the prime end P.

Theorem 5. *Let $P_0\in E_D$, and A be a continuum in D. Assume that D is a regular domain, and the following conditions are fulfilled:*

1) for each $y_0\in\partial D$ there exists $\varepsilon_0=\varepsilon_0(y_0)$, $0<\varepsilon_0<\min\{{\rm dist}\,(z_0, A), 1\}$, such that the set $B(y_0, r)\cap D$ is finitely connected for all $0<r<\varepsilon_0$;

2) for any $0<r<\varepsilon_0$ and for each component K of $\overline{\text{the set }B(y_0, r)\cap D\text{, any }z, y}\in K$ may be joined by a path $\gamma:[a, b]\rightarrow {\Bbb C}$ such that $|\gamma|\in K\cap \overline{B(z_0, \max\{|z-z_0|, |y-z_0|\})}$ with $|\gamma|=\{z\in {\Bbb C}: \exists\,t\in[a, b]: \gamma(t)=z\}$;

3) the condition $Q\in FMO(z_0)$ holds for any $z_0\in \partial D$.

Then $f\in\mathfrak{F}^{A_0, R_0}_{Q, A, \delta}(D)$ has a continuous extension to P_0 and, in addition, there are $\widetilde{C}=\widetilde{C}(\delta, R_0, z_0, Q)>0$ and a neighborhood $U=U(P_0)$ in $\overline{D}_P$ such that the inequality

$$|f(z)-f(P_0)|\leqslant\frac{\widetilde{C}}{\left(\log\frac{1}{|z-z_0|}\right)^{\frac{2\pi}{A_0K_0}}}$$

holds for any $z\in U\cap D$ and any $f\in\mathfrak{F}^{A_0, R_0}_{A, \delta}(D)$, where $K_0>0$ is some constant depending only on the function Q and $\{z_0\}=I(P_0)$.

Theorem 6. *Assume that under the conditions of Theorem 5 instead of the condition 3) the following holds: for every $z_0\in\partial D$ there is $\delta_0=\delta_0(z_0)>0$ such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{z_0}(t)}=\infty.$$

Then f has a continuous extension to P_0 and, in addition, there are $\widetilde{\varepsilon_0}=\widetilde{\varepsilon_0}(P_0)>0$ and a neighborhood U of P_0 in $\overline{D}_P$ such that the inequality

$$|f(z)-f(P_0)|\leqslant\frac{32(1+R_0^2)}{\delta}\exp\left\{-\frac{1}{A_0}\int\limits_{|z-z_0|}^{\delta_0}\frac{dt}{tq_{z_0}(t)}\right\}$$

holds for any $z\in U\cap D$ and any $f\in\mathfrak{F}^{A_0, R_0}_{Q, A, \delta}(D)$, where $\{z_0\}=I(P_0)$.

Theorem 7. *Let $P_0\in E_D$ and δ be some positive number. Assume that D is a regular domain, and the following conditions are fulfilled:*

1) for each $y_0\in\partial D$ there exists $\varepsilon_0=\varepsilon_0(y_0)$, where $0<\varepsilon_0<\min\{\delta, 1\}$, such that the set $B(y_0, r)\cap D$ is finitely connected for all $0<r<\varepsilon_0$;

2) for any $0<r<\varepsilon_0$ and for each component K of $\overline{\text{the set }B(y_0, r)\cap D\text{, any }z, y}\in K$ may be joined by a path $\gamma:[a, b]\rightarrow {\Bbb C}$ such that $|\gamma|\in K\cap \overline{B(z_0, \max\{|z-z_0|, |y-z_0|\})}$ with $|\gamma|=\{z\in {\Bbb C}: \exists\,t\in[a, b]: \gamma(t)=z\}$;

3) the condition $Q\in FMO(z_0)$ holds for any $z_0\in \partial D$.

Then $f\in\mathfrak{R}^{A_0, R_0}_{Q, \delta, N}(D)$ has a continuous extension to P_0 and, in addition, there are $\widetilde{C}=\widetilde{C}(\delta, R_0, z_0, Q, N)>0$ and a neighborhood $U=U(P_0)$ in $\overline{D}_P$ such that the inequality

$$|f(z)-f(P_0)|\leqslant\frac{\widetilde{C}}{\left(\log\frac{1}{|z-z_0|}\right)^{\frac{2\pi}{A_0K_0N}}}$$

holds for any $z\in U\cap D$ and any $f\in\mathfrak{R}^{A_0, R_0}_{Q, \delta, N}(D)$, where $K_0>0$ is some constant depending only on the function Q and $\{z_0\}=I(P_0)$.

Theorem 8. *Let us assume that under the conditions of Theorem 7 instead of condition 3) the following is true: for every* $z_0\in\partial D$ *there is* $\delta_0=\delta_0(z_0)>0$ *such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{z_0}(t)}=\infty.$$

Then f *has a continuous extension to* P_0 *and, in addition, there are* $\widetilde{C}=\widetilde{C}(\delta, R_0, z_0, Q, N)>0$ *and a neighborhood* U *of* P_0 *in* $\overline{D}_P$ *such that the inequality*

$$|f(z)-f(P_0)|\leqslant\frac{32(1+R_0^2)}{\delta}\exp\left\{-\frac{1}{A_0N}\int\limits_{|z-z_0|}^{\varepsilon_0}\frac{dt}{tq_{z_0}(t)}\right\}$$

holds for any $z\in U\cap D$ *and any* $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$*, where* $\{z_0\}=I(P_0)$*.*

2 Preliminaries

Let $a>0$ and $\varphi\colon[a,\infty)\rightarrow[0,\infty)$ be a nondecreasing function such that for some constants $\gamma>0$, $T>0$ and all $t\geqslant T$ the inequality

$$\varphi(2t)\leqslant\gamma\cdot\varphi(t)\tag{9}$$

is fulfilled. We say that such functions *satisfy the doubling condition.*

Let $\varphi\colon[a,\infty)\rightarrow[0,\infty)$ be a function with the doubling condition, then the function $\widetilde{\varphi}(t):=\varphi(1/t)$ does not increase and is defined on a half-interval $(0,1/a]$.

Proposition 1 ([16, Lemma 3.1])**.** *Let* $a>0$ *and* $\varphi\colon[a,\infty)\rightarrow[0,\infty)$ *be a nondecreasing function with a doubling condition* (9)*. Let* $z_0\in\mathbb{C}$ *and* $Q:\mathbb{C}\rightarrow[0,\infty]$ *be a Lebesgue measurable function for which there exists* $0<C<\infty$ *such that*

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{\varphi(1/\varepsilon)}{\pi\cdot\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}Q(z)\,dm(z)\leqslant C.$$

Then there exists $\varepsilon_0^{\prime}>0$ *such that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0^{\prime}}\frac{\varphi(1/|z-z_0|)Q(z)\,dm(z)}{|z-z_0|^2}\leqslant C_1\cdot\left(\log\frac{1}{\varepsilon}\right)\qquad as\ \varepsilon\rightarrow 0,$$

where $C_1:=\frac{4\gamma\pi C}{\log 2}$*.*

A domain R in $\overline{\mathbb{C}}$ is called *a ring,* if $\overline{\mathbb{C}}\setminus R$ consists of exactly two components E and F. In this case, we write $R=R(E,F)$. The following statement is true (see [8, ratio (7.29)]).

Proposition 2. *If* $R=R(E,F)$ *is a ring, then*

$$M(\Gamma(E,F,\overline{\mathbb{C}}))\geqslant\frac{2\pi}{\log\frac{32}{h(E)h(F)}},$$

where $h(E)$ *denotes the chordal diameter of the set* E *defined in* (4)*.*

Let $D\subset\mathbb{C}$, $f:D\rightarrow\mathbb{C}$ be an open discrete mapping, let $\beta:[a,\,b)\rightarrow\mathbb{C}$ be a path and let $z\in f^{-1}(\beta(a))$. A path $\alpha:[a,\,c)\rightarrow D$ is called a *maximal f-lifting* of β with the origin at the point z, if $\alpha(a)=z$; $f\circ\alpha=\beta|_{[a,\,c)}$; and for every $c<c^{\prime}\leqslant b$, there is no a path $\alpha^{\prime}:[a,\,c^{\prime})\rightarrow D$ such that $\alpha=\alpha^{\prime}|_{[a,\,c)}$ and $f\circ\alpha^{\prime}=\beta|_{[a,\,c^{\prime})}$.

The following statement is true (see [12, Lemma 3.12], cf. [20, Lemma 3.7]).

Proposition 3. *Let $f : D \to \mathbb{C}$ be an open discrete mapping. Let $z_0 \in D$ and $\beta : [a, b) \to \mathbb{C}$ be a path such that $\beta(a) = f(z_0)$ and either $\lim\limits_{t\to b}\beta(t)$ exists, or $\beta(t) \to \partial f(D)$ as $t \to b$. Then β has a maximal f-lifting $\alpha : [a, c) \to D$ starting at the point z_0. If $\alpha(t) \to z_1 \in D$ as $t \to c$, then $c = b$ and $f(z_1) = \lim\limits_{t\to b}\beta(t)$. Otherwise, $\alpha(t) \to \partial D$ as $t \to c$.*

Proposition 4 ([17, Theorem 3]). *Let $f : D \to \mathbb{C}$ be an open, discrete and closed bounded mapping of a finite distortion. Assume that $K_{\mu_f}(z) \in L^1_{\rm loc}(D)$. Then for any $z_0 \in \partial D$, any $\varepsilon_0 < d_0 := \sup\limits_{z\in D}|z - z_0|$ and any compactum $C_2 \subset D \setminus B(z_0, \varepsilon_0)$ there is ε_1, $0 < \varepsilon_1 < \varepsilon_0$, such that the relation*

$$M(f(\Gamma(C_1, C_2, D))) \leqslant \int\limits_{A(z_0,\varepsilon,\varepsilon_1)\cap D} Q(z) \cdot \eta^2(|z - z_0|)\, dm(z) \tag{10}$$

holds for any $\varepsilon \in (0, \varepsilon_1)$ and any $C_1 \subset \overline{B(z_0, \varepsilon)} \cap D$, where $Q(z) := N(f, D)K_{\mu_f}(z)$, $A(z_0, \varepsilon, \varepsilon_1)$ is defined in (1), and $\eta : (\varepsilon, \varepsilon_1) \to [0, \infty]$ is arbitrary Lebesgue measurable function satisfying the relation

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \tag{11}$$

Here $N(f, D)$ is defined in (2). Note that, $N(f, D) < \infty$ for open, discrete and closed mappings (see, e.g., [9, Lemma 3.3]).

3 Main lemmas for the case of good boundaries

We should note that the proof of the main results of the article is based on auxiliary statements of a more general plan. Each of the main results corresponds to exactly one such statement, and thus we have to consider each of them separately. We will start with the simplest case, when the domain of the mapping is locally connected at its boundary and the mappings themselves are homeomorphisms. The following statement holds.

Lemma 1. *Let $z_0 \in \partial D$, $z_0 \ne \infty$, $1 \leqslant p < 2$, and A be a continuum in D. Let $Q : \mathbb{C} \to [0, \infty]$ be a Lebesgue measurable function in $\mathbb{C}$ vanishing outside D. Assume that there is $\varepsilon_0 = \varepsilon_0(z_0) > 0$, $0 < \varepsilon_0 < \min\{{\rm dist}\,(z_0, A), 1\}$, such that the following conditions hold:*

1) the set $B(z_0, r) \cap D$ is connected for any $0 < r < \varepsilon_0$;

2) there is a Lebesgue measurable function $\psi : (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, such that the relation

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\, dm(z) \leqslant K_0 \cdot I^p(\varepsilon, \varepsilon_0) \tag{12}$$

holds as $\varepsilon \to 0$ for some $K_0 > 0$, where

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \tag{13}$$

for sufficiently small $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then there is $\widetilde{\varepsilon_0} = \widetilde{\varepsilon_0}(z_0) > 0$ such that the relation

$$|f(z) - f(y)| \leqslant \frac{32(1 + R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{K_0} I^{p-2}(|z - z_0|, \varepsilon_0)\right\} \tag{14}$$

holds for any $f \in \mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ and any $z, y \in B(z_0, \widetilde{\varepsilon_0}) \cap D$ such that $|z - z_0| \geqslant |y - y_0|$.

Proof. Let $\varepsilon_1 > 0$ be a number from Proposition 4 corresponding to ε_0 mentioned above. By the assumption, we may find $0 < \varepsilon_2 < \varepsilon_1$ such that the relations (12)–(13) hold for any $\varepsilon \in (0, \varepsilon_2)$. Set $\widetilde{\varepsilon} := \varepsilon_2$.

Let $z, y \in B(z_0, \widetilde{\varepsilon})$ and let $f \in \mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$. Now $z, y \in B(z_0, \varepsilon_0)$. Without loss of generality we may assume that $|z - z_0| \geqslant |y - z_0|$.

By the definition of ε_0, we obtain $A \subset D \setminus B(z_0, \varepsilon_0)$. Since the points of $S(z_0, r) \cap D$, $0 < r < \varepsilon_0$, are accessible from the domain D by means of some path γ, and the set $B(z_0, r) \cap D$ is connected for all $0 < r < \varepsilon_0$, we may join points z and y by a path K which completely belongs to the ball $\overline{B(z_0, |z - z_0|)}$ and lies in D. We may assume that K is a Jordan path. Let $z, w \in f(A) \subset D'_f = f(D)$ and $u, v \in A$ be such that

$$\operatorname{diam} f(A) = |z - w| = |f(u) - f(v)| \geqslant \delta\,. \tag{15}$$

Since the continuum A is path connected, one can join the points u, v by a path K' inside A. Due to (15), we obtain that

$$\operatorname{diam} f(|K'|) \geqslant |f(u) - f(v)| \geqslant \delta\,. \tag{16}$$

We also may consider that K' is Jordan.

According to Antoine's theorem on the absence of wild arcs (see [7, Theorem II.4.3]), there exists a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, which maps $f(|K|)$ onto some segment I. It follows that any points $z, y \in \mathbb{R}^2 \setminus f(|K|)$ may be joined by a path γ in $\mathbb{R}^2 \setminus f(|K|)$. Reasoning similarly, we may show that any points $z, y \in \mathbb{R}^2 \setminus (f(|K|) \cup f(|K'|))$ may be joined by a path γ in $\mathbb{R}^2 \setminus (f(|K|) \cup f(|K'|))$.

Therefore, $R = R(f(|K|), f(|K'|))$ is a ring domain. We set $\Gamma = \Gamma(f(|K|)), f(|K'|), \mathbb{C})$. By Proposition 2 we have

$$M(\Gamma(f(|K|), f(|K'|), \overline{\mathbb{C}})) \geqslant \frac{2\pi}{\log \frac{32}{h(f(|K|))h(f(|K'|))}}\,. \tag{17}$$

By (16) and (17), and by the definition of K, we obtain

$$M(\Gamma(f(|K|), f(|K'|), \overline{\mathbb{C}})) \geqslant \frac{2\pi}{\log \frac{32}{\delta \cdot h(f(z), f(y))}}\,. \tag{18}$$

Since D'_f is a QED-domain with a constant A_0 in (5), by the condition (18) we get

$$M(\Gamma(f(|K|), f(|K'|), D'_f)) \geqslant \frac{2\pi}{A_0 \log \frac{32}{\delta \cdot h(f(z), f(y))}}\,. \tag{19}$$

Recall that $|K| \subset B(z_0, |z - z_0|)$, and $|K'| \subset A \subset D \setminus B(z_0, |z - z_0|)$. Now, by Proposition 4 and due to the condition $K_{\mu_f}(z) \leqslant Q(z)$ a.e., we obtain

$$\begin{aligned} M(\Gamma(f(|K|), f(|K'|), D'_f)) &= M(f(\Gamma(|K|, |K'|, D))) \\ &\leqslant \int_{A(z_0, |z - z_0|, \varepsilon_0)} Q(x) \cdot \eta^2(|x - z_0|)\, dm(x), \end{aligned} \tag{20}$$

where η is an arbitrary Lebesgue measurable function satisfying the condition (11) for $r_1 = |z - z_0|$, $r_2 = \varepsilon_0$. Now, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|z - z_0|, \varepsilon_0)}, & t \in (|z - z_0|, \varepsilon_0), \\ 0, & t \notin (|z - z_0|, \varepsilon_0). \end{cases}$$

Observe that the function η satisfies the condition (11) for $r_1 = |z - z_0|$ and $r_2 = \varepsilon_0$. Now, by (20), due to (12) we obtain

$$\begin{aligned} M(\Gamma(f(|K|), f(|K'|), D_f')) \\ &\leqslant \frac{1}{I^2(|z-z_0|,\varepsilon_0)} \int_{A(z_0,|z-z_0|,\varepsilon_0)} \frac{Q(x)}{|x-z_0|^2}\, dm(x) \\ &\leqslant K_0 \cdot I^{p-2}(|z-z_0|,\varepsilon_0). \end{aligned} \tag{21}$$

Combining (19) and (21), we obtain

$$\frac{2\pi}{A_0 \log \frac{32}{\delta \cdot h(f(z),f(y))}} \leqslant K_0 \cdot I^{p-2}(|z-z_0|,\varepsilon_0). \tag{22}$$

By direct calculations from (22) we get

$$h(f(z),f(y)) \leqslant \frac{32}{\delta} \exp\left\{-\frac{2\pi}{A_0K_0} I^{2-p}(|z-z_0|,\varepsilon_0)\right\}. \tag{23}$$

By the definition of the chordal distance in (3) and due to the condition $f(D) = D_f' \subset B(0, R_0)$, we obtain

$$h(f(z),f(y)) \geqslant \frac{|f(z)-f(y)|}{1+R_0^2}. \tag{24}$$

Then, by (23), we have that

$$|f(z)-f(y)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0K_0} I^{2-p}(|z-z_0|,\varepsilon_0)\right\}. \tag{25}$$

This relation (25) completes the proof. $\square$

Corollary 1. *Let* $1 < p < 2$. *Let, in the notations of Lemma 1,* $I(\varepsilon, \varepsilon_0) \to \infty$ *as* $\varepsilon \to 0$. *Then, under the conditions of Lemma 1, any* $f \in \mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *has a continuous extension to* z_0 *and, in addition, we have*

$$|f(z)-f(z_0)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0K_0} I^{2-p}(|z-z_0|,\varepsilon_0)\right\} \tag{26}$$

for any $f \in \mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *and any* $z \in B(z_0, \widetilde{\varepsilon_0}) \cap D$.

Proof. If f has no limit as $z \to z_0$, then we may construct at least two sequences $z_m \to x_0$ and $y_m \to z_0$, $m \to \infty$, such that $|f(z_m) - f(y_m)| \geqslant \delta > 0$ for some $\delta > 0$ and all $m = 1, 2, \ldots$. But this contradicts the inequality (14) because $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$ by the assumption. Therefore, the limit of f as $z \to z_0$ exists. To prove the inequality (26), it remains to pass in the relation (14) to the limit as $y \to z_0$. $\square$

Now let us move on to the study of the case when the considered mapping may have branch points. At first, we will also limit ourselves to the case of good domains, i.e. when the domain is locally connected at its boundary. The following statement is similar to Lemma 1, but applies to mappings with branching.

Lemma 2. *Let $z_0 \in \partial D$, $z_0 \neq \infty$, $1 \leqslant p < 2$ and δ be some positive number. Assume that there is $\varepsilon_0 = \varepsilon_0(z_0) > 0$, $0 < \varepsilon_0 < \min\{\delta, 1\}$, such that the following conditions hold:*

1) the set $B(z_0, r) \cap D$ is connected for any $0 < r < \varepsilon_0$;

2) there is a Lebesgue measurable function $\psi : (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, such that the relation

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\cdot\psi^2(|z-z_0|)\,dm(z) \leqslant K_0 \cdot I^p(\varepsilon, \varepsilon_0) \tag{27}$$

holds as $\varepsilon \to 0$ for some $K_0 > 0$, where

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt < \infty \tag{28}$$

for sufficiently small $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then there is $\widetilde{\varepsilon_0} = \widetilde{\varepsilon_0}(z_0) > 0$ such that the relation

$$|f(z) - f(y)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{K_0 N} I^{p-2}(|z-z_0|, \varepsilon_0)\right\}$$

holds for any $f \in \mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ and any $z, y \in B(z_0, \widetilde{\varepsilon_0}) \cap D$ such that $|z - z_0| \geqslant |y - y_0|$.

Proof. Let $\varepsilon_1 > 0$ be a number from Proposition 4 corresponding to ε_0 mentioned above. By the assumption, we may find $0 < \varepsilon_2 < \varepsilon_1$ such that the relations (27)–(28) hold for any $\varepsilon \in (0, \varepsilon_2)$. Set $\widetilde{\varepsilon} := \varepsilon_2$.

Let $z, y \in B(z_0, \widetilde{\varepsilon_0})$ and let $f \in \mathfrak{R}^{A_0,R_0}_{Q,\delta}(D)$. Now, $z, y \in B(z_0, \varepsilon_0)$. Without loss of a generalization, we may assume that $|z - z_0| \geqslant |y - z_0|$.

Let $K_f \subset D_f^{\prime}$ be a path connected continuum such that $\operatorname{diam}(K_f) \geqslant \delta$ and, in addition, $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$ (such a continuum exists by the definition of the class $\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$). By the definition of ε_0, we have $f^{-1}(K_f) \subset D \setminus B(z_0, \varepsilon_0)$. Since the points of $S(z_0, r) \cap D$, $0 < r < \varepsilon_0$, are accessible from the domain D by means of some path γ, and the set $B(z_0, r) \cap D$ is connected for all $0 < r < \varepsilon_0$, the points z and y may be joined by a path K, which belongs entirely to the ball $\overline{B(z_0, |z - z_0|)}$ and belongs to D. We may consider that K is a Jordan path. Let $z, w \in K_f \subset D_f^{\prime}$ be such that

$$\operatorname{diam} K_f = |z - w| \geqslant \delta\,. \tag{29}$$

Since K_f is path connected, we may join points z, w by a Jordan path K' inside K_f. Due to (29) we obtain that

$$\operatorname{diam}|K'| \geqslant |z - w| \geqslant \delta\,. \tag{30}$$

If the path $f(|K|))$ is not Jordan, we discard from $f(|K|))$ no more than a finite number of its loops. Let $\Delta \subset f(|K|))$ be a locus of the Jordan path, which is obtained by such rejection. Just as in the proof of Lemma 1, it may be shown that $R = R(\Delta, |K'|)$ is a ring in $\mathbb{C}$.

In this case, we denote $\Gamma = \Gamma(\Delta, |K'|, \mathbb{C})$. Now, by Proposition 2, we get

$$M(\Gamma(\Delta, |K'|, \mathbb{C})) \geqslant \frac{2\pi}{\log\frac{32}{h(|K'|)h(\Delta)}}\,. \tag{31}$$

Due to (30) and (31), and by the definition of the path K', we obtain

$$M(\Gamma(\Delta, |K'|, \mathbb{C})) \geqslant \frac{2\pi}{\log\frac{32}{\delta\cdot h(f(z), f(y))}}\,. \tag{32}$$

Since D_f' is a QED-domain with a constant A_0 in (5), by (32) we obtain that

$$M(\Gamma(\Delta, |K'|, D_f')) \geqslant \frac{2\pi}{A_0 \log \frac{32}{\delta \cdot h(f(z), f(y))}}. \tag{33}$$

Let Γ^* be a family $\gamma : [0,1) \to D$ of maximal f-liftings of paths $\gamma' : [0,1] \to D_f'$ from $\Gamma = \Gamma(\Delta, |K'|, D_f')$ starting at $|K|$ (such liftings exist by Proposition 3). By the same proposition, due to the closeness of f, we obtain that each path $\gamma \in \Gamma^*$ has an extension $\gamma : [0,1] \to D$ to $b = 1$. Then $\gamma(1) \in f^{-1}(K_f)$, that is, $\Gamma^* \subset \Gamma(|K|, f^{-1}(K_f), D)$.

Observe that $f(\Gamma^*) = \Gamma(\Delta, |K'|, D_f')$. In this case, by Proposition 4 and due to the condition $K_{\mu_f}(z) \leqslant Q(z)$ a.e., we obtain

$$M(f(\Gamma^*)) = M(\Gamma(\Delta, |K'|, D_f')) \leqslant N \int_{A(z_0, |z-z_0|, \varepsilon)} Q(x) \eta^2(|x - z_0|)\, dm(x), \tag{34}$$

where η is an arbitrary nonnegative Lebesgue measurable function satisfying the relation (11) for $r_1 = |z - z_0|$, $r_2 = \varepsilon_0$, and $A(z_0, |z - z_0|, \varepsilon)$ is defined in (1). Now, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|z-z_0|, \varepsilon_0)}, & t \in (|z - z_0|, \varepsilon_0), \\ 0, & t \notin (|z - z_0|, \varepsilon_0). \end{cases}$$

Arguing similarly to the proof of the relations (21)–(25), by (34) we obtain

$$M(\Gamma(\Delta, |K'|, D_f')) \leqslant K_0 N I^{p-2}(|z - z_0|, \varepsilon_0). \tag{35}$$

Combining (33) and (35), we get

$$\frac{2\pi}{A_0 \log \frac{32}{\delta \cdot h(f(z), f(y))}} \leqslant K_0 N I^{p-2}(|z - z_0|, \varepsilon_0). \tag{36}$$

By direct calculations from (36) we obtain that

$$h(f(z), f(y)) \leqslant \frac{32}{\delta} \exp\left\{ -\frac{2\pi}{A_0 K_0 N} I^{2-p}(|z - z_0|, \varepsilon_0) \right\}. \tag{37}$$

Due to (24), by (37) we get

$$|f(z) - f(y)| \leqslant \frac{32(1 + R_0^2)}{\delta} \exp\left\{ -\frac{2\pi}{A_0 K_0 N} I^{2-p}(|z - z_0|, \varepsilon_0) \right\},$$

as required. □

Corollary 2. *Let $1 < p < 2$. Let, in the notations of Lemma 2, $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$. Then, under conditions of Lemma 2, f has a continuous extension to z_0 and*

$$|f(z) - f(z_0)| \leqslant \frac{32(1 + R_0^2)}{\delta} \exp\left\{ -\frac{2\pi}{A_0 K_0 N} I^{2-p}(|z - z_0|, \varepsilon_0) \right\}$$

for any $f \in \mathfrak{R}_{Q,\delta,N}^{A_0,R_0}(D)$ and any $z \in B(z_0, \widetilde{\varepsilon_0}) \cap D$.

The proof of Corollary 2 is similar to the proof of Corollary 1.

4 Domains with prime ends

The statements similar to Lemmas 1–2 also hold for domains with not very good boundaries. However, their wording will change accordingly. The following assertions are true.

Lemma 3. *Let* $P_0 \in E_D$ *and* A *be a continuum in* D. *Assume that* D *is a regular domain, and the following conditions are fulfilled:*

1) for each $y_0 \in \partial D$ *there exists* $\varepsilon_0 = \varepsilon_0(y_0)$, $0 < \varepsilon_0 < \min\{\operatorname{dist}(z_0, A), 1\}$, *such that the set* $B(y_0, r) \cap D$ *is finitely connected for all* $0 < r < \varepsilon_0$;

2) for any $0 < r < \varepsilon_0$ *and for each component* K *of the set* $B(y_0, r) \cap D$, *any* $z, y \in K$ *may be joined by a path* $\gamma : [a, b] \to \mathbb{C}$ *such that* $|\gamma| \in K \cap \overline{B(z_0, \max\{|z - z_0|, |y - z_0|\})}$ *with*

$$|\gamma| = \{z \in \mathbb{C} : \exists\, t \in [a, b] : \gamma(t) = z\};$$

3) for each $z_0 \in I(P_0)$ *there exist a real* $p < 2$ *and a Lebesgue measurable function* $\psi : (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, *such that the relation*

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z) \cdot \psi^2(|z - z_0|)\, dm(z) \leqslant K_0 \cdot I^p(\varepsilon, \varepsilon_0) \tag{38}$$

holds as $\varepsilon \to 0$ *for some* $K_0 > 0$, *where*

$$0 < I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \tag{39}$$

for sufficiently small $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then for each $P \in E_D$ *there exists* $y_0 \in \partial D$ *such that* $I(P) = \{y_0\}$, *where* $I(P)$ *denotes the impression of* P. *In addition, there exists a neighborhood* $U = U(P_0)$ *of* P_0 *such that the relation*

$$|f(z) - f(y)| \leqslant \frac{32(1 + R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0 K_0} I^{2-p}(|z - z_0|, \varepsilon_0)\right\} \tag{40}$$

holds for any $f \in \mathfrak{F}_{Q,A,\delta}^{A_0,R_0}(D)$ *and any* $z, y \in U \cap D$ *such that* $|z - z_0| \geqslant |y - z_0|$, *where* $I(P_0) = \{z_0\}$.

Proof. Since the set $B(y_0, r) \cap D$ is finitely connected for all $0 < r < \varepsilon_0$, the domain D is finitely connected on its boundary. Therefore, the domain D is uniform in the sense of R. Näkki (see [13, Theorem 3.2]). In other words, for every $r > 0$ there exists a number $\delta > 0$ such that the inequality

$$M(\Gamma(F^*, F, D)) \geqslant \delta$$

holds for all continua $F, F^* \subset D$ such that $h(F) \geqslant r$ and $h(F^*) \geqslant r$, where h is a chordal metric defined in (3).

Let us prove that for any $P \in E_D$ there exists $y_0 \in \partial D$ such that $I(P) = \{y_0\}$. We prove this statement from the opposite, namely, suppose that there is $P \in E_D$ such that $I(P)$ contains two points $z, y \in \partial D$, $z \neq y$. In this case, there are at least two sequences $z_m, y_m \in d_m$, $m = 1, 2, \ldots$, which converge to z and y as $m \to \infty$, respectively (here d_m denotes the sequence of decreasing domains formed by some sequence of cuts σ_m, $m = 1, 2, \ldots$, corresponding to the prime end P). Let us join the points z_m and y_m with the path γ_m in the domain d_m. Since $z \neq y$, there exists $m_0 \in \mathbb{N}$ such that $h(|\gamma_m|) \geqslant d(z, y)/2$, $m > m_0$. Choose any nondegenerate continuum $C \subset D \setminus d_1$. Then, due to the uniformity of the domain D, we get

$$M(\Gamma(|\gamma_m|, C, D)) \geqslant \delta_0 > 0 \tag{41}$$

for some $\delta_0 > 0$ and all $m > m_0$. The relation (41) contradicts with the definition of the prime end P. Indeed, by the definition of a cut σ_m, we obtain that $\Gamma(|\gamma_m|, C, D) > \Gamma(\sigma_m, C, D)$. Now, due to (7), we have that

$$M(\Gamma(|\gamma_m|, C, D)) \leqslant M(\Gamma(\sigma_m, C, D)) \to 0$$

as $m \to \infty$. The latter relation contradicts with (41). Thus, $I(P) = \{y_0\}$ for some $y_0 \in \partial D$.

It remains to prove the relation (40). Let $\varepsilon_1 > 0$ be a number from Proposition 4 corresponding to ε_0 mentioned above. By the assumption, we may find $0 < \varepsilon_2 < \varepsilon_1$ such that the relations (38)–(39) hold for any $\varepsilon \in (0, \varepsilon_2)$.

By the proving above, there is $z_0 \in \partial D$ such that $I(P_0) = \{z_0\}$. It follows that there is a neighborhood U in $\overline{D}_P$ containing P_0 such that $U \cap D \subset B(z_0, \varepsilon_2)$. By the definition of a regular domain, U may be chosen such that $U \cap D$ is connected. Let $z, y \in U$ and let $f \in \mathfrak{F}^{A_0, R_0}_{Q, A, \delta}(D)$. We may consider that $|z - z_0| \geqslant |y - z_0|$. By the definition of ε_0, the points z and y may be joined by the path K, which is contained in the ball $\overline{B(z_0, |z - z_0|)}$. The further course of the proof is very similar to the proof of Lemma 1. We may assume that the path K is Jordan. Let also $z, w \in f(A) \subset D^{\,\prime}_f$ and $u, v \in A$ be such that

$$\operatorname{diam} f(A) = |z - w| = |f(u) - f(v)| \geqslant \delta\,. \tag{42}$$

Since A is path connected, it is possible to join the points u, v by the path K' inside A. Due to (42), we obtain

$$\operatorname{diam} f(|K'|) \geqslant |f(u) - f(v)| \geqslant \delta\,. \tag{43}$$

We may assume that the K' is also Jordan. Note that, $f(K)$ and $f(K')$ are also Jordan ones, and they do not split the space $\mathbb{C}$ (see the proof of this fact in the course of proving Lemma 1). Therefore, $R = R(f(|K|), f(|K'|))$ is a ring domain. Let us denote $\Gamma = \Gamma(f(|K|), f(|K'|), \mathbb{C})$. Then, by Proposition 2, we get

$$M(\Gamma(f(|K|), f(|K'|), \overline{\mathbb{C}})) \geqslant \frac{2\pi}{\log \frac{32}{h(f(|K|))h(|f(K'|))}}. \tag{44}$$

By (43) and (44), and by the definition of K, we obtain

$$M(\Gamma(f(|K|), f(|K'|), \overline{\mathbb{C}})) \geqslant \frac{2\pi}{\log \frac{32}{\delta \cdot h(f(z), f(y))}}. \tag{45}$$

Since $D^{\,\prime}_f$ is a QED-domain with a constant A_0 in (5), by the condition (45), we get

$$M(\Gamma(f(|K|), f(|K'|), D^{\,\prime}_f)) \geqslant \frac{2\pi}{A_0 \log \frac{32}{\delta \cdot h(f(z), f(y))}}.$$

By the above inequality and due to Proposition 4, we obtain

$$\begin{aligned} M(\Gamma(f(|K|), f(|K'|), D^{\,\prime}_f)) &= M(f(\Gamma(|K|, |K'|, D))) \\ &\leqslant \int_{A(z_0, |z - z_0|, \varepsilon_0)} Q(x)\eta^2(|x - z_0|)\, dm(x), \end{aligned}$$

where η is any nonnegative Lebesgue measurable function satisfying the relation (11) for $r_1 = |z - z_0|$, $r_2 = \varepsilon_0$.

Now, arguing similarly to the proof of Lemma 1, repeating its proof after the relation (20), we obtain the desired conclusion. □

Corollary 3. *Let* $1 < p < 2$. *Let, in the notations of Lemma 3,* $I(\varepsilon, \varepsilon_0) \to \infty$ *as* $\varepsilon \to 0$. *Then, under conditions and notations of Lemma 3,* f *has a continuous extension to* $P_0 \in E_D$, *wherein*

$$|f(z) - f(P_0)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{\pi}{A_0 K_0} I^{2-p}(|z-z_0|, \varepsilon_0)\right\} \tag{46}$$

holds for any $f \in \mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *and any* $z, y \in U \cap D$, *whenever* $|z - z_0| \geqslant |y - y_0|$, *where* $I(P_0) = \{z_0\}$.

Proof. By Lemma 3, $I(P) = \{z_0\}$. It follows that, if $z \to P_0$ and $y \to P_0$, then $z \to z_0$ and $y \to z_0$, as well. If f has no limit as $z \to P_0$, then we may construct at least two sequences $z_m \to P_0$ and $y_m \to P_0$, $m \to \infty$, such that $|f(z_m) - f(y_m)| \geqslant \delta > 0$ for some positive $\delta > 0$ and all $m = 1, 2, \ldots$. But this contradicts the inequality (40), because $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$ by the assumption. Therefore, the limit of f as $x \to P_0$ exists. To prove (46) it remains to pass in (40) to the limit as $y \to P_0$. ☐

Lemma 4. *Let* $P_0 \in E_D$ *and* δ *be some positive number. Assume that* D *is a regular domain, and the following conditions are fulfilled:*

1) for each $y_0 \in \partial D$ *there is* $\varepsilon_0 = \varepsilon_0(y_0)$, $0 < \varepsilon_0 < \min\{\delta, 1\}$, *such that the set* $B(y_0, r) \cap D$ *is finitely connected for all* $0 < r < \varepsilon_0$;

2) for any $0 < r < \varepsilon_0$ *and for each component* K *of the set* $B(y_0, r) \cap D$, *any* $z, y \in K$ *may be joined by a path* $\gamma : [a, b] \to \mathbb{C}$ *such that* $|\gamma| \in K \cap \overline{B(z_0, \max\{|z - z_0|, |y - z_0|\})}$ *with* $|\gamma| = \{z \in \mathbb{C} : \exists\, t \in [a, b] : \gamma(t) = z\}$;

3) for each $z_0 \in I(P_0)$ *there exist a real* $p \leqslant 2$ *and a Lebesgue measurable function* $\psi : (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, *such that the relation*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} Q(z)\psi^2(|z-z_0|)\, dm(z) \leqslant K_0 I^p(\varepsilon, \varepsilon_0)$$

holds as $\varepsilon \to 0$ *for some* $K_0 > 0$, *where*

$$0 < I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty$$

for sufficiently small $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

Then for each $P \in E_D$ *there exists* $y_0 \in \partial D$ *such that* $I(P) = \{y_0\}$, *where* $I(P)$ *denotes the impression of* P. *In addition, there exists a neighborhood* U *of* P_0 *in* $\overline{D}_P$ *such that the relation*

$$|f(z) - f(y)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0 K_0 N} I^{2-p}(|z-z_0|, \varepsilon_0)\right\} \tag{47}$$

holds for any $f \in \mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ *and any* $z, y \in U \cap D$ *such that* $|z - z_0| \geqslant |y - y_0|$, *where* $I(P_0) = \{z_0\}$.

Proof. Since the proof of lemma is very similar to all the previous ones, we will limit ourselves only to the sketch of the proof. The fact that for each $P \in E_D$ there exists $y_0 \in \partial D$ such that $I(P) = \{y_0\}$, may be established in the same way as under the proof of Lemma 2. Let us establish the relation (47). Let $\{z_0\} = I(P_0)$ and let ε_1 and ε_2 be defined in the same way as in the proof of Lemma 2. It follows that there is a neighborhood U of P_0 such that $U \cap D$ is connected and $U \cap D \subset B(z_0, \varepsilon_2)$. Let $z, y \in U \cap D$, $f \in \mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ and $|z - z_0| \geqslant |y - z_0|$. By the definition of ε_0, the points z and y may be joined by the path K, which is contained in the ball $\overline{B(z_0, |z-z_0|)}$. Let $K_f \subset D_f'$ be a path connected continuum such that $\operatorname{diam}(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$ (such a continuum exists by the definition of the class $\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$). Let also $z, w \in K_f \subset D_f'$ be such that

$$\operatorname{diam} K_f = |z-w| \geqslant \delta\,. \tag{48}$$

Since K_f is path connected, we may join the points z, w by a Jordan path K' inside K_f. Due to (48), we obtain that $\operatorname{diam}|K'| \geqslant \delta$.

Next, we reason similarly to the proof of Lemma 2 after the relation (30). We obtain that

$$|f(z)-f(y)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0K_0N} I^{2-p}(|z-z_0|, \varepsilon_0)\right\}.$$

This relation completes the proof. □

Corollary 4. *Let* $1 < p < 2$. *Let, in the notations of Lemma 4,* $I(\varepsilon, \varepsilon_0) \to \infty$ *as* $\varepsilon \to 0$. *Then, under conditions of Lemma 4,* f *has a continuous extension to* $P_0 \in E_D$, *wherein*

$$|f(z)-f(P_0)| \leqslant \frac{32(1+R_0^2)}{\delta} \exp\left\{-\frac{2\pi}{A_0K_0N} I^{2-p}(|z-z_0|, \varepsilon_0)\right\}$$

for any $f \in \mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ *and any* $z \in U \cap D$, *where* $I(P_0) = \{z_0\}$.

The proof of Corollary 4 is similar to the proof of Corollary 3 and therefore is omitted.

5 Proof of the main results. Consequences

The following statement holds (see [8, Corollary 6.3]).

Proposition 5. *Let* $Q \in FMO(z_0)$, $z_0 \in \mathbb{C}$. *Then* Q *satisfies the relation* (12) *for some* $\varepsilon_0 > 0$ *with* $\psi(t) = \frac{1}{t\log\frac{1}{t}}$, $p = 1$, $I(\varepsilon, \varepsilon_0) = \log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}$ *and some constant* K_0 *depending only on* Q.

Combining Proposition 5 with Lemmas 1–4, we obtain the statements of Theorems 1, 3, 5 and 7.

In what follows, given $0 < \varepsilon < \varepsilon_0 < \infty$, $z_0 \in \mathbb{C}$ and a Lebesgue measurable function $Q : \mathbb{C} \to [0, \infty]$ we set

$$J(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \frac{dt}{tq_{z_0}(t)}, \tag{49}$$

where q_{z_0} is defined in (6). The following statement holds (see [8, relation (7.13)]).

Proposition 6. *Let* $0 < \varepsilon < \varepsilon_0 < \infty$, $z_0 \in \mathbb{C}$ *and* $Q : \mathbb{C} \to [0, \infty]$ *be a Lebesgue measurable function. Put*

$$\psi(t) = \begin{cases} 1/(tq_{z_0}(t)), & t \in (\varepsilon, \varepsilon_0), \\ 0, & t \notin (\varepsilon, \varepsilon_0), \end{cases}$$

where $q_{z_0}(t)$ *is defined in* (6). *Let* $J(\varepsilon, \varepsilon_0)$ *be defined in* (49). *If* $J(\varepsilon, \varepsilon_0) < \infty$ *for any* $\varepsilon > 0$ *and some* $\varepsilon_0 > 0$ *and, in addition,* $J(\varepsilon, \varepsilon_0) > 0$ *as* $\varepsilon \to 0$, *then the function* ψ *satisfies the relation* (12) *with* $p = 1$ *and* $K_0 = 2\pi$.

Observe that if f is a homeomorphism satisfying the relations (10)–(11), then $J(\varepsilon, \varepsilon_0) < \infty$ for some $\varepsilon_0 > 0$ and any $\varepsilon \in (0, \varepsilon_0)$. Now, combining Proposition 6 with Lemmas 1–4, we immediately obtain the statements of Theorems 2, 4, 6 and 8.

Next we formulate statements about the Hölder continuity of mappings.

Corollary 5. *Let* $z_0\in\partial D$, $z_0\ne\infty$. *Assume that the following conditions hold:*

1) there is $\varepsilon_0=\varepsilon_0(z_0)>0$ *such that the set* $B(z_0,r)\cap D$ *is connected for any* $0<r<\varepsilon_0$;

2) there is $0<C=C(z_0)<\infty$, *such that*

$$\limsup_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_{B(z_0,\varepsilon)\cap D}Q(z)\,dm(z)\leqslant C. \tag{50}$$

Then any $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *has a continuous extension to* z_0 *and, in addition, there exist* $\widetilde{C}=\widetilde{C}(\delta,R_0,z_0)>0$ *and* $\widetilde{\varepsilon_0}=\widetilde{\varepsilon}_0(z_0)>0$ *such that*

$$|f(z)-f(z_0)|\leqslant\widetilde{C}\cdot|z-z_0|^{\frac{\log 2}{2CA_0}}$$

holds for any $z\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ *and any* $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$.

Remark. *In particular, condition 1) of Corollary 5 is satisfied if* D *is a convex domain.*

In the case of mappings with branching, we have the following.

Corollary 6. *Let* $z_0\in\partial D$, $z_0\ne\infty$, *and* δ *be some positive number. Assume that there exists* $\varepsilon_0=\varepsilon_0(z_0)>0$, $0<\varepsilon_0<\min\{\delta,1\}$, *such that the following conditions hold:*

1) the set $B(z_0,r)\cap D$ *is connected for any* $0<r<\varepsilon_0$;

2) the condition (50) *holds.*

Then every $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$ *has a continuous extension to* z_0 *and, in addition, there exist numbers* $\widetilde{C}=\widetilde{C}(\delta,R_0,z_0)>0$ *and* $\widetilde{\varepsilon_0}=\widetilde{\varepsilon}_0(z_0)>0$ *such that*

$$|f(z)-f(z_0)|\leqslant\widetilde{C}\cdot|z-z_0|^{\frac{\log 2}{2CNA_0}}$$

holds for any $z,y\in B(z_0,\widetilde{\varepsilon}(z_0))\cap D$ *and any* $f\in\mathfrak{R}^{A_0,R_0}_{Q,\delta,N}(D)$.

Corollary 7. *Assume that* D *is a regular domain,* $P_0\in E_D$, *and the following conditions are fulfilled:*

1) for each $y_0\in\partial D$ *there exists* $\varepsilon_0=\varepsilon_0(y_0)>0$ *such that the set* $B(y_0,r)\cap D$ *is finitely connected for all* $0<r<\varepsilon_0$;

2) for any $0<r<\varepsilon_0$ *and for each component* K *of the set* $B(y_0,r)\cap D$, *any* $z,y\in K$ *may be joined by a path* $\gamma:[a,b]\to\mathbb{C}$ *such that* $|\gamma|\in K\cap\overline{B(z_0,\max\{|z-z_0|,|y-z_0|\})}$ *with*

$$|\gamma|=\{z\in\mathbb{C}:\exists\, t\in[a,b]:\gamma(t)=z\};$$

3) for each $z_0\in I(P_0)$ *there exists* $0<C=C(z_0)<\infty$ *such that*

$$\limsup_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_{B(z_0,\varepsilon)\cap D}Q(z)\,dm(z)\leqslant C. \tag{51}$$

Then for any $P\in E_D$ *there exists* $y_0\in\partial D$ *such that* $I(P)=\{y_0\}$, *where* $I(P)$ *denotes the impression of* P. *In addition, any* $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *has a continuous extension to* $P_0\in E_D$, *and there exist* $\widetilde{C}=\widetilde{C}(\delta,R_0,z_0)>0$ *and a neighborhood* $U=U(P_0)$ *of* P_0 *such that*

$$|f(z)-f(P_0)|\leqslant\widetilde{C}\cdot|z-z_0|^{\frac{\log 2}{2CA_0}}$$

holds for any $f\in\mathfrak{F}^{A_0,R_0}_{Q,A,\delta}(D)$ *and* $z\in U\cap D$, *where* $I(P_0)=\{z_0\}$.

Corollary 8. *Let $P_0 \in E_D$, and A be a continuum in D. Assume that D is a regular domain, and the following conditions are fulfilled:*

1) for each $y_0 \in \partial D$ there exists $\varepsilon_0 = \varepsilon_0(y_0)$, $0 < \varepsilon_0 < \min\{\mathrm{dist}\,(z_0, A), 1\}$, such that the set $B(y_0, r) \cap D$ is finitely connected for all $0 < r < \varepsilon_0$;

2) for any $0 < r < \varepsilon_0$ and for each component K of the set $B(y_0, r) \cap D$, any $z, y \in K$ may be joined by a path $\gamma : [a, b] \to \mathbb{C}$ such that $|\gamma| \in K \cap \overline{B(z_0, \max\{|z - z_0|, |y - z_0|\})}$ with $|\gamma| = \{z \in \mathbb{C} : \exists\, t \in [a, b] : \gamma(t) = z\}$;

3) the condition (51) holds for each $z_0 \in I(P_0)$ and some $0 < C = C(z_0) < \infty$.

Then for every $P \in E_D$ there exists $y_0 \in \partial D$ such that $I(P) = \{y_0\}$, where $I(P)$ denotes the impression of P. In addition, any $f \in \mathfrak{R}^{A_0, R_0}_{Q, \delta, N}(D)$ has a continuous extension to $P_0 \in E_D$, and there exist $\widetilde{C} = \widetilde{C}(\delta, R_0, z_0) > 0$ and a neighborhood $U = U(P_0)$ of P_0 such that the relation

$$|f(z) - f(P_0)| \leqslant \widetilde{C} \cdot |z - z_0|^{\frac{\log 2}{2CNA_0}}$$

holds for any $f \in \mathfrak{R}^{A_0, R_0}_{Q, \delta, N}(D)$ and $z \in U \cap D$, where $I(P_0) = \{z_0\}$.

Proof of Corollaries 5–8 directly follows Lemmas 1–4 and Proposition 1 by the choosing in this proposition $\psi(t) := \frac{1}{t}$ and $\varphi \equiv 1$.

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Досліджені відображення класів Соболєва, визначених у деякій плоскій області. Отримані оцінки спотворення відстані при цих відображеннях на межі. Зокрема доведено, що якщо інтегральні середні значення від характеристики відображень є скінченними, то ці відображення є неперервними за Гельдером.

Ключові слова і фрази: квазіконформне відображення, відображення зі скінченним спотворенням, соболівське відображення, неперервність за Гельдером, неперервність за Ліпшицем.