On existence and continuation of mild solutions of functional-differential equations of neutral type in Banach spaces

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The main object of research of this work is infinite-dimensional functional-differential equations of neutral type in Banach spaces. The conditions for the existence of a mild solution to the initial problem and its extension to the boundary of the domain are established. The presence of a delay in the derivative leads to the appearance of singular terms. The research methods are related to the technique of strong semigroups and fractional powers of the operator. The existence proof is based on the representation of the initial problem in abstract operator form with the further use of Krasnoselsky's theorem on a fixed point. For this purpose, the original operator is represented as the sum of the compression operator and the compact operator. The obtained abstract result is applied to the functional-differential equation in partial derivatives of the reaction-diffusion type.

Key words and phrases: delay, strong semigroup, compactness, fixed point, fractional degree of the operator.

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Introduction

We consider the initial value problem

$$\begin{cases} \dfrac{d}{dt}(u(t) + g(t, u_t)) = Au + f(t, u_t), \\ u(t) = \varphi(t), \quad t \in [-h, 0], \end{cases} \tag{1}$$

for the functional-differential equation of neutral type in a Banach space X. The designation u_t will be shown below. For some $h > 0$ let $C = C([-h, 0]; X)$ be the space of continuous functions $\varphi : [-h, 0] \to X$ with the norm $\|\varphi\|_C = \sup_{t \in [-h,0]} \|\varphi(t)\|_X$. Throughout the article, $\|\varphi(t)\|_X$ is the norm in X. Let A be the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, in X, $u_t = u(t + \theta) \in C$ for any $t \in [0, T]$, $\theta \in [-h, 0]$. Let D be some domain in $[-h, T] \times C$, ∂D be its boundary in the sense of [8, p. 18], and $\overline{D} = D \cup \partial D$. Mappings f and g act from D to X. In the sequel, a solution of the problem (1) we will understand in the mild sense.

Definition 1. *A function $u : [0, T] \mapsto X$ is called a mild solution of the initial value problem* (1) *on $[0, T]$ if:*

1) $u(t) = \varphi(t), t \in [-h, 0]$;

2) $u \in C([0, T], X)$;

3) $u(t)$ *satisfies the integral equation*

$$u(t) = S(t)(\varphi(0) + g(0, \varphi)) - g(t, u_t)$$
$$- \int_0^t AS(t - s)g(s, u_s)\, ds + \int_0^t S(t - s)f(s, u_s)\, ds. \tag{2}$$

We will study the local existence of a mild solution of (1) on interval $[-h, t_1)$. The latter means that if $(t, \varphi) \in D$ for $t \in [-h, 0]$, then $(t, u_t) \in D$ for $t \in [0, t_1)$.

A similar question in the finite-dimensional case was considered in [5]. The author pays detailed attention to the functional-differential equation of the ordinary type. Regarding equations of the neutral type, similar results were obtained there for atomic mappings. Regarding equations in infinite-dimensional spaces, it is worth pointing to the paper [7] for the mild resolvability of differential equations, and the monograph [16], where for functional-differential equations of the ordinary type conditions for the local existence of mild solutions were obtained under the condition of continuity of the right-hand sides. However, cases are often encountered, for example, in optimal control problems [9, 10, 12, 13], when the conditions for the continuity of the right-hand sides are unnatural. Therefore, it is important to obtain the conditions for the existence and continuation of solutions under the conditions of the t-measurability of the coefficients of the equation (such as the Carathédory conditions). For functional-differential equations of the ordinary type, such conditions were obtained in [9]. Note, that even in the finite-dimensional case the solution cannot always be extended to the boundary of the domain (see the example of A. Myshkis [5, p. 60]).

The aim of this work is to obtain similar results for equations of neutral type. Note that the neutral type of the equation causes the presence in the formula (2) of two new terms $g(t, u_t)$ and

$$\int_0^t AS(t - s)g(s, u_s)\, ds,$$

which significantly complicates the study of (1). The work consists of an introduction and three parts. The introduction describes the problem statement and a literature review. Section 2 presents the necessary concepts and formulates the main results. Section 3 is devoted to the proof of the main results. Section 4 gives an example of the obtained results for partial differential equations of parabolic type.

1   Problem statement and main results

Let X be a reflexive Banach space, $A : X \to X$ be a linear operator with domain $D(A)$, $\sigma(-A)$ be a spectrum of $(-A)$.

1.1 Assumptions on the operator A

Assumption (H1). $Re(\sigma(-A)) > \delta > 0$ and A^{-1} is compact operator in X.

Then for every $\alpha \in [0,1]$ we can define fractional power $(-A)^\alpha$, which is a closed linear operator with domain $D((-A)^\alpha$ (see [14] for details). Let us denote by X_α the Banach space $D((-A)^\alpha)$ supplied with the norm $\|u\|_\alpha := \|(-A)^\alpha u\|$, which is equivalent to the graph norm of $(-A)^\alpha$. Let us denote $X_0 = X$. From [6, Section 1.4] we deduce that if A^{-1} is a compact operator, then the semigroup $S(t)$ is compact for $t > 0$. Then under assumption (H1) [14, Theorem 3.2] implies that the semigroup $S(t)$ is continuous in the uniform operator topology for $t > 0$. Thus, due to [14, Theorem 3.3], we can conclude that the operator A has compact resolvent. The last property guarantees the following result (see [6, Theorem 1.4.8]).

Lemma 1. *Under assumption (H1) the embedding $X_\alpha \subset X_\beta$ is compact for $0 \leq \beta < \alpha \leq 1$.*

Lemma 2 ([6, Theorem 1.4.3]). *Under assumption (H1) for every $\alpha \geq 0$ there exists $C_\alpha > 0$ such that $\|(-A)^\alpha S(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t}$, $t > 0$.*

In particular, $\|S(t)\| \leq C_0 e^{-\delta t}$, $t > 0$.

1.2 Assumptions on nonlinear terms

Let U be an open subset of C and $f : [0,T] \times U \to X$.

Assumption (H2):

1) for every $t \in [0,T]$ the mapping f is continuous with respect to φ;

2) for every $\varphi \in U$ the mapping f is measurable with respect to t;

3) for any $R > 0$ there exists integrable function $m_R(t) \in L^p(0,T)$, $p > 1$, such that $\|f(t,\varphi)\| \leq m_R(t)$, if $\|\varphi\| \leq R$.

Assumption (H3):

1) there exist $\alpha \in [0,1]$ and $M_g \in (0,1)$ such that $g : [0,T] \times U \to X_\alpha$;

2) $\|g(t,\varphi_1) - g(t,\varphi_2)\|_\alpha \leq M_g \|\varphi_1 - \varphi_2\|_C$;

3) function g is continuous with respect to t in X_α uniformly over $\varphi \in U$.

Theorem 1 (Local Existence). *Suppose that assumptions (H1)–(H3) hold. Then for every $\varphi \in U$ there exist $t_1 = t_1(\varphi) \in [0,T]$ and continuous function $u \in [-h,t_1] \to X$ such that $u(t)$ is a mild solution of (1) on $[0,t_1]$ with the initial function φ.*

Note that this theorem takes place for every initial point $t_0 \in (0,T)$. So, we can continue the solution on the maximal interval $[t_0 - h, t_0]$.

We consider $D = (0,T) \times U$, its boundary ∂D, and its closure $\overline{D} = D \bigcup \partial D$.

Theorem 2 (Continuation). *Assume that conditions of Theorem 1 are valid. Then the solution with initial data $(t_0, \varphi) \in D$, i.e. on the interval $[t_0 - h, t_0]$, exists on the maximal interval $[t_0, \tau)$, $\tau > t_0$, and $(\tau, u_\tau) \in \partial D$.*

Further, we will use the following fixed point theorem.

Theorem 3 (Krasnoselskii's Theorem [1]). *Let M be a non-empty closed convex subset of a Banach space $(S, \|\cdot\|)$. Assume that operators A and B act from M into S such that:*

(i) *$Ax + By \in M \ \forall \, x, y \in M$;*

(ii) *A is a continuous operator and AM belongs to a compact set;*

(iii) *B is a contraction operator with the constant $\alpha < 1$.*

Then there exists $y \in M$ such that $Ay + By = y$.

2   Proofs of the main results

The proof of Theorem 1. We want to apply the Krasnoselskii fixed point theorem. It should be noted that due to assumptions on A there exists $M > 0$ such that $\|S(t)\| \leq M \ \forall \, t \in [0, T]$. We choose $\beta > 0$ such that $B_\beta = \{\psi \in C : \|\varphi - \psi\|_C \leq \beta\} \subset U$. Analogously to [5, 16], we introduce closed convex bounded set

$$\mathcal{A}(\delta, \beta) = \{y \in C([-h, \delta]; X) : y_0 = \varphi, y_t \in B_\beta, t \in [0, \delta]\},$$

where δ is sufficiently small. On this set we consider the operator

$$G(y)(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ S(t)(\varphi(0) + g(0, \varphi)) - g(t, y_t) \\ \quad - \int_0^t AS(t-s)g(s, y_s)\,ds + \int_0^t S(t-s)f(s, y_s)\,ds, & t \in [0, \delta]. \end{cases}$$

We split this mapping as $G(y) = \Phi(y) + \Psi(y)$, where

$$\Phi(y)(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ S(t)(\varphi(0) + g(0, \varphi)) \\ \quad - \int_0^t AS(t-s)g(s, y_s)\,ds + \int_0^t S(t-s)f(s, y_s)\,ds, & t \in [0, \delta]. \end{cases} \tag{3}$$

and

$$\Psi(y)(t) = \begin{cases} 0, & t \in [-h, 0], \\ -g(t, y_t), & t \in [0, \delta]. \end{cases}$$

Let us check for the given mappings whether the conditions of the Krasnoselskii theorem are fulfilled.

Let us show the continuity of $\Phi(y)$. Let $y^{(n)}(t) \in \mathcal{A}(\delta, \beta)$ and $\sup_{t \in [0, \delta]} \|y^{(n)}(t) - y(t)\| \to 0$ as $n \to \infty$.

Then

$$\Phi(y^{(n)})(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ S(t)(\varphi(0) + g(0, \varphi)) \\ \quad - \int_0^t AS(t-s)g(s, y_s^{(n)})\,ds + \int_0^t S(t-s)f(s, y_s^{(n)})\,ds, & t \in [0, \delta]. \end{cases}$$

So,

$$\sup_{t\in[-h,\delta]} \|\Phi(y^{(n)}(t) - \Phi(y)(t)\| \le \sup_{t\in[0,\delta]} \left\| \int_0^t AS(t-s)(g(s,y_s^{(n)} - g(s,y_s))\,ds \right\|$$
$$+ \sup_{t\in[0,\delta]} \left\| \int_0^t S(t-s)(f(s,y_s^{(n)} - f(s,y_s))\,ds \right\|.$$

For the first summand we have

$$\left\| \int_0^t AS(t-s)(g(s,y_s^{(n)}) - g(s,y_s))\,ds \right\| \le \int_0^t \|(-A)^{1-\alpha}S(t-s)\|\|(-A)^\alpha(g(s,y_s^{(n)}) - g(s,y_s))\|\,ds$$
$$\le \int_0^t \|(-A)^{1-\alpha}S(t-s)\|\,ds \sup_{s\in[0,\delta]} \|g(s,y_s^{(n)}) - g(s,y_s\|_{X_\alpha}$$
$$\le \int_0^t C_{1-\alpha}t^{\alpha-1}ds M_g \sup_{s\in[0,\delta]} \|y_s^{(n)} - y_s\|$$
$$= C_{1-\alpha} \sup_{s\in[0,\delta]} \|y_s^{(n)} - y_s\| \to 0 \quad \text{as} \quad n \to +\infty.$$

It should be noted that $\sup_{t\in[0,\delta]} \|y_t^{(n)} - y_t\|_C \le \sup_{t\in[0,\delta]} \|y_t^{(n)} - y_t\|$ as $n \to +\infty$. Then for every $t \in [0,\delta]$ we get $f(t,y_t^{(n)}) \to f(t,y_t)$ as $n \to +\infty$ due to the continuity of $\varphi \mapsto f(t,\varphi)$. Then the definition of $\mathcal{A}(\delta,\beta)$ and Lebesgue's dominated convergence theorem yield

$$\sup_{t\in[0,\delta]} \left\| \int_0^t S(t-s)(f(s,y_s^{(n)}) - f(s,y_s))\,ds \right\| \le \int_0^\delta M\|f(s,y_s^{(n)}) - f(s,y_s)\|\,ds \to 0.$$

Let us prove that $\Phi(\mathcal{A}(\delta,\beta))$ is contained in a compact set. To this end, we will use the following statements.

Proposition 1 ([2, Proposition 8.4]). *If $S(t), t > 0$, is a compact operator, then for any $p \ge 1$ the mapping*

$$F(f)(t) = \int_0^t S(t-s)f(s)\,ds, \quad f \in L^p(0,T;X), \quad t \in [0,T],$$

is a compact operator from $L^p(0,T;X)$ to $C([0,T];X)$.

Proposition 2 ([15]). *Under assumptions on the operator A the mapping*

$$(Bz)(t) = \int_0^t AS(t-s)z(s)\,ds$$

is a compact operator from $C([0,T],X_\alpha)$ to $C([0,T],X)$.

Further, we will use the infinite dimensional analogous of the Arzela-Askoli theorem. We need to prove that

1) for every $t \in [0,\delta]$ the set $\{\Phi(y)(t) : y \in \mathcal{A}(\delta,\beta)\}$ is precompact in X;

2) the family of functions $\Phi(y)(t)$ is equicontinuous, i.e. for any $\varepsilon > 0$ there exists $\sigma > 0$ such that $\|\Phi(y)(t_1) - \Phi(y)(t_2)\| \le \varepsilon$, if $y \in \mathcal{A}(\delta,\beta), |t_1 - t_2| < \sigma, t_1, t_2 \in [0,\delta]$.

Note that from (H3) we get

$$\sup_{t\in[0,\delta]} \|g(t,y_t)\|_\alpha \leq \sup_{t\in[0,\delta]} \|(g(t,y_t^{(n)}) - g(t,\varphi))\|_\alpha + \|g(t,\varphi)\|_\alpha \leq M_g\beta + \|g(t,\varphi)\|_\alpha \leq C_1 \quad (4)$$

for $t \in [0,\delta]$. Then the compactness of the first integral summand in (3) is the consequence of Proposition 2.

Further, due to the definition of $\mathcal{A}(\delta,\beta)$, we get

$$\sup_{t\in[0,\delta]} \|y(t)\|_C \leq \sup_{t\in[0,\delta]} \|y_t - \varphi\|_C + \|\varphi\|_C \leq \beta + \|\varphi\|_C.$$

Using condition 3) from (H2), we obtain that $\|f(t,y_t)\| \leq m_R(t)$ for $R = \beta + \|\varphi\|_C$. Then the set $f(t,y_t)$ is bounded in $L^p(0,T;X)$. Therefore, Proposition 1 yields the compactness of the second integral summand in (3). Thus, property 1) is proved.

Now let us prove the equicontinuity property. Let us fix an arbitrary ε. We note that the continuity of $S(t)(\varphi(0) + g(0,\varphi))$, $t \in [0,\delta]$, implies its uniform continuity on $[0,\delta]$. Therefore, there exists σ_1 such that for all $t_1, t_2 \in [0,\delta]$ with $|t_1 - t_2| < \sigma_1$ we have

$$\left\|(S(t_1) - S(t_2))(\varphi(0) + g(0, g(0,\varphi)))\right\| < \frac{\varepsilon}{3}. \tag{5}$$

After that, using Lemma 2 and estimate (4), we get

$$\left\|\int_{t_1}^{t_2} AS(t-s)g(s,y_s)\, ds\right\| \leq \int_{t_1}^{t_2} \|(-A)^{1-\alpha}S(t-s)\|\, ds \sup_{t\in[0,\delta]} \|g(s,y_s)\|$$
$$\leq \frac{C_{1-\alpha}}{\alpha}|t_2 - t_1|C_1 < \frac{\varepsilon}{3}, \tag{6}$$

if $|t_1 - t_2| < \sigma_2$.

Due to the absolutely continuity of Lebesgue's integral we deduce that

$$\left\|\int_{t_1}^{t_2} S(t-s)f(s,y_s)\, ds\right\| \leq M \int_{t_1}^{t_2} m_R(t) < \frac{\varepsilon}{3}, \tag{7}$$

if $|t_2 - t_1| < \sigma_3$ and $R = \beta + \|\varphi\|_C$. Then, choosing $\sigma = \min\{\sigma_1,\sigma_2,\sigma_3\}$, from (5)–(7) we deduce equicontinuity of the family $\Phi(y)(t)$. As $\Phi(\mathcal{A}(\delta,\beta))$ is a compact set in $C([-h,0],X)$, so we have that assumption *(ii)* of the Krasnoselskii theorem is fulfilled. Let us verify assumption *(iii)*. For every $y^{(1)}(t)$ and $y^{(2)}(t)$, which belongs to $\mathcal{A}(\delta,\beta)$, we have

$$\sup_{t\in[-h,\delta]} \|\Phi(y^{(1)}(t)) - \Phi(y^{(2)}(t))\| = \sup_{t\in[0,\delta]} \|g(t,y_t^{(1)}) - g(t,y_t^{(2)})\|$$
$$\leq \sup_{t\in[0,\delta]} \|g(t,y_t^{(1)}) - g(t,y_t^{(2)})\|_\alpha \leq M_g \sup_{t\in[0,\delta]} \|y_t^{(1)} - y_t^{(2)}\| = M_g \sup_{t\in[-h,\delta]} \|y_t^{(1)} - y_t^{(2)}\|.$$

Because of the inequality $M_g < 1$ we get *(iii)*.

Let us prove *(i)*. For arbitrary $y \in \mathcal{A}(\delta,\beta)$ and $z \in \mathcal{A}(\delta,\beta)$ we get

$$\Phi(y)(t) + \Psi(z)(t) = \begin{cases} \varphi(t), & t \in [-h,0], \\ S(t)(\varphi(0) + g(0,\varphi)) - g(t,z_t) \\ \quad - \int_0^t AS(t-s)g(s,y_s)\, ds + \int_0^t S(t-s)f(s,y_s)\, ds, & t \in [0,\delta]. \end{cases}$$

Then for every $t \in [0, \delta]$ we have

$$
\begin{aligned}
\|\Phi(y)_t + \Psi(z)_t - \varphi_0\| &= \sup_{\theta \in [-h,0]} \|\Phi(y)_t + \Psi(y)_t - \varphi(\theta)\| \\
&\leq \sup_{\theta \in [-h,-t]} \|\Phi(y)(t+\theta) + \Psi(y)(t+\theta) - \varphi(\theta)\| \\
&\quad + \sup_{\theta \in [-t,0]} \|\Phi(y)(t+\theta) + \Psi(y)(t+\theta) - \varphi(\theta)\|.
\end{aligned}
\tag{8}
$$

From the uniform continuity of φ we deduce

$$
\sup_{\theta \in [-h,-t]} \|\Phi(y)(t+\theta) + \Psi(y)(t+\theta) - \varphi(\theta)\| = \sup_{\theta \in [-h,-t]} \|\varphi(t+\theta) - \varphi(\theta)\| \leq \frac{\beta}{k}
\tag{9}
$$

for all $t \in [0, \delta]$ and for sufficiently small δ. Here k is some natural number, which we choose latter, and $\delta = \delta(k)$. Further, for $t + \theta \in [0, \alpha]$ we get

$$
\begin{aligned}
\Phi(y)(t+\theta) + \Psi(z)(t+\theta) - \varphi(\theta) &= S(t+\theta)(\varphi(0) + g(0,\varphi)) - g(t+\theta, z_{t+\theta}) \\
&\quad - \int_0^{t+\theta} AS(t+\theta-s)g(s,y_s)ds + \int_0^{t+\theta} AS(t+\theta-s)f(s,y_s)ds - \varphi(\theta) \\
&= S(t+\theta)\varphi(0) - \varphi(0) - \varphi(\theta) + \varphi(0) + S(t+\theta)g(0,\varphi) - g(t+\theta, z_{t+\theta}) \\
&\quad - \int_0^{t+\theta} AS(t+\theta-s)g(s,y_s)ds + \int_0^{t+\theta} AS(t+\theta-s)f(s,y_s)ds.
\end{aligned}
\tag{10}
$$

Here $\theta \in [-\alpha, 0]$. Then

$$
\|S(t+\theta)\varphi(0) - \varphi(0)\| \leq \frac{\beta}{k}, \quad t \in [0, \delta], \quad \delta = \delta(k),
$$

due to the C_0-continuity of $S(t)$.

Choosing δ (depending on k) sufficiently small, we get

$$
\|\varphi(\theta) - \varphi(0)\| \leq \frac{\beta}{k}.
\tag{11}
$$

Further,

$$
S(t+\theta)g(0,\varphi) - g(t+\theta, z_{t+\theta}) = S(t+\theta)g(0,\varphi) - g(0,\varphi) + g(0,\varphi) - g(t+\theta, z_{t+\theta}).
\tag{12}
$$

But

$$
\|S(t+\theta)g(0,\varphi) - g(0,\varphi)\| \leq \frac{\beta}{k}, \quad t \in [0, \delta(k)].
\tag{13}
$$

We have

$$
\begin{aligned}
\|g(t+\theta, z_{t+\theta}) - g(0,\varphi)\| &\leq \|g(t+\theta, z_{t+\theta}) - g(t+\theta, \varphi)\|_\alpha + \|g(t+\theta, \varphi) - g(0,\varphi)\|_\alpha \\
&\leq M_g \|z_{t+\theta} - \varphi\|_C + \|g(t+\theta, \varphi) - g(0,\varphi)\|_\alpha \leq M_g \beta + \frac{\beta}{k}.
\end{aligned}
\tag{14}
$$

The last estimate follows from the definition of $\mathcal{A}(\delta, \beta)$ and continuity of the mapping g with respect to t.

Let us estimate the first integral summand in (10). We have

$$
\left\| \int_0^{t+\theta} AS(t+\theta-s)g(s,y_s)ds \right\|
$$
$$
\leq \int_0^{t+\theta} C_{1-\alpha}(t+\theta-s)^{\alpha-1}ds \sup_{s\in[0,t+\theta]} (M_g\|y_s-\varphi\| + \|-g(s,\varphi)\|_\alpha) \qquad (15)
$$
$$
\leq \frac{C_{1-\alpha}}{\alpha}(t+\theta)^\alpha (M_g\beta + \sup_{s\in[0,\alpha]} \|g(s,\varphi)\|_\alpha) \leq \frac{\beta}{k}
$$

for sufficiently small $\delta = \delta(k)$. To estimate the last integral in (10) we have

$$
\left\| \int_0^{t+\theta} S(t+\theta-s)f(s,y_s)ds \right\| \leq M \int_0^\delta m_\beta(s) \leq \frac{\beta}{k}, \qquad (16)
$$

due to the absolutely continuity of Lebesgue's integral. Thus, from (8), (9), (11)–(16) we deduce that

$$
\sup_{t\in[0,\delta(k)]} \|\Phi(y)_t + \Psi(z)_t - \varphi_0\| \leq \frac{7\beta}{k} + M_g\beta. \qquad (17)
$$

Now let us choose k such that $7/k + M_g < 1$. Further, we choose δ such that inequalities (9), (11)–(16) are fulfilled. Taking into account continuity of Φ and $g(t,\varphi)$, we get from (17) condition *(i)* of the Krasnoselskii theorem. After applying this theorem, we deduce that there exists $y \in \mathcal{A}(\delta,\beta)$ such that $G(y) = y$. Theorem 1 is proved. $\square$

The proof of Theorem 2. We need the following lemmas.

Lemma 3. *Under conditions of Theorem 1 if $W \subset D$ and W is a compact set, then there exists $\delta > 0$ such that for arbitrary initial data $(t_0,\varphi) \in W$ the solution of (1) exists on the interval $[t_0, T_0 + \delta]$.*

Proof. As W is a compact set, so there exists an open set V such that $W \subset V \subset D$. Therefore, there exists a function $m(t)$ such that the condition 3) from Assumption (H2) takes place simultaneously for all $(t_0,\varphi) \in W$.

Compactness of W implies the inequality (4) for all φ with $(t_0,\varphi) \in W$. By standard considerations related to the existence for every $\varepsilon > 0$ a finite ε-net in W, we can get that the inequalities (9), (11)–(16) take place uniformly over all φ with $(t_0,\varphi) \in W$. So, there exists $\delta > 0$ such that all mentioned above inequalities take place on $[0,\delta]$ uniformly over $(t_0,\varphi) \in W$. The rest of the proof is the consequence of Theorem 1. $\square$

Lemma 4. *If $u(t)$ is a non-continuing solution of (1) on $[0,\tau]$, then for every compact set $W \subset D$ there exists t_W such that $(t,u_t) \notin W$ for $t \in [t_W, \tau)$.*

Proof. Since W is a compact set in D, Lemma 1 implies that for every point (c,φ) equation (1) has a solution $u(t)$ such that $u(t) = \varphi(t)$ for $t \in [c-h,c]$, which exists at least on the interval $[c, c+\delta]$. Arguing by contradiction, we can find a sequence $t_k \to \tau$, $k \to \infty$, and $\psi \in C$ such that $(t_k, u_{t_k}) \in W$, $(\tau,\psi) \in W$, $(t_k, u_{t_k}) \to (\tau,\psi)$ as $k \to \infty$. The last statement is a consequence of the compactness of W. So, for arbitrary $\varepsilon \in (0,h)$ we have

$$
\lim_{k\to\infty} \sup_{\theta\in[-h,-\varepsilon]} \|u(t_k+\theta) - \psi(\theta)\| = 0.
$$

Therefore, $u(t_k - \varepsilon) \to \psi(-\varepsilon)$ as $k \to +\infty$. Thus, $u(\tau - \varepsilon) = \psi(-\varepsilon)$ and we can conclude that there exists $\lim\limits_{t \to \tau - 0} u(t)$ and $u(t)$ can be continued to continuous function on $[-h, \tau]$, if we put $u(\tau) = \psi(0)$. But $(\tau, u_\tau) \in W$. So, due to Lemma 1 there exists a solution with the initial data (τ, u_τ) on the interval $[\tau, \tau + \delta]$, which is a contradiction. Lemma is proved. $\qquad \square$

Now we are in position to prove Theorem 2. We have that the solution of (1) with initial data $(t_0, \varphi) \in D$ exists on some interval $[t_0, t_0 + \delta]$. Due to Zorn's lemma this solution can be continued on the maximal interval (t_0, τ). Let K be a closed bounded set in D. Let us show that there exists t_k such that $(t, u_t) \notin K$ for $t \in (t_k, \tau)$. Assume the contrary. Then there exists a sequence $t_k \to \tau - 0$ such that $(t_k, u_{t_k}) \in K$ for all k. Then for every $\varepsilon > 0$ the continuity of the solution $u(t)$ in the space X on $[t_0 - h, \tau - \varepsilon]$ implies that uniform continuity of u_t in the space C for $t \in [t_0, \tau - \varepsilon]$. Then the set $P = \{(t, u_t) : t \in [t_0, \tau)\}$ is bounded. Indeed, if it is not bounded, then there exists a sequence (s_k, u_{s_k}) such that $s_k \to \tau - 0$, and $\|u_{s_k}\|_C \to \infty, k \to \infty$. But $(t_k, u_{t_k}) \in K$, so u_{t_k} are bounded in C, which contradicts the uniform continuity of u_t on $[t_0, \tau - \varepsilon]$. Therefore, P is bounded and $\overline{P}$ belongs to K.

Let us show that P belongs to some compact set in D. It is sufficient to prove precompactness of the set $R = \{u_t, t \in [t_0, \tau)\}$. According to the Arzela-Askoli theorem we need to prove that

1) for every $\theta \in [-h, 0]$ the set $R(\theta) = \{u(t + \theta), t \in [t_0, \tau)\}$ is precompact in X;

2) the family of functions $\{u(t + \theta), \theta \in [-h, 0], t \in [t_0, \tau)\}$ is equicontinuous.

For proving the precompactness of $R(\theta)$ we will construct a finite δ-net for every $\delta > 0$.

Without loss of generality we assume that $t_0 = \theta$. Let us fix sufficiently small $\mu < \tau$. We split the set $R(\theta)$ into two parts $R(\theta) = R_1(\theta) \cup R_1(\theta)$, where

$$R_1(\theta) = \{u(t + \theta), t + \theta \in [-h, \mu]\}, \quad R_2(\theta) = \{u(t + \theta), t + \theta \in (\mu, \tau)\}.$$

Uniform continuity of $u(t + \theta)$ on $[-h, \mu]$ implies the existence of a finite δ-net for $R_1(\theta)$.

Let us consider $R_2(\theta)$. As $t + \theta > 0$, then for elements of $R_2(\theta)$ we have

$$u(t + \theta) = S(t + \theta)(\varphi(0) + g(0, \varphi)) - g(t + \theta, u_{t+\theta})$$
$$- \int_0^{t+\theta} AS(t + \theta - s)g(s, u_s)\, ds + \int_0^{t+\theta} S(t + \theta - s)f(s, u_s)\, ds. \tag{18}$$

For every $\varepsilon \in (0, \mu)$ we consider the set $R_\varepsilon(\theta)$, whose elements have the following representation

$$u^{(\varepsilon)}(t + \theta) = S(t + \theta)(\varphi(0) + g(0, \varphi)) - g(t + \theta, u_{t+\theta})$$
$$- \int_0^{t+\theta-\varepsilon} AS(t + \theta - s)g(s, u_s)\, ds + \int_0^{t+\theta-\varepsilon} S(t + \theta - s)f(s, u_s)\, ds. \tag{19}$$

As $\overline{P} \subset K$ is bounded, so due to the condition 3) from Assumption (H2) there exists an integrable function $m(t)$ such that $|f(t, u_t)| \leq m(t), t \in [0, \tau)$ and

$$\sup_{s \in [0,\tau)} \|g(s, u_s)\|_\alpha \leq \sup_{s \in [0,\tau)} \|g(s, u_s - g(s, \varphi)\|_\alpha + \sup_{s \in [0,T]} \|g(s, \varphi)\|_\alpha$$
$$\leq M_g\left(\sup_{s \in [0,\tau)} \|u_s\| + \|\varphi\|\right) + \sup_{s \in [0,T]} \|g(s, \varphi)\| \leq C \tag{20}$$

for some $C > 0$.

Applying the semigroup property, we get

$$u^{(\varepsilon)}(t+\theta) = S(\varepsilon)(\varphi(0) + g(0,\varphi)) - g(t+\theta, u_{t+\theta})$$
$$- S(\varepsilon)\int_0^{t+\theta-\varepsilon} AS(t+\theta-\varepsilon-s)g(s,u_s)\,ds + S(\varepsilon)\int_0^{t+\theta-\varepsilon} S(t+\theta-s-\varepsilon)f(s,u_s)\,ds.$$

Since $t + \theta - \varepsilon > 0$, we have $\|S(t+\theta-\varepsilon)(\varphi(0) + g(0,\varphi))\| \leq M\|\varphi(0) + g(0,\varphi)\|$. As the set $S(t + \theta - \varepsilon)(\varphi(0) + g(0,\varphi))$ is uniformly bounded for $t \in [0,\tau)$, then the set $S(\varepsilon)S(t+\theta-\varepsilon)(\varphi(0) + g(0,\varphi))$ is precompact in X.

Let us prove precompactness of the set $\{g(t+\theta, u_{t+\theta}), t+\theta \in [\mu,\tau)\}$ in X. From (20) and compact embedding $X_\alpha \subset X$ we derive that the set $G(t_1) = \{g(t_1, u_{t+\theta}), t+\theta \in [\mu,\tau)\}$ is precompact in X for every $t_1 \in [0,T]$. So, for every $t_1 \in [0,T]$ it has a finite ε-net $\{z_1, \ldots, z_p\}$ with $p = p(\varepsilon, t_1)$.

Let us construct such a net for the set $G = \{g(s, u_{t+\theta}), t+\theta \in [\mu,\tau), s \in [0,T]\}$. Due to the continuity with respect to $t \in [0,T]$ of the mapping $g(t,\varphi)$, uniformly over $\varphi \in U$, we deduce that for every $\varepsilon > 0$ there exist a finite set $\{t_1(\varepsilon), \ldots, t_r(\varepsilon)\}, r = r(\varepsilon)$, and $\delta > 0$ such that if $|t - t_i| < \delta$, then $\|g(t,\varphi) - g(t_i,\varphi)\| < \varepsilon$. Since each set $G(t_i(\varepsilon))$ is precompact, so it has a finite ε-net. Therefore, the union over t_i of these ε-nets will be 2ε-net for the set G. Then the set G is precompact in X and we can conclude that the set $\{g(t+\theta, u_{t+\theta}), t+\theta \in [\mu,\tau)\}$ is precompact in X.

From (19) we get

$$\left\| \int_0^{t+\theta-\varepsilon} S(t+\theta-\varepsilon-s)f(s,u_s)\,ds \right\| \leq \int_0^{t+\theta-\varepsilon} \|S(t+\theta-\varepsilon-s)\|\|f(s,u_s)\|\,ds$$
$$\leq M\int_0^{t+\theta-s} m(s)\,ds \leq \int_0^T m(s)\,ds \leq C_1$$

for some positive constant C_1.

Then, since $S(\varepsilon)$ is a compact operator, we conclude that the set

$$\left\{ S(\varepsilon)\int_0^{t+\theta-s} S(t+\theta-\varepsilon-s)f(s,u_s)\,ds, \; t+\theta \in [\mu,\tau) \right\}$$

is precompact in X. In the same way, using (15) and (20), we show that the set

$$\left\{ S(\varepsilon)\int_0^{t+\theta-s} AS(t+\theta-\varepsilon-s)g(s,u_s)\,ds, \; t+\theta \in [\mu,\tau) \right\}$$

is precomapct in X. So, for every $\varepsilon \in (0,\mu)$ the set $R_\varepsilon(\theta)$ is precompact in X, therefore, for every $\delta > 0$ it has a finite δ-net. Let us denote it by $\{u^{(\varepsilon)}(t_1 + \theta), \ldots, u^{(\varepsilon)}(t_p + \theta)\}$, $p = p(\delta)$.

From (18) and (19) we get

$$\|u(t+\theta) - u^{(\varepsilon)}(t+\theta)\| \leq \left\| \int_{t+\theta-\varepsilon}^{t+\theta} AS(t+\theta-s)g(s,u_s)\,ds \right\| + \left\| \int_{t+\theta-\varepsilon}^{t+\theta} S(t+\theta-s)f(s,u_s)\,ds \right\|.$$

Moreover,

$$\left\| \int_{t+\theta-\varepsilon}^{t+\theta} AS(t+\theta-s)g(s,u_s)\,ds \right\| \leq \int_{t+\theta-\varepsilon}^{t+\theta} C_{1-\alpha}(t+\theta-s)^{\alpha-1}\,ds \sup_{s\in[0,\tau)} \|g(s,u_s\|$$
$$= \frac{C_{1-\alpha}}{\alpha}\varepsilon^\alpha C_1 \to 0 \quad \text{as} \quad \varepsilon \to 0$$

uniformly over $t + \theta$. Analogously,

$$\left\| \int_{t+\theta-\varepsilon}^{t+\theta} S(t+\theta-s)f(s,u_s)\,ds \right\| \leq \int_{t+\theta-\varepsilon}^{t+\theta} m(s)\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0 \qquad (21)$$

uniformly over $t + \theta$. Finally,

$$\sup_{t+\theta \in [\mu,\tau)} \|u^\varepsilon(t+\theta) - u(t+\theta)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Using the δ-net for $R_\varepsilon(\theta)$, we can construct 3σ-net $\{u(t_1 + \theta), \ldots, u(t_p + \theta)\}$ for $R_2(\theta)$, where we choose elements $u(t_i + \theta)$ from $R_2(\theta)$ such that $\sup_{t \in (0,\tau)} \|u(t+\theta) - u_\varepsilon(t+\theta)\| < \delta$ (it is possible due to (21)). Therefore, the finite δ-net both for $R_1(\theta)$ and $R_2(\theta)$ has been constructed. It means that $R(\theta)$ is precomapct in X, and property 1) is proved.

Let us show equicontinuity of the family $\{u(t+\theta),\ \theta \in [-h,0],\ t \in [0,\tau)\}$. We need to prove that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u(t+\theta_1) - u(t+\theta_2)\| < \varepsilon$ for all $t \in [0,\tau)$ and for all $\theta_1, \theta_2 \in [-h,0]$ with $|\theta_1 - \theta_2| < \delta$. Let us put $\theta_2 = \theta_1 + r$. For sufficiently small $\mu \in (0,\tau)$ we consider three cases.

Case 1. $t + \theta_2 \leq \mu$. In this case equicontinuity of $u(t+\theta)$ follows from the uniform continuity of $u(t)$ on $[-h,\mu]$.

Case 2. $t + \theta_1 < \mu, t + \theta_2 \geq \mu$. In this case we have

$$\|u(t+\theta_2) - u(t+\theta_1)\| \leq \|u(t+\theta_2) - u(\mu)\| + \|u(t+\theta_1) - u(\mu)\|.$$

Choosing sufficiently small δ we can guarantee that the points $t + \theta_1$ and $t + \theta_2$ belongs to small left and right neighbourhoods of μ such that

$$\|u(t+\theta_2) - u(\mu)\| \leq \frac{\varepsilon}{2}, \quad \|u(t+\theta_1) - u(\mu)\| \leq \frac{\varepsilon}{2}.$$

So, in this case the required equicontinuity follows from the continuity of $u(t)$ at μ.

Case 3. $t + \theta_1 \geq \mu$. In this case the semigroup $S(t), t > 0$, is compact. So, $S(t)$ is continuous in the uniform operator topology, and $S(t)$ is uniformly continuous on $[\mu, T]$. Then we get

$$\begin{aligned}
\|u(t+\theta_1) - u(t+\theta_2)\| &\leq \|S(t+\theta_1) - S(t+\theta_1+r)\| \\
&\quad \times \left(\|\varphi(0)\| + \|g(0,\varphi)\| + \|g(t+\theta_1, u_{t+\theta_1}) - g(t+\theta_1+r, u_{t+\theta_1+r})\| \right) \\
&\quad + \int_0^{t+\theta_1} \|A(S(t+\theta_1+r-s) - S(t+\theta_1-s))g(s,u_s)\|\,ds \\
&\quad + \int_{t+\theta_1}^{t+\theta_1+r} \|A(S(t+\theta_1+r-s))g(s,u_s)\|\,ds \\
&\quad + \int_0^{t+\theta_1} \|(S(t+\theta_1+r-s) - S(t+\theta_1-s))\|\|f(s,u_s)\|\,ds \\
&\quad + \int_{t+\theta_1}^{t+\theta_1+r} \|(S(t+\theta_1+r-s))\|\|f(s,u_s)\|\,ds \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \qquad (22)$$

Uniform continuity implies that $I_1 \to 0$ as $r \to 0$ uniformly over $t + \theta_1$. Let us estimate I_2. We get

$$\begin{aligned}
\|g(t + \theta_1, u_{t+\theta_1}) &- g(t+\theta_1+r, u_{t+\theta_1+r})\| \\
&\leq \|g(t+\theta_1, u_{t+\theta_1}) - g(t+\theta_1+r, u_{t+\theta_1})\| + \|g(t+\theta_1+r, u_{t+\theta_1}) - g(t+\theta_1+r, u_{t+\theta_1+r})\| \\
&\leq M\|u_{t+\theta_1} - u_{t+\theta_1+r}\|_C + \|g(t+\theta_1, u_{t+\theta_1}) - g(t+\theta_1+r, u_{t+\theta_1})\|.
\end{aligned}$$

The last summand in the above formula tends to zero as $r \to 0$ uniformly over t, θ_1 and $u_{t+\theta_1}$ due to condition 3) from (H3).

Let us estimate $\|u_{t+\theta_1} - u_{t+\theta_1+r}\|_C$. We have

$$\|u_{t+\theta_1} - u_{t+\theta_1+r}\| \leq \sup_{\theta \in [-h,0]} \|u(t+\theta_1+\theta) - u(t+\theta_1+\theta+r)\|. \tag{23}$$

Again, we consider three cases.

Case 1. $t + \theta_1 + \theta + r < 0$. In this case, uniform convergence to zero of (23) as $r \to 0$ follows from the uniform continuity of $\varphi(t), t \in [-h, 0]$.

Case 2. $t + \theta_1 + \theta < 0, t + \theta_1 + \theta + r > 0$. Then uniform convergence to zero of (23) as $r \to 0$ follows from the uniform continuity of $\varphi(t)$ and $u(t)$ in a neighbourhood of zero point.

Case 3. $t + \theta_1 + \theta > 0$. Then

$$\sup_{\theta \in [-h,0]} \|u(t+\theta_1+\theta) - u(t+\theta_1+\theta+r)\| \leq \sup_{t \in [\mu,\tau]} \|u(t) - u(t+r)\|. \tag{24}$$

Let us estimate I_3 in (22). As the semigroup $S(t)$ is analytic, then for every $x \in X$ we have $S(t)x \in D(A)$ (see [14, Lemma 4.2]) and $AS(t_1)S(t)x = S(t_1)AS(t)x$ for $t_1 > 0, t_2 > 0, x \in X$. Thus, we have

$$A(S(t+\theta_1+r-s) - S(t+\theta_1-s))g(s,u_s) = A\left(S\left(\frac{t+\theta_1-s}{2}+r\right) - S\left(\frac{t+\theta_1-s}{2}\right)\right),$$

$$S\left(\frac{t+\theta_1-s}{2}\right)g(s,u_s) = \left(S\left(\frac{t+\theta_1-s}{2}+r\right) - S\left(\frac{t+\theta_1-s}{2}\right)\right)AS\left(\frac{t+\theta_1-s}{2}\right)g(s,u_s).$$

Therefore,

$$I_3 \leq \int_0^{t+\theta_1} \left\|S\left(\frac{t+\theta_1-s}{2}+r\right) - S\left(\frac{t+\theta_1-s}{2}\right)\right\| \left\|A^{1-\alpha}S\left(\frac{t+\theta_1-s}{2}\right)\right\| \|A^\alpha g(s,u_s)\| \, ds$$

$$\leq \int_0^{t+\theta_1} \left\|S\left(\frac{t+\theta_1-s}{2}+r\right) - S\left(\frac{t+\theta_1-s}{2}\right)\right\| \left\|A^{1-\alpha}S\left(\frac{t+\theta_1-s}{2}\right)\right\| \, ds \sup_{s \in [0,\tau)} \|g(s,u_s)\|.$$

Making a change of variables $(t+\theta_1-s)/2 = s_1$ and taking into account inequality (20), we get

$$I_3 \leq 2 \int_0^\tau \|S(s_1+r) - S(s_1)\| \|A^{1-\alpha}S(s_1)\| ds_1 \cdot C. \tag{25}$$

The continuity of $S(t)$ in the uniform operator topology implies that the function under integral in (25) tends to zero as $r \to 0$. Lemma 2 guarantees that the right-hand part of (25) is an integrable function. So, applying the dominated convergence theorem, we get

$$\int_0^\tau \|S(s_1+r) - S(s_1)\| \|A^{1-\alpha}S(s_1)\| ds_1 \to 0, \quad r \to 0. \tag{26}$$

Uniform over $t + \theta_1$ convergence of I_4 to zero is a consequence of (20) and (15). To estimate I_5, we will use Holder's inequality. We get

$$I_5 \leq \left(\int_0^{t+\theta_1} \|S(t+\theta_1+r-s) - S(t+\theta_1-s)\|^q \, ds\right)^{\frac{1}{q}} \left(\int_0^\tau m^p(s) \, ds\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^\tau \|S(s+r) - S(s)\|^q \, ds\right)^{\frac{1}{q}} \left(\int_0^\tau m^p(s) \, ds\right)^{\frac{1}{p}} \to 0, \quad r \to 0. \tag{27}$$

Using uniform continuity of $S(t)$ and the Lebesgue dominated convergence theorem, we get

$$I_5 \leq M r^{\frac{1}{q}} \left(\int_0^\tau m^p(s) \, ds \right)^{\frac{1}{p}} \to 0, \quad r \to 0. \tag{28}$$

Combining (22)–(28) and using inequality $M_g < 1$, we get

$$\sup_{t+\theta_1 \in [\mu, \tau)} \| u(t + \theta_1) - u(t + \theta_1 + r) \| \to 0, \quad r \to \infty,$$

which means the uniform continuity over θ of the family $\{ u(t + \theta), \ \theta \in [-h, 0], \ t \in [0, \tau) \}$. Thus, the set $P = \{ (t, u_t) : t \in [t_0, \tau) \}$ belongs to a compact set in D, which is a contradiction with Lemma 4. Theorem 2 is proved. $\qquad \square$

3 Applications

Functional-differential equations of parabolic type.

Let Q be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary ∂D. We consider a symmetric elliptic operator

$$A = A(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial}{\partial x_i \partial x_j} = div(a(x), \nabla),$$

where a_{ij} are Holder continuous functions with the Holder exponent $\beta \in (0, 1)$, bounded, and for some C_0 we have

$$\sum_{i,j=1}^d a_{ij} \eta_i \eta_j \geq C_0 \| \eta \|^2, \quad \eta \in \mathbb{R}^d. \tag{29}$$

We denote $X = L^2(Q) = H$, $D(A) = H^2(Q) \cap H_0^1(Q)$.

Consider the following initial boundary-value problem

$$\frac{d}{dt} \left[u(t, x) + b \left(t, x, \int_{-h}^0 \| u(t + \theta) \| \, d\theta \right) \right]$$

$$= div \left[a(x), \nabla_x u(t, x) + f_1 \left(t, x, \int_{-h}^0 \| u(t + \theta) \| \, d\theta \right) \right], \tag{30}$$

$$u(t, x) = \varphi(t, x), \quad t \in [t_0 - h, t_0], \quad x \in Q, \quad u(t, x) = 0, \quad x \in \partial Q, \quad t \in [0, T].$$

Here $\varphi(t, \cdot) \in C([0, T], L^2(Q))$, $t_0 \in [0, T]$. Real-valued functions $b(t, x, y)$, $f_1(t, x, y)$ are given for $t \in [0, T]$, $x \in Q$, $y \in [0, l]$, $l > 0$.

The set $D \subset [0, T) \times C$ is the set $\{ (t, \varphi) : t \in [0, T), \varphi \in U \}$, where U consists of functions $\varphi \in C$ such that

$$\int_{-h}^0 \| \varphi(\theta, \cdot) \| \, d\theta \in (0, l),$$

and ∂U consists of $\varphi \in C$ for which either $\int_{-h}^0 \| \varphi(\theta, \cdot) \| \, d\theta = l$ or $\varphi(\theta, x) = 0$ a.e. We assume that the initial function $\varphi(t, x)$ in (29) also belongs to U. Then $\partial D = ([0, T] \times \partial U) \cup \{ T \} \times \overline{U}$.

It is well-known that A^{-1} is a compact operator (see [3, Section 6.2]), eigenvalues λ_k of A are real numbers with $0 > \lambda_1 \geq \lambda_2 \geq \ldots$ (see [6, Section 1.4]), and the corresponding semigroup $S(t)$ is compact for $t > 0$ and analytic [11] (A is a sectorial self-adjoint operator).

In the sequel, as in [4], we introduce the interpolation space $D_A(1/2, 2) = H_0^1$. According to [2, A.17], $D_A(1/2, 2)$ is isomorphic to $D((-A))^{1/2}$. It means that $X_{1/2} = H_0^1$.

Let us consider assumptions on $b(t, x, y)$ and $f_1(t, x, y)$. Assume that $b(t, x, y)$ is a continuous function, and continuity on t is uniform with respect to x and y. Assume that there exist constants $L > 0, M_g > 0$ such that

$$|\nabla_x b(t, x, y)| \le L, \ |b(t, x, y_1) - b(t, x.y_2)| + |\nabla_x b(t, x, y_1) - \nabla_x b(t, x, y_2)| \le M_g |y_1 - y_2|$$

for all $t \in [0, T]$, $x \in Q$, $y_1, y_2 \in [0, l]$, and $2hLM_g^2 \operatorname{meas}(Q) < 1$, here $\operatorname{meas}(\cdot)$ is a Lebesgue measure in $\mathbb{R}^d$. Assume that $f_1(t, x, y)$ is measurable with respect to t and continuous with respect to $x \in Q$, $y \in [0, l]$, and there exists $m(t) \in L^p(0, T)$, $p > 1$, such that $|f_1(t, x, y)|^2 \le m(t)(1 + |x|^p + |y|^p)$. After introducing the mappings

$$g(t, \varphi)(x) := b\left(t, x, \int_{-h}^0 \|\varphi(\theta, \cdot)\| \, d\theta\right), \tag{31}$$

$$f(t, \varphi)(x) := f_1\left(t, x, \int_{-h}^0 \|\varphi(\theta, \cdot)\| \, d\theta\right), \tag{32}$$

problem (30) can be rewritten in the abstract form (1). Let us verify for (31) and (32) conditions of Theorem 1. From the definition of the uniform metric in C we get continuity of $f(t, \varphi)$ with respect to φ. Its measurability with respect to t is a consequence of t-measurability of f_1. Further,

$$\|f(t, \varphi)\|^2 = \int_Q f_1^2\left(t, x, \int_{-h}^0 \|\varphi(\theta, \cdot)\| \, d\theta\right) dx \le \int_Q m(t)\left(1 + |x|^p + h \sup_{\theta \in [-h, 0]} \|\varphi(\theta, \cdot)\|^2\right) dx.$$

So, assumption (H2) holds.

Let us verify assumtions on $g(t, \varphi)$. Its continuity with respect to t, uniformly over φ, is a consequence of continuity of b. Further,

$$\|g(t, \varphi)\|_{\frac{1}{2}}^2 = \|g(t, \varphi)\|_{H_2^1}^2 = \int_Q |g(t, \varphi)(x)|^2 \, dx + \int_Q |\nabla_x g(t, \varphi)(x)|^2 \, dx.$$

But

$$\int_Q |g(t, \varphi)(x)|^2 \, dx = \int_Q b^2\left(t, x, \int_{-h}^0 \|\varphi(\theta, \cdot) d\theta\|\right) dx < \infty$$

for $t \in [0, T]$, $x \in Q$, $\varphi \in U$, due to the continuity of b. So,

$$\int_Q |\nabla_x g(t, \varphi)(x)|^2 \, dx = \int_Q \left|\nabla_x b\left(t, x, \int_{-h}^0 \|\varphi(\theta, \cdot) d\theta\|\right)\right|^2 dx < L^2 \operatorname{meas}(Q).$$

Finally, we get

$$
\|g(t,\varphi_1) - g(t,\varphi_2)\|_{1/2}^2
$$
$$
= \int_Q |(g(t,\varphi_1)(x) - g(t,\varphi_2)(x))|^2 \, dx + \int_Q |(\nabla_x g(t,\varphi_1)(x) - \nabla_x g(t,\varphi_2)(x))|^2 \, dx
$$
$$
= \int_Q \left| b\left(t,x,\int_{-h}^0 \|\varphi_1(\theta,\cdot)\| \, d\theta\right) - b\left(t,x,\int_{-h}^0 \|\varphi_2(\theta,\cdot)\| \, d\theta\right) \right|^2
$$
$$
+ \int_Q \left| \nabla_x \left(b\left(t,x,\int_{-h}^0 \|\varphi(\theta_1,\cdot)\| \, d\theta\right) - b\left(t,x,\int_{-h}^0 \|\varphi(\theta_2,\cdot)\| \, d\theta\right) \right) \right|^2 dx
$$
$$
\leq 2M_g^2 \int_Q \left(\int_{-h}^0 \left(\|\varphi_1(\theta,\cdot)\| - \|\varphi_2(\theta,\cdot)\| \right) d\theta \right)^2
$$
$$
\leq 2M_g^2 \int_Q \left(\int_{-h}^0 \left(\|\varphi_1(\theta,\cdot) - \varphi_2(\theta,\cdot)\| \right) d\theta \right)^2
$$
$$
\leq 2h M_g^2 \operatorname{meas}(Q) \sup_{\theta \in [-h,0]} \|\varphi_1(\theta,\cdot) - \varphi_2(\theta,\cdot)\|^2.
$$

So, all assumptions of Theorems 1, 2 are fulfilled.

(B) *Parabolic equation with maximum.*

Consider the following initial boundary-value problem

$$
\frac{d}{dt}\left(u(t,x) + b\left(t,x,\max_{s \in I(t)} \|u(s)\|\right) \right) = Au + f_1(t,x,\max_{s \in I(t)} \|u(s)\|), \tag{33}
$$
$$
u(t,x) = \varphi(t,x), \quad t \in [-h,0], \quad x \in Q, \quad u(t,x) = 0, \quad x \in \partial Q, \quad t \in [0,T].
$$

Here $I(t) = [\beta(t), \alpha(t)]$, $\beta(t), \alpha(t)$ are continuous functions on $[0,T]$ such that $\beta(t) \leq \alpha(t) \leq t$ and $\min_{t \in [-h,0]}(\beta(t) - t) = -h$.

Assumptions on A, b, f_1 are the same as in the previous example. Problem (33) can be rewritten in the abstract form (1) if we put

$$
g(t,x)(x) = b\left(t,x,\max_{\theta \in [\beta(t)-t,\alpha(t)-t]} \|\varphi(\theta,\cdot)\|\right), \quad f(t,x)(x) = f_1\left(t,x,\max_{\theta \in [\beta(t)-t,\alpha(t)-t]} \|\varphi(\theta,\cdot)\|\right).
$$

The set $U \subset C$ is the set of functions $\varphi \in C$ such that $\|\varphi(\theta,\cdot)\| \in (0,l)$, $\theta \in [-h,0]$, and ∂U consists of functions from C such that $\|\varphi(\theta,\cdot)\| \in (0,l)$ and either there exists a point $\theta \in [-h,0]$ such that $\|\varphi(\theta,\cdot)\| = l$ or $\varphi(\theta,x) = 0$ for almost all $x \in Q$.

Using the inequality

$$
\left| \sup_\theta \|\varphi_1(\theta,\cdot)\| - \sup_\theta \|\varphi_2(\theta,\cdot)\| \right| \leq \sup_\theta \|\varphi_1(\theta,\cdot) - \varphi_2(\theta,\cdot)\|,
$$

we can verify all assumptions of Theorems 1, 2.

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Перегуда О.В., Станжицький А.О., Мартинюк О.В. *Про існування та продовження м'яких розв'язків функціонально-диференціальних рівнянь нейтрального типу в банахових просторах* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 631–646.

Основним об'єктом дослідження цієї роботи є нескінченновимірні функціонально-диференціальні рівняння нейтрального типу в банахових просторах. Встановлено умови існування м'якого розв'язку початкової задачі та його продовження до межі області визначення. Наявність запізнення в похідній призводить до появи сингулярних доданків. Методи дослідження пов'язані з технікою сильно неперервних напівгруп та дробових степенів оператора. Доведення існування базується на поданні початкової задачі в абстрактній операторній формі з подальшим застосуванням теореми Красносельського про нерухому точку. Для цього вихідний оператор подається у вигляді суми оператора стискання та компактного оператора. Отриманий абстрактний результат застосовано до функціонально-диференціального рівняння в частинних похідних типу "реакція-дифузія".

Ключові слова і фрази: запізнення, сильно неперервна напівгрупа, компактність, нерухома точка, дробовий степінь оператора.