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Non-existence of co-spectral simple connected graphs with small number of edges

Boyko O.[1], **Kaliuzhnyi-Verbovetskyi D.**[1], **Pivovarchik V.**[1,2]

We show that the spectrum of the Sturm-Liouville problem on a connected simple equilateral graph with the Dirichlet boundary conditions at the pendant vertices is related with the spectrum of the discrete Laplacian of the corresponding combinatorial graph. It enables us to compare the spectra of discrete Laplacians to find co-spectral combinatorial graphs and finally co-spectral quantum graphs. Using this method we prove that there are no co-spectral (in our sense) graphs with the number of edges less or equal 7. Thus, in this case the inverse problem of recovering the shape of a quantum graph possesses a unique solution.

Key words and phrases: graph, edge, vertex, eigenvalue, potential, adjacency matrix, characteristic function.

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[1] K.D. Ushinsky South Ukrainian National Pedagogical University, 26 Staroportofrankivska str., 65026, Odesa, Ukraine
[2] University of Vaasa, Wolffintie 32, 65200, Vaasa, Finland
E-mail: `boykohelga@gmail.com` (Boyko O.), `dmitry2k@yahoo.com` (Kaliuzhnyi-Verbovetskyi D.), `vpivovarchik@gmail.com` (Pivovarchik V.)

Introduction

In quantum graph theory, the problem of recovering the shape of a graph was stated in [1, 9]. In [9], it was shown that if the lengths of the edges are non-commensurate, then the spectrum of the spectral Sturm-Liouville problem on a graph with the standard conditions at its vertices (the Neumann conditions at the pendant vertices and the continuity conditions + Kirchhoff's conditions at the interior vertices) uniquely determines the shape of this graph. In [1], it was shown that in the case of commensurate lengths of the edges there exist co-spectral quantum graphs. In [3, 10], it was shown that the spectrum of the Neumann problem with zero potential on P_2 uniquely determines the shape of the graph. In [6], it was shown that if the graph is simple connected equilateral with the number of vertices less than or equal 5 and the potentials on the edges are real L_2 functions, then the spectrum of the Sturm-Liouville problem with the standard conditions at the vertices uniquely determines the shape of the graph. For trees, the minimal number of vertices in a co-spectral pair is 9 (see [4, 7, 15]). In the case of the standard conditions at all the vertices, the asymptotics of the spectrum shows whether the graph is a tree. If the number of vertices does not exceed 8, then to find the shape of a tree we need just to find in [6] the characteristic polynomial corresponding to the given spectrum.

For the case of the Dirichlet conditions at the pendant vertices, it was shown in [4] that there are no co-spectral trees with the number of vertices ≤ 8. However, in the case of the Dirichlet conditions at the pendant vertices, the asymptotics of the spectrum do not indicate whether

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the graph is a tree. In this case, the possibility that the spectrum of a tree coincides with that of a non-tree graph (or that the spectra of two graphs with different cyclomatic numbers coincide) cannot be excluded *a priori*. Such a possibility is excluded in the case of the Neumann conditions at the pendant vertices.

It should be mentioned that an attempt to use two spectra to find the shape of a tree was made in [16], where expansions into branched continued fractions of certain polynomials related to the Neumann and Dirichlet problems were used. For branched continued fractions, see [8].

In the paper, we prove that if the number of edges $g \leq 7$, then there are no co-spectral simple connected equilateral graphs and we conclude that the asymptotics of the spectrum uniquely determine the graph shape. Examples of co-spectral graphs with the Dirichlet conditions at the pendant vertices with $g = 8$ are given in [15, Fig. 1]. In Section 2, we describe the Sturm-Liouville problem on a simple connected equilateral graph with the Dirichlet conditions at each pendant vertex and the standard conditions at all the interior vertices. We expose the known theorem relating the characteristic function (the function whose set of zeros coincides with the spectrum) of the above described Sturm-Liouville problem with the determinant of the normalized Laplacian of the corresponding combinatorial graph. In Section 3, we show all the simple connected equilateral graphs on 7 or less edges and calculate the corresponding characteristic functions. We compare them and find that there are no co-spectral pairs.

1 Statement of the problem

Let G be an equilateral simple connected graph with p vertices and g edges each of the length l. We direct the edges incident with the pendant vertices away from these vertices. Orientation of the rest of the vertices is arbitrary. Let us describe the spectral problem on G. We consider the Sturm-Liouville equations on the edges

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \ldots, g, \tag{1}$$

where $q_j \in L_2(0, l)$ are real.

At the beginning of each edge e_j incident with a pendant vertex, we impose the Dirichlet condition

$$y_j(0) = 0. \tag{2}$$

At each interior vertex, we impose the standard conditions, i.e. the continuity conditions

$$y_j(l) = y_k(0) \tag{3}$$

for the incoming into v_i edges e_j and for all e_k outgoing from v_i, and the Kirchhoff's conditions

$$\sum_j y_j'(l) = \sum_k y_k'(0), \tag{4}$$

where the sum in the right-hand side is taken over all edges e_k outgoing from v_i and the sum in the left-hand side is taken over all edges e_j incoming to v_i.

We call the above conditions (continuity together with the Kirchhoff's or Neumann's conditions) standard.

In the sequel, if the potentials are the same on all the edges we omit the index in q_j and y_j.

In order to find the characteristic function of our Sturm-Liouville problems, we look for coefficients A_j, B_j such that the solution of (1) can be expressed in the form

$$y_j = A_j s_j(\lambda, x) + B_j c_j(\lambda, x), \quad x \in (0, l),$$

where $s_j(\lambda, x)$ are the solutions of (1), which satisfy the conditions $s_j(\lambda, 0) = s_j'(\lambda, 0) - 1 = 0$ and $c_j(\lambda, x)$ are the solutions of (1), which satisfies the conditions $c_j(\lambda, 0) - 1 = c_j'(\lambda, 0) = 0$.

Substituting this into the continuity conditions, as well as into Kirchhoff's condition at each interior vertex and into the Dirichlet conditions at all pendant vertices, we obtain a system of $2g$ linear algebraic equations with unknowns A_j, B_j. Denote the $2g \times 2g$ matrix of this system by $\|\Phi_D(\lambda)\|$, we call it the *characteristic matrix* of our problem. Observe that it involves the values $s_j(\lambda, l)$, $s_j'(\lambda, l)$, $c_j(\lambda, l)$, $c_j'(\lambda, l)$. Then the equation $\det \|\Phi_D(\lambda)\| = 0$ completely determines the spectrum of problem (1)–(4).

Let A be the adjacency matrix of G, and

$$D = \mathrm{diag}(d(v_0), d(v_1), \dots, d(v_p)),$$

the degree matrix. Here $d(v_i)$ is the degree of the vertex v_i. Denote by $-z\hat{D} + \hat{A}$ the submatrix of $-zD + A$ obtained by deleting the rows and the columns corresponding to those pendant vertices (where the Dirichlet conditions are imposed).

The following theorem was proved in [13].

Theorem 1 ([13, Theorem 6.4.2]). *Let G be a simple connected graph with $p \geq 2$. Assume that all edges have the same length l and the same real potential $q(x) \in L_2(0, l)$ symmetric with respect to the midpoint of an edge ($q(x) = q(l - x)$). Then the spectrum of problem* (1)–(4) *coincides with the set of zeros of the characteristic function*

$$\phi(\lambda) = s^{g-p+r}(\lambda, l)\psi(c(\lambda, l)), \tag{5}$$

where r is the number of pendant vertices and

$$\psi(z) = \det(-z\hat{D} + \hat{A}).$$

It is clear that, in case of identically zero potential, we get

$$\phi_0(\lambda) = \left(\frac{\sin\sqrt{\lambda}l}{\sqrt{\lambda}}\right)^{g-p+r} \psi(\cos\sqrt{\lambda}l). \tag{6}$$

Corollary 1. *Let G be a simple connected graph with $p \geq 2$. Assume that the edges have the same length l and the potentials on the edges $q_j(x) \in L_2(0, l)$ are real. Then the characteristic function of problem* (1)–(4) *satisfies*

$$\phi(\lambda) \overset{\lambda\to+\infty}{=} \phi_0(\lambda) + O(1). \tag{7}$$

Proof. We use the following asymptotics [14]:

$$s_j(\lambda, l) \overset{\lambda\to+\infty}{=} \frac{\sin\sqrt{\lambda}l}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right), \qquad c_j(\lambda, l) \overset{\lambda\to+\infty}{=} \cos\sqrt{\lambda}l + O\left(\frac{1}{\sqrt{\lambda}}\right), \tag{8}$$

$$s_j''(\lambda, l) \overset{\lambda\to+\infty}{=} \cos\sqrt{\lambda}l + O\left(\frac{1}{\sqrt{\lambda}}\right), \qquad c_j'(\lambda, l) \overset{\lambda\to+\infty}{=} -\sqrt{\lambda}\sin\sqrt{\lambda}l + O\left(1\right).$$

Suppose first that all the potentials on the edges are the same and symmetric with respect to the midpoint of an edge. Then using (5) and (8), we obtain (7). ☐

Now let the potentials be diffirent and not symmetric but real $L_2(0, l)$-functions. Then we apply [5, Theorem 5.4] and obtain (7). This means that if two graphs are co-spectral, then they must have not only the same $\phi(\lambda)$, but also the same $\psi(z)$. Thus, we need to investigate $\psi(z)$.

2 Cospectrality

The spectrum of problem (1)–(4) consists of normal (isolated Fredholm) eigenvalues of finite multiplicity. The corresponding operator is selfadjoint, therefore these eigenvalues are real. For the main term of the asymptotics we have

$$\lim_{k\to\infty}\frac{\lambda_k}{k^2}=\frac{\pi^2}{g^2l^2} \tag{9}$$

(see [2,11,12] or [1, Corollary 1]). In the paper, by co-spectral we mean simple connected equilateral (with the same length of the edges) graphs with the same spectrum of problem (1)–(4).

According to (9), co-spectral graphs must have the same number of edges g.

Equations (6) and (7) imply that co-spectral graphs must have the same value of $g-p+r$. Let us explain it. The following theorem was proved in [4].

Theorem 2 ([4]). *Let T be an equilateral tree. The eigenvalues of problem* (1)–(4) *can be presented as the union of subsequences* $\{\lambda_k\}_{k=1}^{\infty}=\bigcup\limits_{i=1}^{2p-r-1}\{\lambda_k^{(i)}\}_{k=1}^{\infty}$ *with the following asymptotics:*

$$\sqrt{\lambda_k^{(i)}}\underset{k\to\infty}{=}\frac{2\pi(k-1)}{l}\pm\frac{1}{l}\arccos\alpha_i+O\left(\frac{1}{k}\right)\quad \text{for } i=1,2,\dots,p-p_{pen},\ k=1,2,\dots, \tag{10}$$

$$\sqrt{\lambda_k^{(i)}}\underset{k\to\infty}{=}\frac{\pi k}{l}+O\left(\frac{1}{k}\right)\quad \text{for } i=p-p_{pen}+1,\dots,p-1,\ k=1,2,\dots. \tag{11}$$

Here $\alpha_1,\alpha_2,\dots,\alpha_{p-p_{pen}}$ are the zeros of the polynomial $\psi(z)$, and $p_{pen}=r$ is the number of pendant vertices.

Here the subsequences (11) correspond to the factor $\left(\frac{\sin\sqrt{\lambda}l}{\sqrt{\lambda}}\right)^{q-p+r}$ in (6) while the subsequences (10) correspond to the factor $\psi(\cos\sqrt{\lambda}l)$. Of course, the polynomial $\psi(z)$ may contain a factor $(1-z^2)$ and, consequently, $\psi(\cos\sqrt{\lambda}l)$ may contain $(1-\cos^2\sqrt{\lambda})=\sin^2\sqrt{\lambda}l$. However, this factor gives a subsequence

$$\sqrt{\lambda_k^{(i)}}\underset{k\to\infty}{=}\frac{\pi k}{l}+O\left(\frac{1}{k}\right)\quad \text{for } k=0,1,\dots,$$

which starts with $k=0$ and therefore differs from (11). Similar arguments lead to the assertion that to check cospectrality it is sufficient to compare only graphs with the same g and the same $\Delta\equiv g-p+r$.

Theorem 3. *There are no co-spectral graphs among simple connected equilateral graphs of seven or less edges in case of the Dirichlet conditions at the pendant vertices and standard conditions at the interior vertices.*

Proof. All simple connected graphs of 1, 2, 3 and 4 edges are presented in Figure 1. We denote the graphs by $G^i_{g,g-p+r}$, where the upper index enumerate the graphs with the same g and same $g-p+r$ given as the lower indices.

Among these graphs there are only two $G^1_{4,1}$ and $G^2_{4,1}$ with the same g and $g-p+r$. However, the corresponding characteristic polynomials are

$$\psi^1_{4,1}(z)=-8z^3+4z,\qquad \psi^2_{4,1}(z)=-12z^3+7z+2.$$

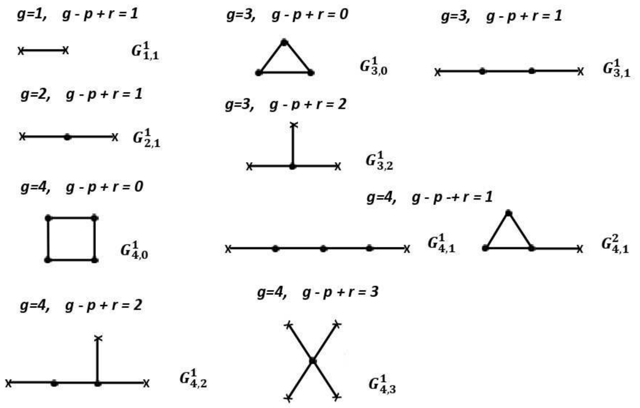


Figure 1. Simple connected graphs of 1, 2, 3 and 4 edges

The sets of zeros of these polynomials are different. Thus there are no co-spectral graphs among the graphs of Figure 1.

Now let us consider the graphs of 5 edges. All simple connected graphs of 5 edges are presented in Figure 2.

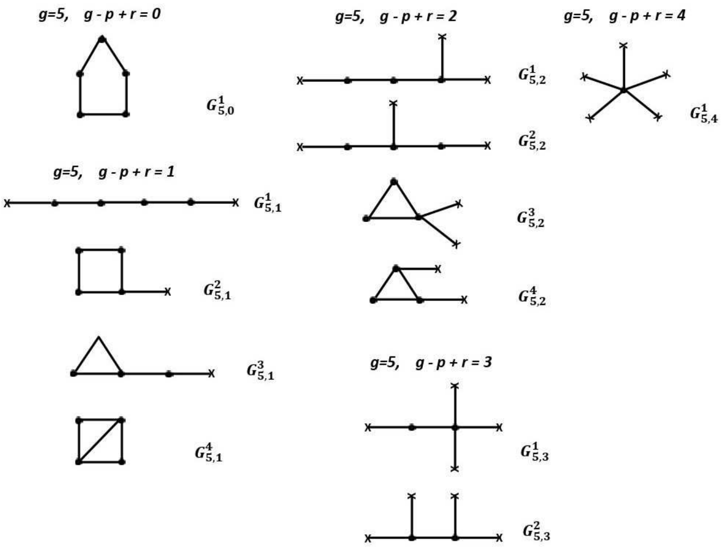


Figure 2. Simple connected graphs with $g = 5$

Among these graphs, there are four graphs, $G^1_{5,1}$, $G^2_{5,1}$, $G^3_{5,1}$, and $G^4_{5,1}$, with the same values $g = 5$ and $g - p + r = 1$. However, the corresponding characteristic polynomials are

$$\psi^1_{5,1}(z) = 16z^4 - 12z^2 + 1, \qquad \psi^2_{5,1}(z) = 24z^4 - 20z^2,$$

$$\psi^3_{5,1}(z) = 24z^4 - 18z^2 - 4z + 1, \qquad \psi^4_{5,1}(z) = 24z^4 - 24z^2 - 8z.$$

It is clear that these polynomials have different sets of zeros.

There are four graphs, $G^1_{5,2}$, $G^2_{5,2}$, $G^3_{5,2}$, and $G^4_{5,2}$ with the same $g = 5$ and $g - p + r = 2$. Their polynomials are

$$\psi^1_{5,2}(z) = -12z^3 + 5z, \qquad \psi^2_{5,2}(z) = -12z^3 + 4z,$$

$$\psi^3_{5,2}(z) = -16z^3 + 8z + 2, \qquad \psi^4_{5,2}(z) = -18z^3 + 8z + 2.$$

The sets of zeros of these polynomials are different.

There are two graphs, $G^1_{5,3}$ and $G^2_{5,3}$, shown in Figure 2 with $g = 5$ and $g - p + r = 3$. Their characteristic polynomials

$$\psi^1_{5,3}(z) = 8z^2 - 1, \qquad \psi^2_{5,3}(z) = 9z^2 - 1$$

have different sets of zeros.

There are 29 simple connected graphs with 6 edges. They are shown in Figures 3 and 4.

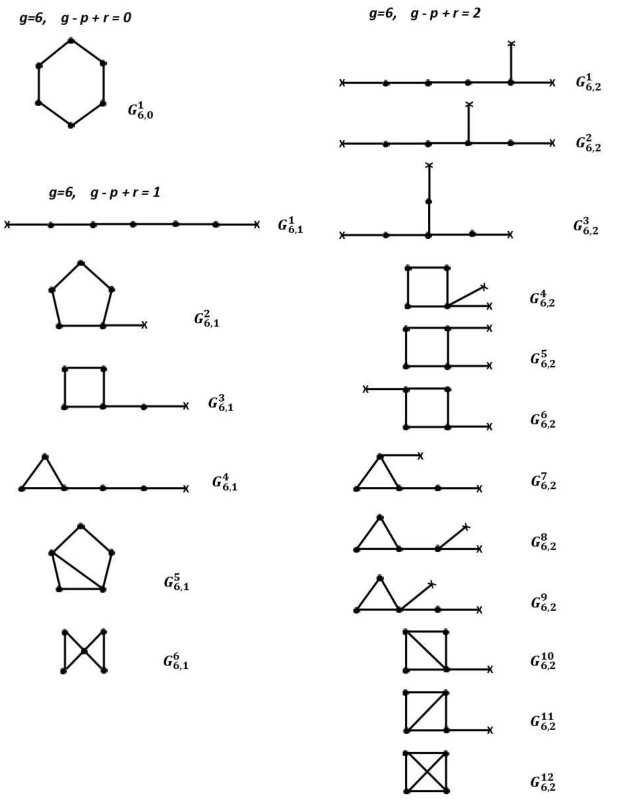


Figure 3. Simple connected graphs of 6 edges

We see that there are six simple connected graphs $G^1_{6,1}$, $G^2_{6,1}$, $G^3_{6,1}$, $G^4_{6,1}$. $G^5_{6,1}$, $G^6_{6,1}$ with $g = 6$ and $g - p + r = 1$. Their characteristic polynomials are

$$\psi^1_{6,1}(z) = -32z^5 + 32z^3 - 6z, \qquad \psi^2_{6,1}(z) = -48z^5 + 52z^2 - 11z + 2,$$

$$\psi^3_{6,1}(z) = -48z^5 + 48z^3 - 4z, \qquad \psi^4_{6,1}(z) = -48z^5 + 48z^3 + 8z^2 - 9z - 2,$$

$$\psi^5_{6,1}(z) = -48z^5 + 60z^3 + 8z^2 - 9z, \qquad \psi^6_{6,1}(z) = -64z^5 + 64z^3 + 16z^2 - 12z - 4.$$

We see that there are no polynomials with the same set of zeros among these.

There are 12 graphs, $G^1_{6,2}$, $G^2_{6,2}$, $G^3_{6,2}$, $G^4_{6,2}$, $G^5_{6,2}$, $G^6_{6,2}$, $G^7_{6,2}$, $G^8_{6,2}$, $G^9_{6,2}$, $G^{10}_{6,2}$, $G^{11}_{6,2}$, $G^{12}_{6,2}$ with $g = 6$ and $g - p + r = 2$. The corresponding polynomials are

$$\psi^1_{6,2}(z) = 24z^4 - 16z^2 + 1, \qquad \psi^2_{6,2}(z) = 24z^5 - 14z^2 + 1,$$

$$\psi^3_{6,2}(z) = 24z^4 - 12z^2, \qquad \psi^4_{6,2}(z) = 32z^4 - 24z^2,$$

$$\psi^5_{6,2}(z) = 36z^4 - 25z^2, \qquad \psi^6_{6,2}(z) = 36z^4 - 24z^2,$$

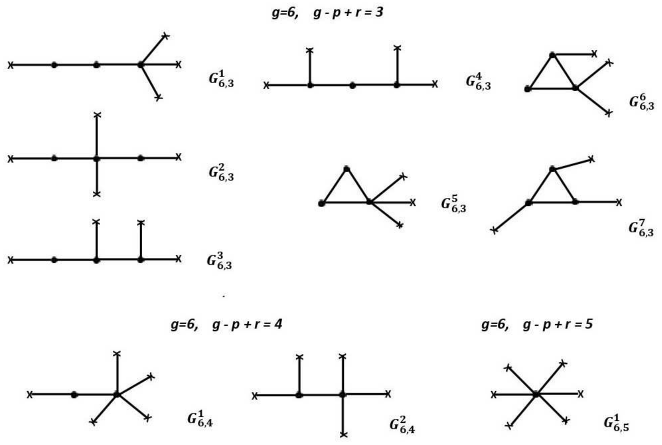


Figure 4. Simple connected graphs of 6 edges

$$\psi^7_{6,2}(z) = 36z^4 - 22z^2 - 4z + 1, \qquad \psi^8_{6,2}(z) = 36z^5 - 25z^2 - 6z + 1,$$
$$\psi^9_{6,2}(z) = 32z^4 - 20z^2 - 4z + 1, \qquad \psi^{10}_{6,2}(z) = 48z^- - 32z^2 - 8z,$$
$$\psi^{11}_{6,2}(z) = 54z^4 - 36z^2 - 10z, \qquad \psi^{12}_{6,2}(z) = 81z^4 - 54z^2 - 24z - 3.$$

We see that the sets of zeros of these polynomials are different.

There are seven graphs with $g = 6$ and $g - p + r = 3$ (see Figure 4). The corresponding polynomials are

$$\psi^1_{6,3}(z) = -16z^3 + 6z, \qquad \psi^2_{6,3}(z) = -16z^3 + 4z,$$
$$\psi^3_{6,3}(z) = -18z^3 + 5z, \qquad \psi^4_{6,3}(z) = -18z^3 + 6z,$$
$$\psi^6_{6,3}(z) = -20z^3 + 9z + 2, \qquad \psi^7_{6,3}(z) = -24z^3 + 9z + 2,$$
$$\psi^8_{6,3}(z) = -27z^3 + 9z + 2.$$

They have different sets of zeros.

Two graphs of $g = 6$ and $g - p + r = 4$ are shown in Figure 4. Their polynomials $\psi^1_{6,4}(z) = 10z^2 - 1$ and $\psi^2_{6,4}(z) = 12z^2 - 1$ have different sets of zeros.

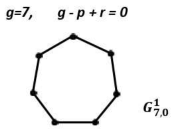
Thus, we conclude that there are no co-spectral graphs of 6 or less edges.

Now we consider graphs with $g = 7$. We look for co-spectral among simple connected graphs. The graph C_7 (the cycle of seven vertices) has $g = 7$ and $g - p + r = 0$. There are no other simple connected graphs with such parameters and consequently C_7 has no co-spectral partner.

Now let $q = 7$ and $g - p + r = 1$. There are 8 simple connected graphs with $g = 7$ and $g - p + r = 1$. These graphs are shown in Figure 5. The corresponding characteristic polynomials are:

$$\psi^1_{7,1}(z) = 64z^6 - 80z^4 + 24z^2 - 1, \qquad \psi(z)^2_{7,1} = 96z^6 - 120z^4 - 16z^3 + 36z^2 + 8z - 1,$$
$$\psi(z)^3_{7,1} = 96z^6 - 120z^4 + 28z^2, \qquad \psi(z)^4_{7,1} = 96z^6 - 112z^4 + 30z^2 - 2,$$
$$\psi(z)^5_{7,1} = 96z^6 - 120z^4 + 34z^2 - 4z - 1, \qquad \psi(z)^6_{7,1} = 144z^6 - 168z^4 + 48z^2 - 4,$$
$$\psi(z)^7_{7,1} = 144z^6 - 168z^4 + 49z^2 - 4, \qquad \psi(z)^8_{7,1} = 128z^6 - 160z^4 - 16z^3 + 40z^2 + 8z.$$

We see that there are no polynomials with the same set of zeros among them.



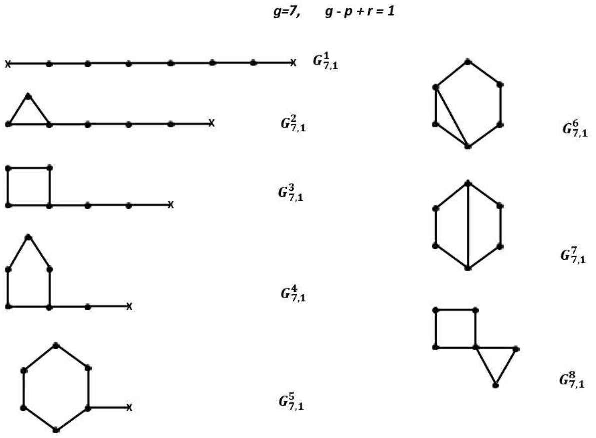


Figure 5. Simple connected graphs with $g = 7$ and $g - p + r = 1$

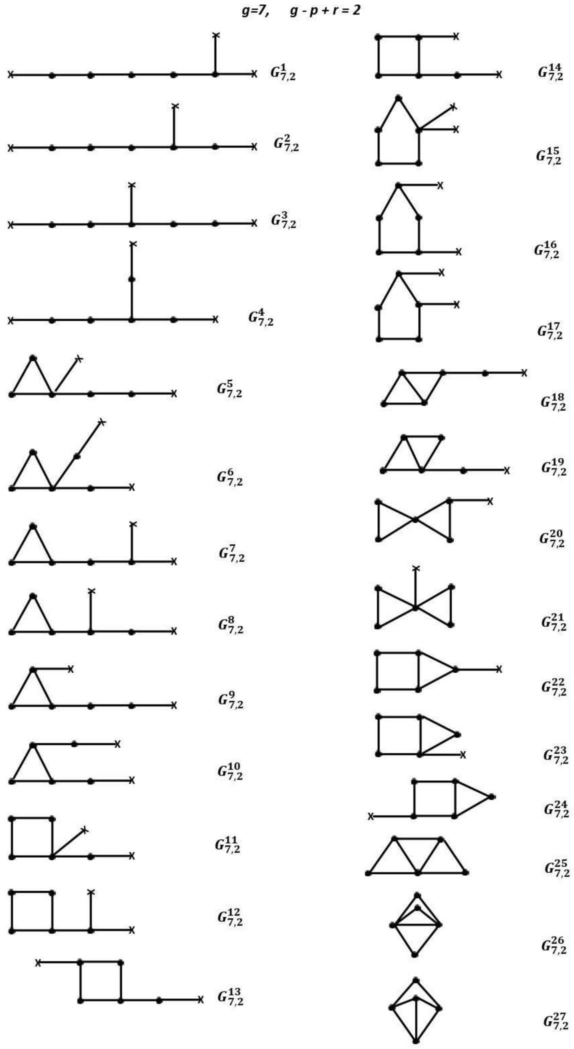


Figure 6. Simple connected graphs with $g = 7$ and $g - p + r = 2$

Let $q = 7$ and $g - p + r = 2$. There are 27 simple connected graphs corresponding to $g = 7$ and $g - p + r = 2$. These graphs are shown in Figure 6. The corresponding characteristic polynomials are

$$\psi^1_{7,2}(z) = -48z^5 + 44z^3 - 7z, \qquad \psi^2_{7,2}(z) = -48z^5 + 40z^3 - 6z,$$

$$\psi^3_{7,2}(z) = -48z^5 + 40z^3 - 7z, \qquad \psi^4_{7,2}(z) = -48z^5 + 36z^3 - 4z,$$

$$\psi(z)^5_{7,2} = -64z^5 + 56z^3 + 8z^2 - 10z - 2, \qquad \psi(z)^6_{7,2} = -64z^5 + 48z^3 + 8z^2 - 4z,$$

$$\psi(z)^7_{7,2} = -72z^5 + 66z^3 + 12z^2 - 10z - 2, \qquad \psi(z)^8_{7,2} = -72z^5 + 62z^3 + 12z^2 - 9z - 2,$$

$$\psi(z)^9_{7,2} = -72z^5 + 62z^3 + 8z^2 - 10z - 2, \qquad \psi(z)^{10}_{7,2} = -72z^5 + 56z^3 + 8z^2 - 6z,$$

$$\psi(z)^{11}_{7,2} = -64z^5 + 56z^3 - 4z, \qquad \psi(z)^{12}_{7,2} = -72z^5 + 68z^3 - 4z,$$

$$\psi(z)^{13}_{7,2} = -72z^5 + 60z^3 - 4z, \qquad \psi(z)^{14}_{7,2} = -72z^5 + 60z^3 - 5z,$$

$$\psi(z)^{15}_{7,2} = -64z^5 + 64z^3 - 12z + 2, \qquad \psi(z)^{16}_{7,2} = -72z^5 + 66z^3 - 12z + 2,$$

$$\psi(z)^{17}_{7,2} = -72z^5 + 68z^3 - 12z + 2, \qquad \psi(z)^{18}_{7,2} = -108z^5 + 90z^3 + 2z^2 - 8z - 2,$$

$$\psi(z)^{19}_{7,2} = -98z^5 + 76z^3 + 16z^2 - 4z, \qquad \psi(z)^{20}_{7,2} = -96z^5 + 84z^3 + 20z^2 - 13z - 4,$$

$$\psi(z)^{21}_{7,2} = -80z^5 + 72z^3 + 16z^2 - 13z - 4, \qquad \psi(z)^{22}_{7,2} = -108z^5 + 994z^3 + 8z^2 - 10z,$$

$$\psi(z)^{23}_{7,2} = -96z^5 + 88z^3 + 8z^2 - 11z, \qquad \psi(z)^{24}_{7,2} = -108z^5 + 96z^3 + 12z^2 - 11z,$$

$$\psi(z)^{25}_{7,2} = -98z^5 + 176z^3 + 16z^2 - 4z, \qquad \psi(z)^{26}_{7,2} = -128z^5 + 104z^3 + 24z^2,$$

$$\psi(z)^{27}_{7,2} = -16z^5 + 144z^3 + 24z^2 - 6z.$$

We see that there are no plynomials with the same set of zeros among them.

We have 21 graphs corresponding to $g = 7$ and $g - p + r = 3$. These graphs are shown in Figure 7. The corresponding characteristic polynomials are

$$\psi(z)^1_{7,3} = 32z^4 - 20z^2 + 1, \qquad \phi(z)^2_{7,3} = 36z^4 - 19z^2 + 1,$$

$$\psi(z)^3_{7,3} = 36z^4 - 18z^2 + 1, \qquad \psi(z)^4_{7,3} = 36z^4 - 21z^2 + 1,$$

$$\psi(z)^5_{7,3} = 36z^4 - 16z^2 + 1, \qquad \psi(z)^6_{7,3} = 36z^4 - 12z^2,$$

$$\psi(z)^7_{7,3} = 36z^4 - 16z^2, \qquad \psi(z)^8_{7,3} = 36z^4 - 12z^2 + 1,$$

$$\psi(z)^9_{7,3} = 40z^4 - 22z^2 - 4z + 1, \qquad \psi(z)^{10}_{7,3} = 48z^4 - 26z^2 - 4z + 1,$$

$$\psi(z)^{11}_{7,3} = 54z^4 - 27z^2 - 4z + 1, \qquad \psi(z)^{12}_{7,3} = 48z^4 - 24z^2 - 4z + 1,$$

$$\psi(z)^{13}_{7,3} = 48z^4 - 30z^2, \qquad \psi(z)^{14}_{7,3} = 48z^4 - 28z^2,$$

$$\psi(z)^{15}_{7,3} = 54z^4 - 30z^2, \qquad \psi(z)^{16}_{7,3} = 40z^4 - 28z^2,$$

$$\psi(z)^{17}_{7,3} = 72z^4 - 41z^2 - 10z, \qquad \psi(z)^{18}_{7,3} = 64z^4 - 36z^2 - 8z,$$

$$\psi(z)^{19}_{7,3} = 81z^4 - 76z^2 - 12z, \qquad \psi(z)^{20}_{7,3} = 108z^4 - 63z^2 - 26z - 3,$$

$$\psi(z)^{21}_{7,3} = 48z^4 - 32z^2 - 8z + 1.$$

We see that there are no polynomials with the same set of zeros among them.

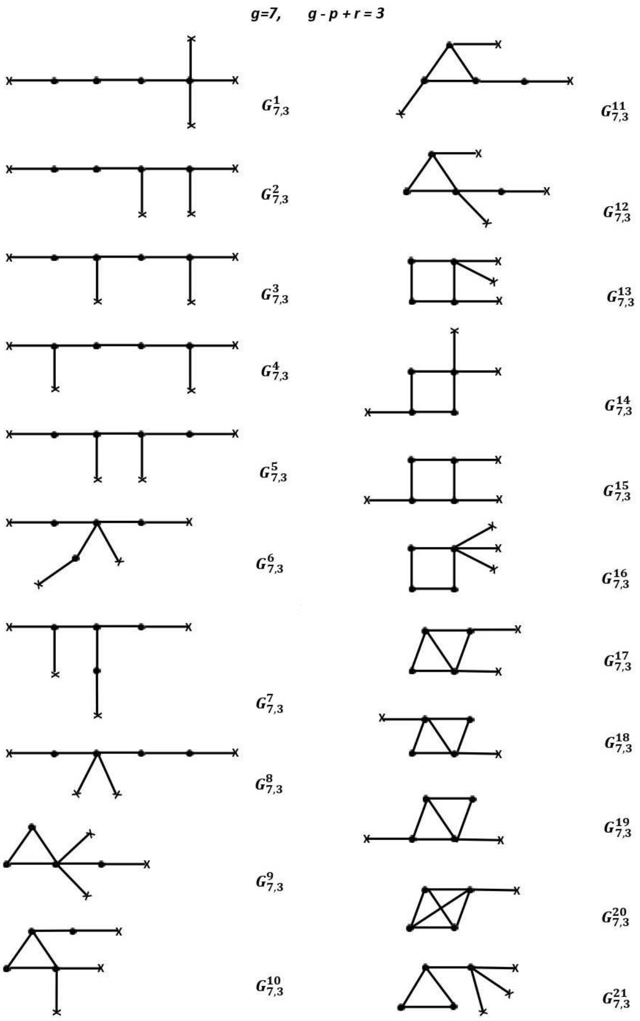


Figure 7. Simple connected graphs with $g = 7$ and $g - p + r = 3$

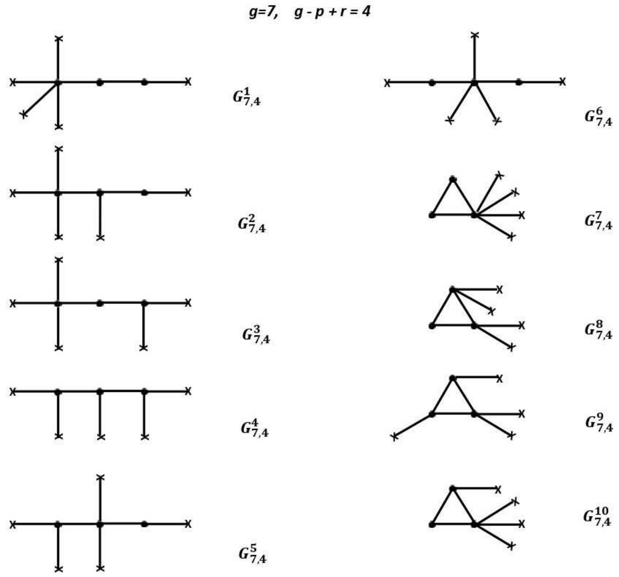


Figure 8. Simple connected graphs with $g = 7$ and $g - p + r = 4$

We have 10 graphs corresponding to $g-p+r=4$, $g=7$. These graphs are shown in Figure 8. The corresponding characteristic polynomials are

$$\psi(z)^1_{7,4}=-20z^3+7z, \qquad \psi(z)^2_{7,4}=-24z^3+6z,$$

$$\psi(z)^3_{7,4}=-24z^3+7z, \qquad \psi(z)^4_{7,4}=-27z^3+6z,$$

$$\psi(z)^5_{7,4}=-24z^3+5, \qquad \psi(z)^6_{7,4}=-20z^3+4z,$$

$$\psi(z)^7_{7,4}=-24z^3+10z+2, \qquad \psi(z)^8_{7,4}=-32z^3+10z+2,$$

$$\psi(z)^9_{7,4}=-36z^3+10z+2, \quad \psi(z)^{10}_{7,4}=-30z^3+10z+2.$$

We see that there are no polynomials with the same set of zeros among them.

There are four graphs with $g=7$ and $g-p+r=5$ or $g-p+r=6$. They are shown in Figure 9.

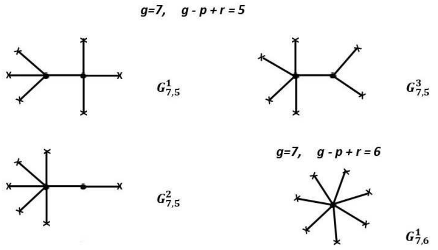


Figure 9. Simple connected graphs with $g=7$ and $g-p+r=5$ and $g-p+r=6$

The corresponding polynomials are

$$\psi^1_{7,5}=16z^2-1, \qquad \psi^2_{7,5}=12z^2-1, \qquad \psi^3_{7,5}=15z^2-1, \qquad \phi^1_{7,6}=-7z.$$

We see that the sets of zeros are different. $\square$

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References

[1] von Below J. Can one hear the shape of a network? In: Mehmeti F., von Below J., Nicaise S. (Eds.) Partial Differential Equations on Multistructures. Lecture Notes in Pure and Appl. Math. Vol. 219, 19–36. Marcel Dekker, New York, 2001.

[2] von Below J. *A characteristic equation associated with an eigenvalue problem on c^2-networks*. Lin. Algebra Appl. 1985, **71**, 309–325. doi:10.1016/0024-3795(85)90258-7

[3] Boman J., Kurasov P., Suhr R. *Schrödinger operators on graphs and geometry II. Spectral estimates for L_1-potentials and Ambartsumian theorem*. Integr. Equ. Oper. Theory 2018, **90**, article 40. doi:10.1007/s00020-018-2467-1

[4] Boyko O., Martynyuk O., Pivovarchik V. *On recovering the shape of a quantum tree from the spectrum of the Dirichlet boundary problem*. Mat. Stud. 2023, **60** (2), 162–172. doi:10.30970/ms.60.2.162-172

[5] Carlson R., Pivovarchik V. *Spectral asymptotics for quantum graphs with equal edge lengths*. J. Phys. A: Math. Theor. 2008, **41**, 145202. doi:10.1088/1751-8113/41/14/145202

[6] Chernyshenko A., Pivovarchik V. *Recovering the shape of a quantum graph*. Integr. Equ. Oper. Theory 2020, **92**, article 23. doi:10.1007/s00020-020-02581-w

[7] Chernyshenko A., Pivovarchik V. *Cospecral quantum graphs with Dirichlet conditions at pendant vertices*. Ukr. Math. J. 2023, **75** (3), 439–455. doi:10.1007/s11253-023-02209-3 (translation of Ukrain. Mat. Zh. 2023, **75** (3), 382–396. doi:10.37863/umzh.v75i3.7351)

[8] Dmytryshyn R., Lustiv I.-A., Dmytryshyn M. *On the analytic extention of the Horn's hypergeometric function H_4*. Carpatian Math. Publ. 2024, **16** (1), 32–39. doi:10.15330/cmp.16.1.32-39

[9] Gutkin B., Smilansky U. *Can one hear the shape of a graph?* J. Phys. A: Math. Gen. 2001, **34**, 6061–6068. doi: 10.1088/0305-4470/34/31/301

[10] Kurasov P., Naboko S. *Rayleigh estimates for differential operators on graphs*. J. Spectr. Theory 2014, **4** (2), 211–219.

[11] Mehmeti F.A. *A characterization of generalized C^∞-notion on nets*. Integr. Equ. Oper. Theory 1986, **9**, 753–766. doi:10.1007/BF01202515

[12] Nicaise S. *Spectre des re'seaux topoogiques finis*. Bull. Sci. Math. 1987, **111** (2), 401–413.

[13] Möller M., Pivovarchik V. Direct and inverse finite-dimensional spectral problems on graphs. In: Ball J.A., Böttcher A., Tretter C. (Eds.) Operator Theory: Adv. and Appl., 283. Birkhäuser, 2020. doi:10.1007/978-3-030-60484-4

[14] Marchenko V.A. Sturm-Liouville operators and applications. In: Ball J.A., Böttcher A., Tretter C. (Eds.) Operator Theory: Adv. and Appl., 22. Birkhäuser, 1986. doi:10.1007/978-3-0348-5485-6

[15] Pistol M.-E. *Generating isospectral but not isomorphic quantum graphs*. arXiv:2104.12885 [math.SP]. doi: 10.48550/arXiv.2104.12885

[16] Pivovarchik V. *Recovering the Shape of an Equilateral Quantum Tree by Two Spectra*. Integr. Equ. Oper. Theory 2024, **96** (2), article 11. doi:10.1007/s00020-024-02759-6

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Показано, що спектр задачі Штурма-Ліувілля на зв'язному простому рівносторонньому графі з умовами Діріхле у висячих вершинах пов'язаний зі спектром дискретного лапласіана відповідного комбінаторного графа. Це дає змогу порівнювати спектри дискретних лапласіанів з метою знаходження коспектральних комбінаторних графів і, зрештою, коспектральних квантових графів. Використовуючи цей метод, ми доводимо, що не існує коспектральних (у нашому розумінні) графів із кількістю ребер, меншою або рівною 7. Отже, у цьому випадку обернена задача відновлення форми квантового графа має єдиний розв'язок.

Ключові слова і фрази: граф, ребро, вершина, власне значення, потенціал, матриця суміжності, характеристична функція.