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A characterization of F-spaces containing an isomorph of ℓ_0

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We prove that an F-space X contains an isomorph of ℓ_0 if and only if there exists a continuous at zero function $T\colon L_0 \to X$ with $T0 = 0$ possessing the following two properties.

(1) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in L_0)(\forall a > 0)(\exists b > 0)\ \big(\|aTx\| \geq \varepsilon \Rightarrow \|T(bx)\| \geq \delta\big)$, which is a weak version of the positive homogeneity.

(2) For every $x, y, z \in L_0$ with $Tx \neq 0$, if x is a disjoint sum of y and z, then either $Ty \neq 0$ or $Tz \neq 0$.

We also provide examples showing that all assumptions on T are essential.

Key words and phrases: F-space, vector lattice, F-lattice, orthogonally additive operator.

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Introduction and preliminaries

The F-spaces L_0 and ℓ_0, which have "maximal" linear structure as a function space and, respectively, a sequence space, possess some extremal properties. For instance, L_0 is prime, that is, every complemented subspace of L_0 is isomorphic to L_0 [3, Theorem 6.2]. Another exotic property is that L_0 has a rigid subspace X, that is, an infinite dimensional subspace such that every continuous linear operator $T\colon X \to X$ is a multiple of the identity I_X of X (see [5, Theorem 3.6]). One more result asserts that every L_0-valued vector measure is bounded (see [4, Theorem 1]).

There are nice characterizations of F-spaces containing a subspace isomorphic to one of these extremal spaces:

1) *an F-space X contains a subspace isomorphic to L_0 if and only if there exists a nonzero continuous linear operator $T\colon L_0 \to X$* (a result by N.J. Kalton [3, Theorem 5.4]);

2) *an F-space X contains a subspace isomorphic to ℓ_0 if and only if X contains arbitrary short lines* (see below for definitions) (a result by C. Bessaga, A. Pełczyński and S. Rolewicz [2], [9, Proposition 4.2.7]).

Since ℓ_0 is isomorphic to a subspace of L_0 (for instance, any subspace of L_0 spanned by a disjoint sequence is isomorphic to ℓ_0), a characterization of spaces containing an isomorph of ℓ_0 in terms of mappings from L_0 to X must involve a wide class of mappings than continuous linear ones. Below we describe such a class of mappings in our main result.

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In this note, we consider real vector spaces only. By an F-space we mean a complete metric linear space X with an invariant metric ρ [9]. In particular, every Banach space and more general, every p-Banach space with $0 < p \leq 1$ is an F-space. By an F-norm on X we mean the function $\|x\| = \rho(x, 0)$. Typical examples of F-spaces, which are not isomorphic to every p-Banach space with $0 < p \leq 1$, are L_0 and ℓ_0. L_0 is the F-space of all equivalence classes of measurable functions $x\colon [0,1] \to \mathbb{R}$ with the F-norm

$$\|x\| = \int_{[0,1]} \frac{|x|}{1+|x|}\, d\mu,$$

and ℓ_0 is the F-space of all sequences $x = (\xi_n)_{n=1}^{\infty}$ with the F-norm

$$\|x\| = \sum_{n=1}^{\infty} 2^{-n} \frac{|\xi_n|}{1+|\xi_n|}.$$

An F-space X is said to contain *arbitrary short lines* provided for every $\varepsilon > 0$ there exists $x \in X \setminus \{0\}$ such that $\|tx\| < \varepsilon$ for all $t \in \mathbb{R}$. For instance, the F-spaces ℓ_0 and L_0 contain arbitrary short lines. It is well known and easily seen that the convergence of sequences in L_0 is equivalent to their convergence in measure.

An F-space, which is a vector lattice with respect to some partial order $\leq$ such that for every $x, y \in X$ the inequality $x \leq y$ implies $\|x\| \leq \|y\|$, is called an F-lattice. Both F-spaces L_0 and ℓ_0 are F-lattices with respect to the natural orders: $x \leq y$ if and only if $x(t) \leq y(y)$ for almost all $t \in [0,1]$ for L_0, and $x \leq y$ if and only if $\xi_n \leq \eta_n$ for all $n \in \mathbb{N}$, where $x = (\xi_n)_{n=1}^{\infty}$ and $y = (\eta_n)_{n=1}^{\infty}$ for ℓ_0.

For standard definitions and notation concerning vector lattices (= Riesz spaces) we refer the reader to [1]. Now we provide some much less known terminology and notation. The equality $x = y \sqcup z$ for elements of a vector lattice means that $x = y + z$ and $y \perp z$.

Let E be a vector lattice and X a vector space. A mapping $T\colon E \to X$ is called an orthogonally additive operator, if for every $x, y \in E$ with $x \perp y$ one has $T(x+y) = Tx + Ty$.

Every vector lattice $(X, \leq)$ is endowed with another non-strict partial order, named the lateral order: $x \sqsubseteq y$ provided $x \perp (y - x)$ (in this case x is called a fragment of y). Obviously, if $e = x + y$ then the following three conditions are equivalent: $x \sqsubseteq e$, $y \sqsubseteq e$ and $x \perp y$. Hence, if $e = \bigsqcup_{k=1}^{m} x_k$, then $(x_k)_{k=1}^{m}$ are pairwise disjoint fragments of e. Observe that every subset of E is laterally bounded from below by zero. The infimum with respect to the lateral order of a two-point subset $\{x, y\}$ of E (if exists) is denoted by $x \Cap y$ (equivalently, $x \Cap y$ is the maximal common fragment of x and y). If E has the principal projection property then every two-point subset of E has a lateral infimum, and moreover, for every fixed $e \in E$ the mapping $E \ni x \mapsto x \Cap e$ is an orthogonally additive operator. One can see for an explicit formula describing $x \Cap y$ in terms of lattice operations in [7, Theorem 3.2]. We refer the reader to [6] and references therein for more information on the lateral order.

1 Main result

By an operator we mean any function $T\colon X \to Y$ between vector spaces X and Y such that $T0 = 0$.

Definition 1. *Let X, Y be F-spaces. A nonzero operator $T\colon X \to Y$ is said to be homogeneously non-vanishing provided for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(\forall x \in X)(\forall a > 0)(\exists b > 0)\, \big(\|aTx\| \geq \varepsilon \Rightarrow \|T(bx)\| \geq \delta\big).$$

To see that the homogeneously non-vanishing assumption on an operator is not too much restrictive, observe that every positively p-homogeneous operator with $p > 0$ (that is, $T(\lambda x) = \lambda^p Tx$ for all $x \in X$ and $\lambda > 0$) is homogeneously non-vanishing. Example 3 below shows that a homogeneously non-vanishing operator need not be positively p-homogeneous for any $p > 0$.

Definition 2. *Let X be a linear space. A nonzero operator $f \colon L_0 \to X$ is said to be horizontally non-vanishing provided for every $x \in L_0$ with $Tx \neq 0$ and every decomposition $x = y \sqcup z$ one has either $Ty \neq 0$ or $Tz \neq 0$.*

Obviously, every nonzero orthogonally additive operator is horizontally non-vanishing.

Theorem 1. *For an F-space X the following assertions are equivalent:*

(i) *there exists a nonzero continuous at zero operator $T \colon L_0 \to X$, which is both homogeneously non-vanishing and horizontally non-vanishing;*

(ii) *X contains an isomorph of ℓ_0.*

Proof. $(i) \Rightarrow (ii)$. We show that X contains arbitrary short lines. Let $e \in L_0$ be such that $Te \neq 0$. Assume, on the contrary, that X does not contain arbitrary short lines. Choose $\varepsilon > 0$ so that for every $x \in X \setminus \{0\}$ there exists $t(x) \in \mathbb{R}$ such that $\|t(x)\, x\| \geq \varepsilon$. Then, using that T is homogeneously non-vanishing, choose $\delta > 0$ so that

$$(\forall x \in L_0)(\forall a > 0)(\exists b > 0) \big(\|aTx\| \geq \varepsilon \Rightarrow \|T(bx)\| \geq \delta\big). \tag{1}$$

Using that T is horizontally non-vanishing, define recursively a sequence $(e_n)_{n=1}^\infty$ of fragments $e_{n+1} \sqsubseteq e_n \sqsubseteq e$ of e with the following properties for every $n \in \mathbb{N}$:

1) $\mu(\operatorname{supp} e_n) = 2^{1-n}\mu(\operatorname{supp} e)$;

2) $Te_n \neq 0$.

Indeed, set $e_1 = e$. Suppose e_n have been constructed for $n = 1, \dots, k$ to satisfy *1)* and *2)*. Using the atomlessness of μ and standard arguments, decompose the Σ-measurable set $A := \operatorname{supp} e_k$ into a disjoint union of Σ-measurable subsets $A = B \sqcup C$ of measure $\mu(B) = \mu(C) = \mu(A)/2$ and set $y := e_k \cdot \mathbf{1}_B$ and $z := e_k \cdot \mathbf{1}_C$. Then $y, z \in L_0$ and $e_k = y \sqcup z$. Since T is horizontally non-vanishing, by *2)* for $n = k$ either $Ty \neq 0$ or $Tz \neq 0$. So we set e_{k+1} to be either y or z respectively. So the desired sequence is constructed.

Now fix any $n \in \mathbb{N}$. By the choice of ε and *2)*, we have $\|t(Te_n)\, Te_n\| \geq \varepsilon$. Then choose by (1) for $a = t(Te_n)$ a number $b_n > 0$ so that $\|T(b_n e_n)\| \geq \delta$. Observe that $\lim\limits_{n\to\infty} \|b_n e_n\|_{L_0} = 0$. By the continuity of T at zero, $\lim\limits_{n\to\infty} \|T(b_n e_n)\| = 0$, which contradicts the choice of b_n's.

$(ii) \Rightarrow (i)$. For simplicity of the notation, we assume that $X = \ell_0$. Given any $x \in L_0$, by $x^* \in L_0$ we denote the nonincreasing rearrangement of $|x|$ given by

$$x^*(t) := \inf\big\{u \geq 0 : \mu\{s \in [0,1] : |x(s)| > u\} < t\big\}, \ t \in [0,1].$$

It is well known that the functions $|x|$ and x^* are equimeasurable (= have the same distribution). Although x^* is an element of L_0, the values $x^*(t)$ are well defined at all points $t \in [0,1]$.

Now we define an operator $T\colon L_0 \to \ell_0$ by setting

$$Tx = \big(x^*(2^{1-n})\big)_{n=1}^{\infty}. \tag{2}$$

Obviously, $T0 = 0$, that is, T is an operator. To show that T is homogeneously non-vanishing, we prove that, moreover, T is positively homogeneous.

Claim 1. *Let $x \in L_0$ and $a > 0$. Then $T(ax) = aTx$.*

Proof. Observe that $(ax)^* = ax^*$. Then

$$T(ax) = \big((ax)^*(2^{1-n})\big)_{n=1}^{\infty} = \big(ax^*(2^{1-n})\big)_{n=1}^{\infty} = aTx.$$

□

As a consequence of Claim 1, T is homogeneously non-vanishing.

The continuity of T at zero is a direct consequence of the next claim.

Claim 2. *For every $x \in L_0$ one has $\|Tx\| \le \|x\|$.*

Proof. Define a countable-valued simple function $x^{**}\colon (0,1] \to \mathbb{R}$ by setting $x^{**}(t) = x^*(2^{1-n})$ for all $t \in (2^{-n}, 2^{1-n}]$ and $n \in \mathbb{N}$. Then $x^* \ge x^{**}$. Hence, taking into account that the function $g(t) = \frac{t}{1+t}$ is increasing on $[0,+\infty)$, we obtain

$$\begin{aligned}\|x\| = \|x^*\| = \int_{[0,1]} \frac{x^*}{1+x^*}\, d\mu &\ge \int_{[0,1]} \frac{x^{**}}{1+x^{**}}\, d\mu \\ &= \sum_{n=1}^{\infty} \int_{\left(2^{-n},2^{1-n}\right]} \frac{x^{**}}{1+x^{**}}\, d\mu = \sum_{n=1}^{\infty} 2^{-n} \frac{x^*(2^{1-n})}{1+x^*(2^{1-n})} = \|Tx\|.\end{aligned}$$

□

T is horizontally non-vanishing, because $Tx = 0$ easily implies $x = 0$. So, Theorem 1 is proved. □

2 Essentiality of the assumptions on operators in the main result

In this section, we provide examples showing that all assumptions on an operator in Theorem 1 are essential. We take the range space $X = \mathbb{R}$ in all examples for simplicity, which obviously contains no copy of ℓ_0.

Example 1. *There exists a nonzero continuous at zero operator $T_1\colon L_0 \to \mathbb{R}$, which is positively homogeneous (and hence, homogeneously non-vanishing).*

Proof. Define an operator $T_1\colon L_0 \to \mathbb{R}$ by setting

$$T_1 x = x^*\left(\frac{1}{2}\right), \quad x \in L_0,$$

where x^* is the nonincreasing rearrangement of $|x|$. The positive homogeneity and continuity of T_1 at zero can be proved using definitions. □

Although the fact that T_1 is not horizontally non-vanishing follows from Theorem 1, we show it directly as follows.

Set $x := \mathbf{1}_{\left[0,\frac{1}{2}\right]}$, $y := \mathbf{1}_{\left[0,\frac{1}{4}\right)}$ and $z := \mathbf{1}_{\left[\frac{1}{4},\frac{1}{2}\right]}$. Then

$$x = y \sqcup z, \quad T_1 x = 1 \quad \text{and} \quad T_1 y = T_1 z = 0.$$

Example 2. *There exists a nonzero continuous at zero operator* $T_2\colon L_0 \to \mathbb{R}$*, which is horizontally non-vanishing.*

We provide two completely different examples. The first operator $T_{2,1}$ is somewhat more involved, however it is easy to see that $T_{2,1}$ fails to be homogeneously non-vanishing. The second operator $T_{2,2}$ is an easy adjustment of the above construction from the proof of Theorem 1, however it is difficult to show directly that $T_{2,2}$ is not homogeneously non-vanishing.

Proof. First example. Put $\mathbf{1} = \mathbf{1}_{[0,1]}$ and define an operator $T_{2,1}\colon L_0 \to \mathbb{R}$ by setting

$$T_{2,1}x = \int_{[0,1]} (x \cap \mathbf{1})\, d\mu = \mu\big(\operatorname{supp}(x \cap \mathbf{1})\big), \quad x \in L_0.$$

Prove that $T_{2,1}$ is continuous at zero. Let $(x_n)_{n=1}^{\infty}$ be any tending to zero sequence in L_0. Hence, $x_n \to 0$ in measure. Given any $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ so that for every $n \geq n_0$ one has

$$\mu(A_n) < \varepsilon, \quad \text{where } A_n = \left\{ t \in [0,1] : |x_n(t)| \geq \frac{1}{2} \right\}.$$

Fix any $n \geq n_0$. Since $\operatorname{supp}(x_n \cap \mathbf{1}) \subseteq A_n$, we obtain

$$0 \leq T_{2,1}(x_n) = \mu\big(\operatorname{supp}(x_n \cap \mathbf{1})\big) \leq \mu(A_n) < \varepsilon.$$

So $T_{2,1}$ is continuous at zero.

Show that $T_{2,1}$ is an orthogonally additive operator and hence, is horizontally non-vanishing. Let $x, y, z \in L_0$ with $x = y \sqcup z$ and $T_{2,1}x \neq 0$. By [8, Theorem 1.6], we get

$$x \cap \mathbf{1} = (y \cap \mathbf{1}) + (z \cap \mathbf{1})$$

and hence $T_{2,1}x = T_{2,1}y + T_{2,1}z$. So, $T_{2,1}$ is an orthogonally additive operator.

Second example. Let $T\colon L_0 \to \ell_0$ be the operator defined by (2). Then define an operator $T_{2,2}\colon L_0 \to \mathbb{R}$ by setting

$$T_{2,2}\, x = \|Tx\|, \quad x \in L_0.$$

By Claim 2, $T_{2,2}$ is continuous at zero. Since T is horizontally non-vanishing, so is $T_{2,2}$. ☐

By Theorem 1, both operators $T_{2,1}$ and $T_{2,2}$ are not homogeneously non-vanishing. The following simple arguments demonstrate that $T_{2,2}$ is not homogeneously non-vanishing.

Assume on the contrary that $T_{2,2}$ is homogeneously non-vanishing. Choose $\delta > 0$ to satisfy the condition for $\varepsilon = 1$. Then for every $n \in \mathbb{N}$ take $x_n = \mathbf{1}_{\left[0,\frac{1}{n}\right]}$ and $a_n = n$, and choose $b_n > 0$ so that $\|T_{2,2}(b_n x_n)\| \geq \delta$. However $\|a_n T_{2,2}\, x_n\| = 1 = \varepsilon$ and $\|T_{2,2}(b_n x_n)\| = \frac{1}{n}$, a contradiction.

Example 3. *There exists a nonzero operator* $T_3\colon L_0 \to \mathbb{R}$*, which is homogeneously non-vanishing and horizontally non-vanishing.*

Proof. Let $T\colon L_0 \to \ell_0$ be the operator defined by (2). Then define an operator $T_3\colon L_0 \to \mathbb{R}$ by setting

$$T_3 x = \begin{cases} 1, & \text{if } x \in L_0 \setminus \{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

All the claimed properties of T_3 are obvious. ☐

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Доведено, що F-простір X містить ізоморфну копію ℓ_0 тоді і лише тоді, коли існує неперервна в нулі функція $T\colon L_0 \to X$ з умовою $T0 = 0$, яка володіє такими двома властивостями.

(1) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in L_0)(\forall a > 0)(\exists b > 0)\ \big(\|aTx\| \geq \varepsilon \Rightarrow \|T(bx)\| \geq \delta\big)$, що є слабкою версією додатної однорідності.

(2) Для довільних $x, y, z \in L_0$ з умовою $Tx \neq 0$ з того, що x є диз'юнктною сумою y та z, випливає, що або $Ty \neq 0$, або $Tz \neq 0$.

Також наведено приклади, які показують, що всі умови на функцію T є істотними.

Ключові слова і фрази: F-простір, векторна ґратка, F-ґратка, ортогонально адитивний оператор.