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Conjugation problem with initial-nonlocal conditions for factorized higher order equations

Nytrebych Z.M.[1], Savka I.Ya.[2,✉], Shevchuk R.V.[1], Symotiuk M.M.[2]

In this paper, we consider a problem in the cylindrical domain $(-\alpha,\beta)\times(\mathbb{R}/2\pi\mathbb{Z})$ separated by the hyperplane $\{0\}\times(\mathbb{R}/2\pi\mathbb{Z})$ into nonoverlapping cylindrical subdomains. In particular, this problem can be interpreted as the problem of finding the solution of two factorized PDEs defined in these subdomains respectively, which satisfies the conjugation conditions on the hyperplane as well as the initial-nonlocal conditions on the bottom and top surfaces of the domain.

By the method of separation of variables, the solution can be represented formally as Fourier series, but there is a question about the convergence of the given series in Sobolev spaces of periodic functions with respect to the spatial variables. This convergence is related to the problem of small denominators and may be unstable with respect to small variations of the coefficients of the problem or the parameters of the domain.

We establish the estimates for the small denominators ensuring the convergence of the solutions, from which we obtain the sufficient conditions for the solvability of the problem in Sobolev spaces. The obtained results show that the solvability of the problem depends on the coefficients of the differential equations.

Key words and phrases: hyperbolic equation, conjugation problem, initial-nonlocal condition, small denominator.

[1] Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine
[2] Pidstrygach Institute for Applied Problems of Mechanics and Mathematics, 3b Naukova str., 79060, Lviv, Ukraine
✉ Corresponding author
E-mail: znytrebych@gmail.com (Nytrebych Z.M.), s-i@ukr.net (Savka I.Ya.), r.v.shevchuk@gmail.com (Shevchuk R.V.), quaternion@ukr.net (Symotiuk M.M.)

Introduction

The mixed type partial differential equations have applications in many fields of science, for example in fluid mechanics [18], gas dynamics [8], electromagnetism [1], and transmission problems [12, 16]. In addition, in the mathematical modeling of practical problems, conjugation conditions and boundary conditions are used together to specify the solutions of the differential equations [4, 6, 9].

In the past years, the conjugation problems for equations of different types with various boundary conditions have been studied by many authors. In particular, some boundary conditions can be described as nonlocal when the data on the domain boundary cannot be measured directly, or when the data on the boundary depends on the data inside the domain. The well-posedness of various conjugation problems for such equations with nonlocal conditions has been also studied in many publications (see [2, 3, 13, 15] and references therein). However, the well-posedness of the problem for higher order equations is not well investigated. Generally

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speaking, in the case of a bounded domain, the problems with nonlocal conditions are ill-posed in the sense of Hadamard and are related to the problem of small denominators [11,17].

In this paper, we study the solvability of the conjugation problem for two nonclassical PDEs with the initial-nonlocal in a time variable t and periodic in a spatial variable x conditions. The investigated higher order hyperbolic equations are considered in two different domains and admit a factorization into first-order equations. In the paper [21], the problem with multipoint conditions in the case of multiple nodes is studied by the method of separation of variables with the use of the metric approach. The solution of the conjugation problem was constructed in each subdomain, and its uniqueness and existence were studied.

This paper is organized as follows. In Section 1, we formulate the problem under consideration and the definition of its solution. In Section 2, we formulate the conditions of uniqueness for the conjugation problem formulated in Section 1 and construct its formal solution. Section 3 deals with the proof of the existence of the solution in Sobolev spaces. The metric approach is used in Section 4 for estimating small denominators occurring while constructing the solution. Finally, based on the obtained results, we establish sufficient conditions for the existence of a unique solution of the problem.

1 Problem statement

Let $\Omega = \mathbb{R}/2\pi\mathbb{Z}$ be a unit circle and let $\mathcal{D} = (-\alpha, \beta) \times \Omega$ be a cylindrical domain of the variables (t, x), that is separated by the hyperplane $\{t = 0\} \times \Omega$ into nonoverlapping cylindrical subdomains $\mathcal{D}_- = (-\alpha, 0) \times \Omega$ and $\mathcal{D}_+ = (0, \beta) \times \Omega$, where α and β are positive real numbers. The problem is to find a pair of functions $u_1 = u(t, x)$ and $u_2 = u_2(t, x)$, defined in subdomains $\mathcal{D}_-$ and $\mathcal{D}_+$, respectively, which satisfy the following differential equations

$$\begin{cases} L_1 u \equiv \prod_{j=1}^{n} \Big(\dfrac{\partial}{\partial t} - \lambda_j \dfrac{\partial}{\partial x}\Big) u_1 = 0, \quad (t, x) \in \mathcal{D}_-, \quad n \in \mathbb{N}, \\ L_2 u \equiv \prod_{j=1}^{m} \Big(\dfrac{\partial}{\partial t} - \mu_j \dfrac{\partial}{\partial x}\Big) u_2 = 0, \quad (t, x) \in \mathcal{D}_+, \quad m \in \mathbb{N}, \end{cases} \tag{1}$$

with conjugation conditions on the hyperplane $t = 0$

$$\lim_{t\to 0_-} \frac{\partial^{j-1} u_1}{\partial t^{j-1}} = \lim_{t\to 0_+} \frac{\partial^{j-1} u_2}{\partial t^{j-1}}, \quad j = 1, \ldots, m, \quad x \in \Omega, \tag{2}$$

nonlocal conditions

$$\frac{\partial^{j-1} u_1}{\partial t^{j-1}}\Big|_{t=-\alpha} - \nu_j \frac{\partial^{j-1} u_2}{\partial t^{j-1}}\Big|_{t=\beta} = \varphi_j(x), \quad j = 1, \ldots, m, \quad x \in \Omega, \tag{3}$$

and initial conditions

$$\frac{\partial^{m+j-1} u_1}{\partial t^{m+j-1}}\Big|_{t=-\alpha} = \varphi_{m+j}(x), \quad j = 1, \ldots, n-m, \quad x \in \Omega, \tag{4}$$

where $1 \leq m \leq n$, $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in \mathbb{R}\backslash\{0\}$, $\nu_1, \ldots, \nu_m \in \mathbb{C}\backslash\{0\}$, $\varphi_1(x), \ldots, \varphi_n(x)$ are given functions. Moreover, we suppose that numbers $\lambda_1, \ldots, \lambda_n$ as well as $\mu_1, \ldots, \mu_m$ are pairwise different, respectively.

Below, we use the following functional spaces.

$\mathbf{H}_q = \mathbf{H}_q(\Omega)$, $q \in \mathbb{R}$, is the Sobolev space of all trigonometric series $\varphi(x) = \sum_{k\in\mathbb{Z}} \varphi_k e^{ikx}$, where $\varphi_k \in \mathbb{C}$, with the finite norm $\|\varphi; \mathbf{H}_q\| = \sqrt{\sum_{k\in\mathbb{Z}} (1 + |k|)^{2q} |\varphi_k|^2}$.

$\mathbf{C}^n([a,b];\mathbf{H}_q)$, $n\in\mathbb{Z}_+$, $q\in\mathbb{R}$, is the space of all series of the form $u(t,x)=\sum_{k\in\mathbb{Z}}u_k(t)e^{ikx}$, where $u_k\in\mathbf{C}^n[a,b]$ and, for any fixed point $t\in[a,b]$, functions $\frac{\partial^j u(t,x)}{\partial t^j}\equiv\sum_{k\in\mathbb{Z}}u_k^{(j)}(t)e^{ikx}$ belong to the space $\mathbf{H}_{q-j}$ for $j=0,1,\dots,n$, respectively, and, as the elements of this space, are continuous in t on $[a,b]$; the norm in $\mathbf{C}^n([a,b];\mathbf{H}_q)$ is defined by

$$\|u;\mathbf{C}^n([a,b];\mathbf{H}_q)\|=\sum_{j=0}^{n}\max_{t\in[a,b]}\|\partial^j u(t,\cdot)/\partial t^j;\mathbf{H}_{q-j}\|.$$

Note that $\mathbf{C}([a,b];\mathbf{H}_q):=\mathbf{C}^0([a,b];\mathbf{H}_q)$.

Definition 1. *Let $\varphi_j\in\mathbf{H}_{q-j+1}$, where $j=1,\dots,n$, q is a fixed arbitrary real number. A pair of functions (u_1,u_2) with the property $(u_1,u_2)\in\mathbf{C}^n([-\alpha,0];\mathbf{H}_q)\times\mathbf{C}^m([0,\beta];\mathbf{H}_q)$, satisfying the conditions*

$$\|L_1u_1;\mathbf{C}([-\alpha,0];\mathbf{H}_{q-n})\|=0,\qquad \|L_2u_2;\mathbf{C}([0,\beta];\mathbf{H}_{q-m})\|=0, \tag{5}$$

$$\lim_{\varepsilon\to0_+}\left\|\frac{\partial^{j-1}u_1(-\varepsilon,\cdot)}{\partial t^{j-1}}-\frac{\partial^{j-1}u_2(\varepsilon,\cdot)}{\partial t^{j-1}};\mathbf{H}_{q-j+1}\right\|=0,\quad j=1,\dots,m, \tag{6}$$

$$\left\|\frac{\partial^{j-1}u_1(t,\cdot)}{\partial t^{j-1}}\Big|_{t=-\alpha}-\nu_j\frac{\partial^{j-1}u_2(t,\cdot)}{\partial t^{j-1}}\Big|_{t=\beta}-\varphi_j;\mathbf{H}_{q-j+1}\right\|=0,\quad j=1,\dots,m, \tag{7}$$

$$\left\|\frac{\partial^{m+j-1}u_1(t,\cdot)}{\partial t^{m+j-1}}\Big|_{t=-\alpha}-\varphi_{m+j};\mathbf{H}_{q-m-j+1}\right\|=0,\quad j=1,\dots,n-m, \tag{8}$$

is called the solution of the problem (1)–(4).

2 Uniqueness of the solution

We look for a solution (u_1,u_2) of the problem (1)–(4) in the form of a Fourier series

$$u_1(t,x)=\sum_{k\in\mathbb{Z}}u_{1,k}(t)e^{ikx},\quad t\in[-\alpha,0],\qquad u_2(t,x)=\sum_{k\in\mathbb{Z}}u_{2,k}(t)e^{ikx},\quad t\in[0,\beta]. \tag{9}$$

From the conditions (5)–(8) it follows that for each vector $k\in\mathbb{Z}$ functions $u_{1,k}(t)$ and $u_{2,k}(t)$ are solutions of the problem for ordinary differential equations

$$\begin{cases}\prod_{j=1}^{n}\left(\frac{d}{dt}-i\lambda_jk\right)u_{1,k}(t)=0, & t\in(-\alpha,0),\\ \prod_{j=1}^{m}\left(\frac{d}{dt}-i\mu_jk\right)u_{2,k}(t)=0, & t\in(0,\beta),\end{cases} \tag{10}$$

$$\lim_{t\to0_-}\frac{d^{j-1}u_{1,k}(t)}{dt^{j-1}}=\lim_{t\to0_+}\frac{d^{j-1}u_{2,k}(t)}{dt^{j-1}},\quad j=1,\dots,m, \tag{11}$$

$$\frac{d^{j-1}u_{1,k}(t)}{dt^{j-1}}\Big|_{t=-\alpha}-\nu_j\frac{d^{j-1}u_{2,k}(t)}{dt^{j-1}}\Big|_{t=\beta}=\varphi_{jk},\quad j=1,\dots,m, \tag{12}$$

$$\frac{d^{m+j-1}u_{1,k}(t)}{dt^{m+j-1}}\Big|_{t=-\alpha}=\varphi_{m+j,k},\quad j=1,\dots,n-m, \tag{13}$$

where φ_{jk} are Fourier coefficients of the function φ_j, $j=1,\dots,n$.

For each fixed $k\in\mathbb{Z}$, we construct a solution of problem (10)–(13). There are two cases to consider: $k=0$ and $k\neq0$.

When $k=0$, the solutions of the equation (10) are polynomials with unknown coefficients A_s and B_s, namely $u_{1,0}(t)=\sum_{s=1}^{n}A_st^{n-s}$, $t\in(-\alpha,0)$, $u_{2,0}(t)=\sum_{s=1}^{m}B_st^{m-s}$, $t\in(0,\beta)$. Then it follows from (11) that $B_s=A_{n-m+s}$ for $s=1,\ldots,m$, and $u_{2,0}(t)=\sum_{s=n-m+1}^{n}A_st^{n-s}$.

Using the conditions (12) and (13), the coefficients $A_1,\ldots,A_n$ are solutions of the following system of linear equations

$$\begin{cases}\displaystyle\sum_{s=1}^{n-m}\mathcal{A}_{n-s}^{j-1}(-\alpha)^{n-s-j+1}A_s+\sum_{s=n-m+1}^{n}\mathcal{A}_{n-s}^{j-1}((-\alpha)^{n-s-j+1}-\nu_j\beta^{n-s-j+1})A_s=\varphi_{j,0},\\ \quad j=1,\ldots,m,\\ \displaystyle\sum_{s=1}^{n}\mathcal{A}_{n-s}^{m+j-1}(-\alpha)^{n-s-m-j+1}A_s=\varphi_{m+j,0},\quad j=1,\ldots,n-m,\end{cases}\tag{14}$$

where

$$\mathcal{A}_{n-s}^{j-1}=\begin{cases}\dfrac{(n-s)!}{(n-s-j+1)!}, & n-s\geq j-1,\\ 0, & n-s<j-1.\end{cases}$$

The determinant Δ_0 of the system (14) has block structure and can be calculated as

$$\Delta_0=\det\left[\frac{\mathbf{A}\,\big|\,\mathbf{B}}{\mathbf{C}}\right]=(-1)^n\prod_{j=1}^{n-1}j!\prod_{j=1}^{m}(1-\nu_j),$$

where $\mathbf{A}$ is the rectangular $m\times(n-m)$ matrix, $\mathbf{B}$ is the square matrix of order m, $\mathbf{C}$ is the rectangular $(n-m)\times n$ matrix, namely

$$\mathbf{A}=[\mathcal{A}_{n-s}^{j-1}(-\alpha)^{n-s-j+1}]_{j,s=1}^{m,n-m},\quad \mathbf{B}=[\mathcal{A}_{m-s}^{j-1}((-\alpha)^{m-s-j+1}-\nu_j\beta^{m-s-j+1})]_{j,s=1}^{m,m},$$
$$\mathbf{C}=[\mathcal{A}_{n-s}^{m+j-1}(-\alpha)^{n-s-m-j+1}]_{j,s=1}^{n-m,n}.$$

If we have $\Delta_0\neq0$, then for $k=0$, the solutions of the problem (10)–(13) can be uniquely determined by the formulas

$$u_{1,0}(t)=\frac{1}{\Delta_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_0^{ij}t^{n-j}\varphi_{i,0},\quad t\in(-\alpha,0),$$
$$u_{2,0}(t)=\frac{1}{\Delta_0}\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta_0^{i,n-m+j}t^{m-j}\varphi_{i,0},\quad t\in(0,\beta),$$

where Δ_0^{ij} is the cofactor of (i,j)-element of the determinant Δ_0.

For the case $k\neq0$ solutions of the equations (10) are quasi-polynomials of the form

$$u_{1,k}(t)=\sum_{s=1}^{n}A_{sk}e^{i\lambda_skt},\quad t\in(-\alpha,0),\qquad u_{2,k}(t)=-\sum_{s=1}^{m}B_{sk}e^{i\mu_skt},\quad t\in(0,\beta).\tag{15}$$

Substituting the obtained solutions (15) in conditions (11)–(13), we find that the coefficients A_{sk},B_{sk} satisfy the system of linear $n+m$ equations

$$\begin{cases}\displaystyle\sum_{s=1}^{n}(i\lambda_s)^{j-1}A_{sk}+\sum_{s=1}^{m}(i\mu_s)^{j-1}B_{sk}=0,\quad j=1,\ldots,m,\\ \displaystyle\sum_{s=1}^{n}(i\lambda_s)^{j-1}e^{-i\lambda_sk\alpha}A_{sk}+\nu_j\sum_{s=1}^{m}(i\mu_s)^{j-1}e^{i\mu_sk\beta}B_{sk}=k^{1-j}\varphi_{jk},\quad j=1,\ldots,m,\\ \displaystyle\sum_{s=1}^{n}(i\lambda_s)^{m+j-1}e^{-i\lambda_sk\alpha}A_{sk}=k^{1-m-j}\varphi_{m+j,k},\quad j=1,\ldots,n-m.\end{cases}\tag{16}$$

The $(n+m)\times(n+m)$ matrix $\mathbf{M}_k$ of the system (16) is given as follows

$$\mathbf{M}_k = \left[\begin{array}{ccc|ccc} 1 & \dots & 1 & 1 & \dots & 1 \\ (i\lambda_1) & \dots & (i\lambda_n) & (i\mu_1) & \dots & (i\mu_m) \\ (i\lambda_1)^2 & \dots & (i\lambda_n)^2 & (i\mu_1)^2 & \dots & (i\mu_m)^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (i\lambda_1)^{m-1} & \dots & (i\lambda_n)^{m-1} & (i\mu_1)^{m-1} & \dots & (i\mu_m)^{m-1} \\ \hline e^{-ik\lambda_1\alpha} & \dots & e^{-ik\lambda_n\alpha} & \nu_1 e^{ik\mu_1\beta} & \dots & \nu_1 e^{ik\mu_m\beta} \\ (i\lambda_1)e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)e^{-ik\lambda_n\alpha} & \nu_2(i\mu_1)e^{ik\mu_1\beta} & \dots & \nu_2(i\mu_m)e^{ik\mu_m\beta} \\ (i\lambda_1)^2e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^2e^{-ik\lambda_n\alpha} & \nu_3(i\mu_1)^2e^{ik\mu_1\beta} & \dots & \nu_3(i\mu_m)^2e^{ik\mu_m\beta} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (i\lambda_1)^{m-1}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{m-1}e^{-ik\lambda_n\alpha} & \nu_m(i\mu_1)^{m-1}e^{ik\mu_1\beta} & \dots & \nu_m(i\mu_m)^{m-1}e^{ik\mu_m\beta} \\ \hline (i\lambda_1)^{m}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{m}e^{-ik\lambda_n\alpha} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (i\lambda_1)^{n-1}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{n-1}e^{-ik\lambda_n\alpha} & 0 & \dots & 0 \end{array}\right].$$

Alternatively, it can be represented as a block matrix of the form

$$\mathbf{M}_k = \left[\begin{array}{c|c} \mathbf{W}_\lambda & \mathbf{W}_\mu \\ \hline \mathbf{E}_k^\alpha & \begin{array}{c}\mathbf{E}_k^\beta \\ \mathbf{O}\end{array} \end{array}\right],$$

where $\mathbf{O}$ is the rectangular zero $(n-m)\times m$ matrix, $\mathbf{W}_\lambda = [(i\lambda_s)^{j-1}]_{j,s=1}^{m,n}$, $\mathbf{W}_\mu = [(i\mu_s)^{j-1}]_{j,s=1}^{m}$, $\mathbf{E}_k^\alpha = [(i\lambda_s)^{j-1}e^{-ik\lambda_s\alpha}]_{j,s=1}^{n,n}$, $\mathbf{E}_k^\beta = [\nu_j(i\mu_s)^{j-1}e^{ik\mu_s\beta}]_{j,s=1}^{m}$.

If $\Delta_k := \det \mathbf{M}_k \neq 0$, then by Cramer's rule, we have

$$A_{sk} = \frac{1}{\Delta_k}\sum_{j=1}^{n} \Delta_k^{j+m,s}\frac{\varphi_{jk}}{k^{j-1}}, \quad s = 1,\dots,n, \tag{17}$$

$$B_{sk} = \frac{1}{\Delta_k}\sum_{j=1}^{n} \Delta_k^{j+m,s+n}\frac{\varphi_{jk}}{k^{j-1}}, \quad s = 1,\dots,m, \tag{18}$$

where Δ_k^{is} denotes the cofactor of (i,s)-element of the determinant Δ_k obtained from Δ_k deleting its the ith row and the sth column and multiplying by $(-1)^{i+s}$.

Substituting the coefficients (17)–(18) into formulas (15), we get the following solutions to the problem (10)–(13) for $k \neq 0$:

$$u_{1,k}(t) = \frac{1}{\Delta_k}\sum_{s=1}^{n}\sum_{j=1}^{n} \Delta_k^{j+m,s}\frac{\varphi_{jk}}{k^{j-1}}e^{ik\lambda_s t}, \qquad u_{2,k}(t) = -\frac{1}{\Delta_k}\sum_{s=1}^{m}\sum_{j=1}^{n} \Delta_k^{j+m,s+n}\frac{\varphi_{jk}}{k^{j-1}}e^{ik\mu_s t}. \tag{19}$$

Theorem 1. *The problem* (1)–(4) *cannot have two distinct solutions if and only if*

$$\Delta_k \neq 0 \quad \forall\, k \in \mathbb{Z}. \tag{20}$$

The proof follows from the uniqueness of the Fourier expansion for periodic functions based on the orthogonal system $\{e^{ikx}\}_{k\in\mathbb{Z}}$. Thus, if the condition (20) is fulfilled, we have formal solution of the problem

$$\begin{aligned} u_1(t,x) &= \frac{1}{\Delta_0}\sum_{s=1}^{n}\sum_{j=1}^{n}\Delta_0^{sj}t^{n-j}\varphi_{s,0} + \sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{s=1}^{n}\sum_{j=1}^{n}\frac{\Delta_k^{j+m,s}\varphi_{jk}}{k^{j-1}\Delta_k}e^{ik(\lambda_s t+x)}, \quad t\le 0, \\ u_2(t,x) &= \frac{1}{\Delta_0}\sum_{s=1}^{n}\sum_{j=1}^{m}\Delta_0^{s,n-m+j}t^{m-j}\varphi_{s,0} - \sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{s=1}^{m}\sum_{j=1}^{n}\frac{\Delta_k^{j+m,s+n}\varphi_{jk}}{k^{j-1}\Delta_k}e^{ik(\mu_s t+x)}, \quad t\ge 0. \end{aligned} \tag{21}$$

3 The conditions of existence of the solution

In what follows, we assume that condition (20) is satisfied. Then there exists a unique solution of the problem (1)–(4), which admits a formal representation (21).

The existence of the solution $u = (u_1, u_2) \in \mathbf{C}^n([-\alpha, 0]; \mathbf{H}_q) \times \mathbf{C}^m([0, \beta]; \mathbf{H}_q)$ of the problem (1)–(4) is related to the so-called problem of small denominator [11], because the terms of the sequence $\{\Delta_k\}_{k\in\mathbb{Z}}$ in the denominator of the formula (21), being different from zero by the condition (20), can arbitrarily rapidly tend to zero if $|k| \to +\infty$. This can lead to a divergence of series (21) in the spaces $\mathbf{C}^n([-\alpha, 0]; \mathbf{H}_q)$ and $\mathbf{C}^m([0, \beta]; \mathbf{H}_q)$ respectively, therefore we get unsolvable problem. The following example demonstrates this and shows that the solvability of the problem depends on the Diophantine approximation of real numbers by rational numbers.

Example 1. *Consider the problem* (1)–(4) *in the special case with* $n = 2$ *and* $m = 1$*:*

$$\begin{aligned} &\Big(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\Big)u_1 = 0, \quad (t,x)\in(-2\pi,0)\times\Omega, \\ &\Big(\frac{\partial}{\partial t}-\lambda\frac{\partial}{\partial x}\Big)u_2 = 0, \quad (t,x)\in(0,\pi)\times\Omega, \quad \lambda>0, \end{aligned} \tag{22}$$

$$\begin{aligned} &u_1|_{t=0_-} = u_2|_{t=0_+}, \quad x\in\Omega, \\ &u_1|_{t=-2\pi} + u_2|_{t=\pi} = \sum_{k\in\mathbb{Z}\setminus\{0\}}|k|^{-\varrho}e^{ikx}, \quad x\in\Omega, \quad \varrho>0, \\ &\frac{\partial u_1}{\partial t}\Big|_{t=-2\pi} = 0, \quad x\in\Omega. \end{aligned} \tag{23}$$

The formal solutions to the problem (22)–(23) are

$$\begin{cases} u_1(t,x) = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{|k|^{-\varrho}\cos(kt)}{1+e^{i\pi k\lambda}}e^{ikx}, & (t,x)\in[-2\pi,0]\times\Omega; \\ u_2(t,x) = \displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{|k|^{-\varrho}e^{ik\lambda t}}{1+e^{i\pi k\lambda}}e^{ikx}, & (t,x)\in(0,\pi)\times\Omega. \end{cases}$$

From the equality for the denominator

$$|1+e^{i\pi k\lambda}| = 2\Big|\sin\Big(\frac{\pi k\lambda}{2}+\frac{\pi}{2}\Big)\Big| = 2\Big|\sin\Big(\frac{\pi k\lambda}{2}+\frac{\pi}{2}-\pi m\Big)\Big|, \quad k\in\mathbb{Z}\setminus\{0\}, \quad m\in\mathbb{Z},$$

and the inequality

$$2|k|\Big|\lambda-\frac{2m_0-1}{k}\Big| \le 2\Big|\sin\Big(\frac{\pi k\lambda}{2}+\frac{\pi}{2}\Big)\Big| \le \pi|k|\Big|\lambda-\frac{2m-1}{k}\Big|, \quad k\in\mathbb{Z}\setminus\{0\}, \quad m\in\mathbb{Z},$$

the two-sided estimate can be derived for the problem's solution

$$\tilde{C}_1\sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{(1+|k|)^{2(q-\varrho-1)}}{\Big|\lambda-\dfrac{2m-1}{k}\Big|^2}} \le \|u_2;\mathbf{C}^1([0,\pi];\mathbf{H}_q)\| \le \tilde{C}_2\sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{(1+|k|)^{2(q-\varrho-1)}}{\Big|\lambda-\dfrac{2m_0-1}{k}\Big|^2}}, \tag{24}$$

where $m \in \mathbb{Z}$, $\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$ and integer $m_0 = m_0(\lambda, k)$ such that

$$\Big|\frac{k\lambda}{2} + \frac{1}{2} - \pi m_0\Big| \le \frac{1}{2}.$$

The Thue-Siegel-Roth's theorem [19] states that for any algebraic number λ and any $\epsilon > 0$, there exists a constant $c = c(\lambda, \epsilon) > 0$ such that $|\lambda - m/n| > c/n^{2+\epsilon}$ for all irreducible rationals m/n. In this case, the series on the right hand side of the inequality (24) converges, when $\varrho = q + 3/2 + \epsilon$, and therefore $u_2 \in \mathbf{C}^1([0,\pi]; \mathbf{H}_q)$ for any algebraic number λ.

Similar results in the metric theory of Diophantine approximation provide statements that are valid for almost all numbers in the sense of the Lebesgue measure, i.e. for all numbers outside a set of Lebesgue measure 0. Let $\mathcal{A}$ be defined as

$$\mathcal{A}(\psi) = \Big\{\lambda \in [0,1] : \Big|\lambda - \frac{m}{n}\Big| < \frac{\psi(n)}{n} \text{ for infinitely many pair } (m,n) \in \mathbb{Z} \times \mathbb{N}\Big\},$$

where $\psi : (0, +\infty) \to (0,1)$. From the convergence case of Khinchin's theorem on metric Diophantine approximation [14], we get $\operatorname{meas}_{\mathbb{R}} \mathcal{A}(\psi) = 0$, if $\sum_{n=1}^{+\infty} \psi(n) < \infty$.

In particular, if $\psi(n) := n^{-(1+\epsilon)}$ and $\varrho = q + 3/2 + \epsilon$ for some positive $\epsilon > 0$, then almost all $\lambda \in [0,1]$ are not very well ψ-approximable and the series on the right hand side of the inequality (24) also converges for almost all (with respect to the Lebesgue measure in $\mathbb{R}$) numbers $\lambda \in [0,1]$.

However, there are irrational numbers that can be approximated arbitrarily well by a particular sequence of finite continued fractions (convergents). According to the Khinchin's theorem (see [14, Theorem 22] or [20, Theorem 2.13]), for any function $\psi(n) : \mathbb{N} \to \mathbb{R}_+$ there exist irrational number $\lambda \in \mathbb{R}$, such that for infinitely many values $n \in \mathbb{N}$ the inequality $|\lambda - m/n| < \psi(n)/n$ holds. It follows that there exists an irrational number $\lambda = \lambda_0$ (ψ-approximable number by rationals) such that the inequality

$$\Big|\lambda_0 - \frac{2m-1}{k}\Big| < \frac{\psi(k)}{k}, \quad \psi(k) := \frac{2^{-k}}{k}, \tag{25}$$

has infinitely many solutions in integers m and $k > 0$.

For example, this number $\lambda_0 \in \mathcal{A}(\psi)$ can be expressed uniquely as an infinite continued fraction (see [20, Theorem 2.8]) $\lambda_0 = [0; a_1, a_2, \ldots, a_{n+1}, \ldots]$ with convergents p_n/q_n and recursive transformation $a_{n+1} = 2^{q_n} + 2$, $q_n = a_n q_{n-1} + q_{n-2}$, $q_0 = 1$, $q_{-1} = 0$. All convergents converge with rate $k^{-2}2^{-k}$. The fast approximation (25) is associated with the growth of the coefficients $a_n, n \in \mathbb{N}$. In this case, the $\mathbf{C}^1([0,\pi]; \mathbf{H}_q)$-norm of the solution u_2 of the problem (22)–(23) with $\lambda = \lambda_0$ is not finite for all possible values of q and ϱ because the series on the left hand side of the inequality (24) diverges. As you can see, the solvability of the problem (22)–(23) depends on the Diophantine approximation of real number λ by rational numbers.

Therefore, to obtain the correct solvability of the given problem with the relevant restrictions on the functions $\varphi_1, \ldots, \varphi_n$, it is important to establish lower bounds for modules of the quantities Δ_k with respect to $k \in \mathbb{Z}$. This is one of the objectives of this paper. Note that the estimates of the small denominators Δ_k, which have a nonlinear structure, have not yet been considered in the scientific literature.

Theorem 2. *Let the condition* (20) *holds. Assume also that there exits some constants* $c_0 > 0$ *and* $\eta \in \mathbb{R}$ *such that for all vectors* $k \in \mathbb{Z}$ *the following inequality*

$$|\Delta_k| \geq c_0(1+|k|)^{-\eta} \tag{26}$$

is satisfied. If $\varphi_j \in \mathbf{H}_{q+\eta+1-j}$, $j = 1,\dots,n$, *then the problem* (1)–(4) *has a unique solution in the space* $\mathbf{C}^n([-\alpha,0];\mathbf{H}_q) \times \mathbf{C}^m([0,\beta];\mathbf{H}_q)$.

Proof. First, we need to estimate the coefficients $u_{1,k}(t)$ and $u_{2,k}(t)$ for every $k \in \mathbb{Z}$. For this end, let us start with the upper estimates of the determinants $\Delta_k^{j+m,s}$, $j = 1,\dots,n$, $s = 1,\dots,n+m$, in the formulas (21).

From Hadamard's inequality for determinants we obtain that there exists some constant $c_4 > 0$ independent of the vector k and indices j and s such that for all determinants we have

$$|\Delta_k^{j+m,s}| \leq c_4 = c_3^{n+m-1}(n+m-1)^{(n+m-1)/2}, \quad s = 1,\dots,n+m, \tag{27}$$

where $c_3 := \max\{c_1, c_2\}$, $c_1 := \max_{j,s\in\{1,\dots,n\}}\{|\lambda_s|^{j-1}\}$, $c_2 := \max_{j,s\in\{1,\dots,m\}}\{|\mu_s|^{j-1}, |\nu_j\mu_s|^{j-1}\}$. Using the estimates (27) for the upper bound and the apriori condition (26), from formula (19) and inequality $|k| > (1+|k|)/2$, for $k \neq 0$ we get

$$\begin{aligned}
|u_{1,k}^{(\ell)}(t)| &\leq \frac{1}{|\Delta_k|}\sum_{s=1}^{n}\sum_{j=1}^{n}\frac{|\lambda_s k|^{\ell}|\Delta_k^{j+m,s}||\varphi_{jk}|}{|k|^{j-1}} \leq \frac{nc_4c_5}{c_0}\sum_{j=1}^{n}2^{j-1}(1+|k|)^{\ell+\eta+1-j}|\varphi_{jk}|,\\
&\ell = 0,1,\dots,n, \quad t<0,\\
|u_{2,k}^{(\ell)}(t)| &\leq \frac{1}{|\Delta_k|}\sum_{s=1}^{m}\sum_{j=1}^{n}\frac{|\mu_s k|^{\ell}|\Delta_k^{j+m,s+n}||\varphi_{jk}|}{|k|^{j-1}} \leq \frac{mc_4c_6}{c_0}\sum_{j=1}^{n}2^{j-1}(1+|k|)^{\ell+\eta+1-j}|\varphi_{jk}|,\\
&\ell = 0,1,\dots,m, \quad t>0,
\end{aligned}$$

where $c_5 := \max\limits_{\substack{s\in\{1,\dots,n\},\\ \ell\in\{0,\dots,n\}}}\{|\lambda_s|^{\ell}\} > 0$, $c_6 := \max\limits_{\substack{s\in\{1,\dots,m\},\\ \ell\in\{0,\dots,m\}}}\{|\mu_s|^{\ell}\} > 0$.

It is easy to see that there exists constant $c_7 > 0$ such that for $k = 0$ we have

$$|u_{s,0}^{(\ell)}(t)| \leq c_7\sum_{j=1}^{n}|\varphi_{j,0}|, \quad s = 1,2.$$

Then

$$\begin{aligned}
\|u_1;\mathbf{C}^n([-\alpha,0];\mathbf{H}_q)\| &= \sum_{\ell=0}^{n}\max_{t\in[-\alpha,0]}\|\partial^{\ell}u_1/\partial t^{\ell};\mathbf{H}_{q-\ell}\| = \sum_{\ell=0}^{n}\max_{t\in[-\alpha,0]}\sqrt{\sum_{k\in\mathbb{Z}}(1+|k|)^{2(q-\ell)}|u_{1,k}^{(\ell)}(t)|^2}\\
&\leq (n+1)\max\{n2^n c_4c_5/c_0, c_7\}\sqrt{n\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}}(1+|k|)^{2(q+\eta+1-j)}|\varphi_{jk}|^2}\\
&\leq (n+1)\max\{n2^n c_4c_5/c_0, c_7\}\sqrt{n\sum_{j=1}^{n}\|\varphi_j;\mathbf{H}_{q+\eta+1-j}\|^2}\\
&\leq c_8\sum_{j=1}^{n}\|\varphi_j;\mathbf{H}_{q+\eta+1-j}\|, \qquad c_8 := (n+1)\sqrt{n}\max\{n2^n c_4c_5/c_0, c_7\};
\end{aligned}$$

$$\|u_2;\mathbf{C}^m([0,\beta];\mathbf{H}_q)\| = \sum_{\ell=0}^{m}\max_{t\in[0,\beta]}\sqrt{\sum_{k\in\mathbb{Z}}(1+|k|)^{2(q-\ell)}|u_{2,k}^{(\ell)}(t)|^2}$$

$$\le (m+1)\sqrt{n}\max\{m2^n c_4c_6/c_0,c_7\}\sqrt{\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}}(1+|k|)^{2(q+\eta+1-j)}|\varphi_{jk}|^2}$$

$$\le c_9\sum_{j=1}^{n}\|\varphi_j;\mathbf{H}_{q+\eta+1-j}\|, \qquad c_9 := (m+1)\sqrt{n}\max\{m2^n c_4c_6/c_0,c_7\}.$$

The proof of the theorem is complete. □

4 The metric estimates of the small denominators

Now we study the conditions of validity of the inequalities (26). For this, we use Borel-Cantelli lemma and the auxiliary assertion on the upper bound for the Lebesgue measures of the exceptional set $E(f,\varepsilon,[a,b])$ of a smooth function defined as

$$E(f,\varepsilon,[a,b]) = \{t\in[a,b] : |f(t)|<\varepsilon\}.$$

Let $\{A_k\}_{k\in\mathbb{N}}$ be a countable family of measurable subsets $\mathbb{R}^m$ and let $\operatorname{meas}A_k := \operatorname{meas}_{\mathbb{R}^m}A_k$ denotes the Lebesgue measure of a set A_k. The limit superior of $\{A_k\}_{k\in\mathbb{N}}$ is the set

$$A =: \limsup_{k\to\infty} A_k = \{a : a\in A_k \text{ for infinitely many } k\} = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k.$$

Lemma 1 (Convergence Borel-Cantelli lemma [7, 10]). *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence such that $\sum_{k=1}^{\infty}\operatorname{meas}A_k<\infty$. Then the set of all elements that belong to infinitely many sets A_k has zero Lebesgue measure in $\mathbb{R}^m$, i.e.* $\operatorname{meas}A=0$.

Let $R(\lambda)$ be a polynomial in the real variable λ of degree n of the form

$$R(\lambda)\equiv\lambda^n+\sum_{j=1}^{n}a_j\lambda^{n-j}, \quad a_1,\dots,a_n\in\mathbb{C}, \tag{28}$$

and let $Q(t)$ be a quasi-polynomial of the form

$$Q(t)\equiv\sum_{j=1}^{m}e^{z_jt}p_j(t), \quad z_j\ne z_q, \quad j\ne q, \tag{29}$$

where $z_1,\dots,z_m\in\mathbb{C}$ and $p_1(t),\dots,p_m(t)$ are polynomials of degrees $n_1-1,\dots,n_m-1$, respectively. Here $n_1,\dots,n_m\in\mathbb{N}$. For $R(t)$ and $Q(t)$, we denote

$$A_R = 1+\max_{0\le j\le n-1}\left|\frac{R^{(j)}(0)}{j!}\right|^{1/(n-j)}, \quad M_Q = 1+\max_{1\le j\le m}|z_j|, \quad n_0\equiv n_1+\ldots+n_m.$$

Lemma 2 ([5]). *Let $R(\lambda)$ and $Q(t)$ be the polynomial and the quasi-polynomial defined by equalities* (28) *and* (29), *respectively. If there exists some constant $\delta\in\mathbb{R}$ such that the condition $\forall t\in[a,b]$ $|R(d/dt)Q(t)|\ge\delta>0$ is fulfilled, then for any $\varepsilon\in(0,\frac{\delta}{(2n+2)A_R^n}]$ the estimate* $\operatorname{meas}_{\mathbb{R}}E(Q,\varepsilon,[a,b])\le C_1M_Q(\varepsilon/\delta)^{1/n}$ *holds, where $C_1=C_1(n,n_0,b-a)>0$.*

Let $\Pi_m=\{\vec{\mu}\equiv(\mu_1,\dots,\mu_m)\in\mathbb{R}^m : \mu_1<\ldots<\mu_m\}$, $W(a_1,\dots,a_n) := \det[a_i^{j-1}]_{i,j=1}^n$ denotes the Vandermonde determinant constructed by the values of $a_1,\dots,a_n$.

Theorem 3. *If $\eta > 2m^2 - m$ and $C_0 = |W(\lambda_1, \dots, \lambda_n)| > 0$, then for almost all (with respect to Lebesgue measure in $\mathbb{R}^m$) vectors $\vec{\mu} \in \Pi_m$ the inequality $|\Delta_k| \geq C_0|k|^{-\eta}$ holds for all (except for finitely many numbers) $k \in \mathbb{Z}$.*

Proof. Let $P_m = \prod_{j=1}^{m}[a_j, b_j]$ be an arbitrary fixed parallelepiped such that $P_m \subset \Pi_m$. In the proof, we will assume that the determinant Δ_k is a function of the variable $\vec{\mu}$, $\Delta_k(\vec{\mu}) := \Delta_k$. For each $k \neq 0$, we introduce the following sets $E_\eta(k) = \{\vec{\mu} \in P_m : |\Delta_k(\vec{\mu})| < C_0|k|^{-\eta}\}$. Let E_η be the set of those vectors that belong to infinitely many sets $E_\eta(k)$, $k \in \mathbb{Z}$, namely

$$E_\eta = \limsup_{|k| \to +\infty} E_\eta(k) = \bigcap_{K=0}^{+\infty} \bigcup_{|k| \geq K} E_\eta(k).$$

To prove the theorem, it suffices to show that $\operatorname{meas}_{\mathbb{R}^m} E_\eta = 0$ for any $P_m \subset \Pi_m$. For this end, we use the Borel-Cantelli lemma and establish the convergence of the series $\sum_{k \in \mathbb{Z}} \operatorname{meas}_{\mathbb{R}^m} E_\eta(k)$. In particular, for its convergence, let us prove the estimate $\operatorname{mes}_{\mathbb{R}^m} E_\eta(k) \leq C_2|k|^{-1-\varepsilon}$ for all (except for, perhaps, finitely many numbers) $k \in \mathbb{Z}\backslash\{0\}$, where ε and C_2 are some positive constants independent of k.

Let us permute blocks in the matrix $\mathbf{M}_k$ so that

$$\tilde{\mathbf{M}}_k = \left[\begin{array}{c|c} \mathbf{E}_k^\alpha & \begin{matrix}\mathbf{E}_k^\beta \\ \mathbf{O}\end{matrix} \\ \hline \mathbf{W}_\lambda & \mathbf{W}_\mu \end{array}\right],$$

namely

$$\tilde{\mathbf{M}}_k = \left[\begin{array}{ccc|ccc}
e^{-ik\lambda_1\alpha} & \dots & e^{-ik\lambda_n\alpha} & \nu_1 e^{ik\mu_1\beta} & \dots & \nu_1 e^{ik\mu_m\beta} \\
(i\lambda_1)e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)e^{-ik\lambda_n\alpha} & \nu_2(i\mu_1)e^{ik\mu_1\beta} & \dots & \nu_2(i\mu_m)e^{ik\mu_m\beta} \\
(i\lambda_1)^2e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^2e^{-ik\lambda_n\alpha} & \nu_3(i\mu_1)^2e^{ik\mu_1\beta} & \dots & \nu_3(i\mu_m)^2e^{ik\mu_m\beta} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(i\lambda_1)^{m-1}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{m-1}e^{-ik\lambda_n\alpha} & \nu_m(i\mu_1)^{m-1}e^{ik\mu_1\beta} & \dots & \nu_m(i\mu_m)^{m-1}e^{ik\mu_m\beta} \\ \hline
(i\lambda_1)^{m}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{m}e^{-ik\lambda_n\alpha} & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(i\lambda_1)^{n-1}e^{-ik\lambda_1\alpha} & \dots & (i\lambda_n)^{n-1}e^{-ik\lambda_n\alpha} & 0 & \dots & 0 \\ \hline
1 & \dots & 1 & 1 & \dots & 1 \\
(i\lambda_1) & \dots & (i\lambda_n) & (i\mu_1) & \dots & (i\mu_m) \\
(i\lambda_1)^2 & \dots & (i\lambda_n)^2 & (i\mu_1)^2 & \dots & (i\mu_m)^2 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(i\lambda_1)^{m-1} & \dots & (i\lambda_n)^{m-1} & (i\mu_1)^{m-1} & \dots & (i\mu_m)^{m-1}
\end{array}\right].$$

It is obvious that $\Delta_k(\vec{\mu}) = (-1)^{(m+n-1)m} \det \tilde{\mathbf{M}}_k$.

We denote by Δ_{jk} the determinant obtained from $\det \tilde{\mathbf{M}}_k$ by crossing out last $(n+m-j)$ rows and last $(n+m-j)$ columns, where $j = n, \dots, n+m$. Moreover, $\Delta_{jk} = \Delta_{jk}(\mu_1, \dots, \mu_{j-n})$, $j = n, \dots, n+m$, $\Delta_{nk} \equiv \det[\mathbf{E}_k^\alpha] = W(i\lambda_1, \dots, i\lambda_n)e^{-i\alpha(\lambda_1+\dots+\lambda_n)k}$, $|\Delta_{nk}| = |W(\lambda_1, \dots, \lambda_n)|$, $\Delta_{n+m,k} = \det \tilde{\mathbf{M}}_k$.

For every $k \in \mathbb{Z}\backslash\{0\}$, consider the sets:

$$F_\eta(k) = \{\vec{\mu} \in P_m : |\Delta_{n+m,k}(\vec{\mu})| < \nu_{n+m}(k, \vec{\lambda})\},$$

$$F_\eta(j,k) = \{\vec{\mu} \in P_m : |\Delta_{jk}(\mu_1, \dots, \mu_{j-n})| < \nu_j(k, \vec{\lambda}),$$

$$|\Delta_{j-1,k}(\mu_1, \dots, \mu_{j-n+1})| \geq \nu_{j-1}(k, \vec{\lambda})\}, \quad j = \overline{n+1, n+m},$$

where $\nu_j(k,\vec{\lambda})$, $\vec{\lambda}=(\lambda_1,\ldots,\lambda_n)$, are defined as follows:

$$\nu_n(k,\vec{\lambda})=|\Delta_{nk}|=|W(\lambda_1,\ldots,\lambda_n)|\neq 0,\quad \nu_j(k,\vec{\lambda})=\nu_n(k,\vec{\lambda})\xi_j(k),\quad j>n,$$
$$\xi_j(k)=\frac{\xi_{j-1}(k)}{|k|^{2(j-n-1+m)-m+\varepsilon_j}},\quad j>n,\quad \xi_n(k)=1,$$

$\varepsilon_j=\varepsilon_0/2^{n+m-j+1}(1-2^{-m})$, $j=n+1,\ldots,n+m$, the number $\varepsilon_0>0$ is arbitrarily small.

Note that

$$F_\eta(k)\subset\bigcup_{j=n+1}^{n+m}F_\eta(j,k)\quad \forall\, k\in\mathbb{Z}\backslash\{0\}.$$

Indeed, let $\vec{\mu}\in F_\eta(k)$. If $\vec{\mu}\notin F_\eta(j,k)$ for all $j\in\{n+1,\ldots,n+m\}$, then one can obtain the contradictory inequality $\nu_n(k,\vec{\mu})\equiv|\Delta_{nk}|<\nu_n(k,\vec{\mu})$, which contradicts the definition of $F_\eta(j,k), n+1\leq j\leq n+m$. Therefore, $\vec{\mu}\in F_\eta(j_0,k)$ for some number $j_0\in\{n+1,\ldots,n+m\}$, and the required inclusion is true.

Thus,

$$\text{meas}_{\mathbb{R}^m}F_\eta(k)\leq\sum_{j=n+1}^{n+m}\text{meas}_{\mathbb{R}^m}F_\eta(j,k).\tag{30}$$

By Fubini's theorem, for each of the sets $F_\eta(j,k)$, $n+1\leq j\leq n+m$, we have

$$\text{meas}_{\mathbb{R}^m}F_\eta(j,k)=\int_{\overline{P}_{j-n}}\text{meas}_{\mathbb{R}}F_\eta(j,k,\vec{\mu}_{j-n})d\vec{\mu}_{j-n},\tag{31}$$

where

$$\overline{P}_s=\prod_{q=1,q\neq s}^{m}[a_q,b_q],\quad \vec{\mu}_s=(\mu_1,\ldots,\mu_{s-1},\mu_{s+1},\ldots,\mu_m),\quad s\in\{1,\ldots,m\},$$
$$F_\eta(j,k,\vec{\mu}_{j-n})=\{\mu_{j-n}\in[a_{j-n},b_{j-n}]:\vec{\mu}\in F_\eta(j,k)\},\quad j=n+1,\ldots,n+m.$$

For upper estimates of the Lebesgue measures of sets $F_\eta(j,k,\vec{\mu}_{j-n})$, we use Lemma 2. For this end, we introduce the polynomials $R_j(\xi,k)=\xi^{j-n-1}(\xi-ik\beta)^m$ with $n+1\leq j\leq n+m$. If we apply the differential operator $R_j(\frac{\partial}{\partial\mu_{j-n}},k)$ to the expansion of the determinant $\Delta_j(k,\mu_1,\ldots,\mu_{j-n})$, $j=\overline{n+1,n+m}$, along the last column, then as a result, we get

$$R_j\Big(\frac{\partial}{\partial\mu_{j-n}},k\Big)\Delta_{jk}(\mu_1,\ldots,\mu_{j-n})=(j-n-1)!i^{j-n-1}(-ik\beta)^m\Delta_{j-1,k}(\mu_1,\ldots,\mu_{j-n-1}).\tag{32}$$

If $\vec{\mu}\in F_\eta(j,k)$, then from the formulas (32) and the definition of the set $F_\eta(j,k)$, it follows that

$$\Big|R_j\Big(\frac{\partial}{\partial\mu_{j-n}},k\Big)\Delta_{jk}(\mu_1,\ldots,\mu_{j-n})\Big|\geq\beta^m|k|^m\nu_{j-1}(k,\vec{\lambda}),\quad n+1\leq j\leq n+m.\tag{33}$$

For fixed $\mu_1,\ldots,\mu_{j-n-1}$ the determinant $\Delta_{jk}(\mu_1,\ldots,\mu_{n-j})$ as a function of the variable μ_{j-n} is a quasi-polynomial with $M_{\Delta_{jk}}=1+\beta|k|$, where $j=n+1,\ldots,n+m$.

The degree of the polynomial $R_j(\xi,k)$ in the variable ξ is equal to $j-n-1+m$, $\deg R_j=j-n-1+m$, and the value A_{R_j} is calculated according to the formula

$$A_{R_j}=1+\max_{0\leq q\leq\deg R_j}\Big|\frac{R_j^{(q)}(0,k)}{q!}\Big|^{\frac{1}{\deg R_j-q}},\quad R_j^{(q)}(0,k)=\frac{d^qR_j(\xi,k)}{d\xi^q}\Big|_{\xi=0}.$$

Since

$$R_j^{(q)}(0,k)=\sum_{\ell=0}^{q}C_q^\ell[\xi^{j-n-1}]^{(\ell)}[(\xi-ik\beta)^m]^{(q-\ell)}\Big|_{\xi=0}=\frac{C_q^{j-n-1}(j-n-1)!m!}{(m+j-n-1-q)!}(-ik\beta)^{\deg R_j-q},$$

we get $A_{R_j} \leq C_3|k|$, $n+1 \leq j \leq n+m$, where C_3 is some positive constant independent of j and k. Based on the the estimates (33) and Lemma 2, we have the following inequalities for $j = n+1, \dots, n+m$ and all $|k|$ large enough:

$$\begin{aligned} &\operatorname{meas}_{\mathbb{R}} F_\eta(j,k,\vec{\mu}_{j-n}) \\ &\leq C_1 M_{\Delta_{jk}} \Big(\frac{\nu_j(k,\vec{\lambda})}{\beta^m |k|^m \nu_{j-1}(k,\vec{\lambda})} \Big)^{\frac{1}{\deg R_j}} \leq C_1(1+\beta|k|) \Big(\frac{\xi_j(k,\vec{\lambda})}{\beta^m |k|^m \xi_{j-1}(k,\vec{\lambda})} \Big)^{\frac{1}{\deg R_j}} \\ &\leq C_1(1+\beta)|k| \Big(\frac{1}{\beta^m |k|^m |k|^{2(j-n-1+m)-m+\varepsilon_j}} \Big)^{\frac{1}{\deg R_j}} \leq C_1(1+\beta) \Big(\frac{|k|^{\deg R_j}}{\beta^m |k|^{2\deg R_j + \varepsilon_j}} \Big)^{\frac{1}{\deg R_j}} \\ &\leq C_1(1+\beta) \Big(\frac{1}{\beta^m |k|^{\deg R_j + \varepsilon_j}} \Big)^{\frac{1}{\deg R_j}} \leq C_4 |k|^{-1-\widetilde{\varepsilon}_j}, \end{aligned} \tag{34}$$

where $C_1 = C_1(\deg R_j, \deg R_j, b_{j-n} - a_{j-n}) > 0$, $|k|^{\deg R_j} > 2(\deg R_j + 1) C_3^{\deg R_j} \beta^{-m}$, $\widetilde{\varepsilon}_j = \varepsilon_j / \deg R_j > 0$, $n+1 \leq j \leq n+m$, and constant

$$C_4 = (1+|\beta|) \max_{j=n+1,\dots,n+m} \beta^{-m/\deg R_j} C_1(\deg R_j, \deg R_j, b_{j-n} - a_{j-n})$$

is independent of the values of k.

Integrating the estimates (34), from the equalities (31) we obtain

$$\operatorname{meas}_{\mathbb{R}^m} F_\eta(j,k) \leq C_5 |k|^{-1-\widetilde{\varepsilon}}, \quad n+1 \leq j \leq n+m, \tag{35}$$

where $C_5 = C_4 \max\limits_{j=1,\dots,m} \prod_{q=1, q\neq j}^m (b_q - a_q)$, $\widetilde{\varepsilon} = \min_{j=n+1,\dots,n+m} \widetilde{\varepsilon}_j$. Then, from estimates (35) and inequalities (30), for all large enough $|k|$ we get

$$\operatorname{mes}_{\mathbb{R}^m} F_\eta(k) \leq m C_5 |k|^{-1-\widetilde{\varepsilon}}. \tag{36}$$

Note that $\varepsilon_{n+1} + \ldots + \varepsilon_{n+m} = \varepsilon_0$,

$$\begin{aligned} \nu_{n+m}(k,\vec{\lambda}) &= \nu_n(k,\vec{\lambda}) \xi_{n+m}(k) = \frac{\nu_n(k,\vec{\lambda})}{|k|^{\sum_{j=n+1}^{n+m}(2(j-n-1+m)-m+\varepsilon_j)}} = \frac{\nu_n(k,\vec{\lambda})}{|k|^{2\sum_{s=1}^m (s-1+m) - m^2 + \varepsilon_0}} \\ &= \frac{\nu_n(k,\vec{\lambda})}{|k|^{2(m-1)m/2 + m^2 + \varepsilon_0}} = \frac{|W(\lambda_1,\dots,\lambda_n)|}{|k|^\eta} = \frac{C_0}{|k|^\eta}, \end{aligned}$$

where $\eta = 2m^2 - m + \varepsilon_0$, $\varepsilon_0 > 0$ can be arbitrarily small.

From the above, we get that for all (except for, perhaps, finitely many numbers) $k \in \mathbb{Z}$ the inclusion $E_\eta(k) \subset F_\eta(k)$ is true if $\eta > 2m^2 - m$. Finally, it follows from (36) that the series $\sum_{k \in \mathbb{Z} \setminus \{0\}} \operatorname{meas}_{\mathbb{R}^m} E_\eta(k)$ is convergent. This completes the proof. □

The next theorem is similar to the previous one.

Theorem 4. *If $\eta > 2n^2 - n$ and $C_0 = |W(\mu_1,\dots,\mu_m)| > 0$, then for almost all (with respect to Lebesgue measure in $\mathbb{R}^n$) vectors $\vec{\lambda} = (\lambda_1,\dots,\lambda_n) \in \Pi_n$ the inequality $|\Delta_k| \geq C_0 |k|^{-\eta}$ holds for all (except for finitely many numbers) $k \in \mathbb{Z}$.*

Proof. To prove this, we can use the representation of the determinant Δ_k in the form

$$\Delta_k = \det \mathbf{M}_k = (-1)^{(m+n-1)m} e^{ik[(\mu_1+\ldots+\mu_m)\beta - (\lambda_1+\ldots+\lambda_n)\alpha]} \det \widehat{\mathbf{M}}_k,$$

where the matrix $\widehat{\mathbf{M}}_k$ is obtained from $\mathbf{M}_k$ by permuting the right and left blocks and extracting exponents $e^{ik\mu_1\beta},\dots,e^{ik\mu_m\beta},e^{-ik\lambda_1\alpha},\dots,e^{-ik\lambda_n\alpha}$ from each column,

$$\widehat{\mathbf{M}}_k = \left[\begin{array}{ccc|ccc} e^{-ik\mu_1\beta} & \dots & e^{-ik\mu_m\beta} & e^{ik\lambda_1\alpha} & \dots & e^{ik\lambda_n\alpha} \\ (i\mu_1)e^{-ik\mu_1\beta} & \dots & (i\mu_m)e^{-ik\mu_m\beta} & (i\lambda_1)e^{ik\lambda_1\alpha} & \dots & (i\lambda_n)e^{ik\lambda_n\alpha} \\ (i\mu_1)^2e^{-ik\mu_1\beta} & \dots & (i\mu_m)^2e^{-ik\mu_m\beta} & (i\lambda_1)^2e^{ik\lambda_1\alpha} & \dots & (i\lambda_n)^2e^{ik\lambda_n\alpha} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (i\mu_1)^{m-1}e^{-ik\mu_1\beta} & \dots & (i\mu_m)^{m-1}e^{-ik\mu_m\beta} & (i\lambda_1)^{m-1}e^{ik\lambda_1\alpha} & \dots & (i\lambda_n)^{m-1}e^{ik\lambda_n\alpha} \\ \hline 1 & \dots & 1 & 1 & \dots & 1 \\ (i\mu_1) & \dots & (i\mu_m) & (i\lambda_1) & \dots & (i\lambda_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (i\mu_1)^{m-1} & \dots & (i\mu_m)^{m-1} & (i\lambda_1)^{m-1} & \dots & (i\lambda_n)^{m-1} \\ \hline 0 & \dots & 0 & (i\lambda_1)^m & \dots & (i\lambda_n)^m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & (i\lambda_1)^{n-1} & \dots & (i\lambda_n)^{n-1} \end{array}\right].$$

Given this structure of the matrix $\widehat{\mathbf{M}}_k$, the proof of Theorem 4 is similar to that of Theorem 3 and is therefore omitted. □

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Нитребич З.М., Савка І.Я., Шевчук Р.В., Симотюк М.М. *Задача спряження з початково-нелокальними умовами для факторизованих рівнянь високого порядку* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 647–660.

У цій статті задача розглядається у циліндричній області $(-\alpha, \beta) \times (\mathbb{R}/2\pi\mathbb{Z})$, яка розділяється гіперплощиною $\{0\} \times (\mathbb{R}/2\pi\mathbb{Z})$ на дві неперетинні циліндричні підобласті. Зокрема, дану задачу можна інтерпретувати як пошук розв'язку для пари факторизованих рівнянь із частинними похідними зі сталими коефіцієнтами, які відповідно визначені в цих підобластях, з умовами спряження на гіперплощині та початково-нелокальними умовами на нижній і верхній поверхні області.

Формально розв'язок можна представити у вигляді рядів Фур'є методом розділення змінних, але виникає питання про збіжність даного ряду в просторах Соболєва, періодичних функцій за просторовою змінною. Ця збіжність пов'язана з проблемою малих знаменників і може бути нестійкою щодо малих змін коефіцієнтів задачі та параметрів області.

Встановлено метричні оцінки для малих знаменників, які гарантують збіжність розв'язків. Таким чином, отримано достатні умови розв'язності задачі в просторах Соболєва. Результати показали, що розв'язність залежить від коефіцієнтів диференціальних рівнянь.

Ключові слова і фрази: гіперболічне рівняння, задача спряження, початково-нелокальна умова, малий знаменник.