ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2026, **18** (2), 504–516
doi:10.15330/cmp.18.2.504-516

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2026, Т.18, №2, С.504–516

Transposed Poisson structures on null-filiform associative algebras

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In this paper, we describe all transposed Poisson structures on associative null-filiform algebras. We consider the unique n-dimensional null-filiform associative algebra and determine all Lie brackets that satisfy the transposed Poisson compatibility identity with the given associative multiplication. First, we derive the general form of all transposed Poisson algebra structures defined on null-filiform associative algebras. Next, by studying the automorphism group of the underlying associative algebra, we establish explicit isomorphism criteria and classify these structures up to isomorphism. As a consequence, we obtain a complete list of pairwise non-isomorphic transposed Poisson algebras in arbitrary dimension. The low-dimensional cases are analyzed separately, yielding complete classifications in dimensions two, three, and four.

Key words and phrases: algebra of polynomials, Lie algebra, Poisson algebra, transposed Poisson algebra.

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1 Introduction

Poisson algebras originated in the study of Poisson geometry in the 1970s and have since found applications in a wide range of mathematical and physical disciplines, including Poisson manifolds, algebraic geometry, operad theory, quantization theory, quantum groups, and classical and quantum mechanics. More recently, the notion of transposed Poisson algebras has been introduced in [5], providing a dual perspective on Poisson structures and leading to novel algebraic frameworks. This concept has been applied in various algebraic structures, including Novikov-Poisson algebras and 3-Lie algebras [7]. In [11,13,23,28], Poisson structures on canonical algebras and the finitary incidence algebras of an arbitrary poset over a commutative unital ring are described. In [8], it is considered a new family of modified double Poisson brackets and mixed double Poisson algebras.

Research on transposed Poisson structures has gained significant traction in recent years. In [6], a comprehensive algebraic and geometric classification of transposed Poisson algebras was provided, further expanding the understanding of their structural properties. In paper [29], authors obtained a rich family of identities for transposed Poisson n-Lie algebras, and classified transposed Poisson 3-Lie algebras of dimension 3. Various works have explored transposed Poisson structures on different classes of Lie algebras. For instance, [14–16] examined such structures on Block Lie algebras and superalgebras, and Witt-type algebras,

УДК 512.554.31
2020 *Mathematics Subject Classification:* 17A30, 17A50, 17B63.

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revealing intricate interactions between transposed Poisson and classical Poisson structures. The study of transposed Poisson structures on low-dimensional Lie algebras was advanced in [3], where authors investigated the transposed Poisson structures on low dimensional quasi-filiform Lie algebras of maximum length.

Further progress has been made in the classification of transposed Poisson structures on specific algebraic frameworks. In [17], these structures were investigated in the context of upper triangular matrix Lie algebras. Additionally, the application of transposed Poisson structures to incidence algebras was explored in [18], extending their relevance to combinatorial algebra. A different perspective was provided in [9], where the connection between transposed Poisson algebras and $\frac{1}{2}$-derivations of Lie algebras was analyzed, uncovering new insights into derivation-based algebraic properties.

In [20, 21], transposed Poisson structures were examined in the context of not-finitely graded Witt-type algebras and Virasoro-type algebras, emphasizing their relevance in theoretical physics and representation theory. Using the connection between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras, descriptions of all transposed Poisson structures on some types of Lie algebras were obtained [2, 19, 22, 24, 26, 30].

The development of transposed Poisson structures continues to be an active area of research, with recent contributions exploring their connections to Jordan superalgebras [10], and Schrödinger algebras [27]. Additionally, applications to classification problems in non-associative algebras have been addressed in [12], further integrating transposed Poisson structures into the broader landscape of algebraic structures.

This paper provides a complete classification of all transposed Poisson algebra structures on null-filiform associative algebras. All Poisson algebra structures on null-filiform associative algebras were constructed in [1]. In Section 2, we introduce the necessary definitions and results that form the basis of our study. Using these foundations, we describe all such structures in Section 3.

2 Preliminaries

In this section, we introduce the relevant concepts and known results. Unless stated otherwise, all algebras considered in this paper are defined over the field $\mathbb{C}$.

We first recall the definition of a Poisson algebra.

Definition 1 ([5]). *Let $\mathfrak{L}$ be a vector space equipped with two bilinear operations*

$$\cdot\,,\ [-,-] : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L},$$

where $(\mathfrak{L}, \cdot\,)$ is a commutative associative algebra and $(\mathfrak{L}, [-,-])$ is a Lie algebra.

The triple $(\mathfrak{L}, \cdot, [-,-])$ is called a Poisson algebra if

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z] \quad \text{for all } x, y, z \in \mathfrak{L}. \tag{1}$$

The triple $(\mathfrak{L}, \cdot, [-,-])$ is called a transposed Poisson algebra if

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]. \tag{2}$$

For any transposed Poisson algebra $(\mathfrak{L}, \cdot, [\,\cdot\,,\,\cdot\,])$ each right multiplication of $(\mathfrak{L}, \cdot\,)$ gives a $\frac{1}{2}$-derivation of $(\mathfrak{L}, [\,\cdot\,,\,\cdot\,])$. Note, $\frac{1}{2}$-derivations are particular case of δ-derivations (about δ-derivations see in [7, 31]). A (transposed) Poisson algebra $\mathfrak{L}$ is called *trivial*, if $\mathfrak{L} \cdot \mathfrak{L} = 0$ or $[\mathfrak{L}, \mathfrak{L}] = 0$.

The next result shows that the compatibility relations of the Poisson algebra and those of the transposed Poisson algebra are independent in the following sense.

Proposition 1 ([5]). *Let $(\mathfrak{L}, \cdot)$ be a commutative associative algebra and $(\mathfrak{L}, [-,-])$ be a Lie algebra. Then $(\mathfrak{L}, \cdot, [-,-])$ is both a Poisson algebra and a transposed Poisson algebra if and only if $x \cdot [y,z] = [x \cdot y, z] = 0$.*

For an algebra $\mathcal{A}$, we consider the series

$$\mathcal{A}^1 = \mathcal{A}, \quad \mathcal{A}^{i+1} = \sum_{k=1}^{i} \mathcal{A}^k \mathcal{A}^{i+1-k}, \quad i \geq 1.$$

We say that an algebra $\mathcal{A}$ is *nilpotent* if $\mathcal{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer i satisfying $\mathcal{A}^i = 0$ is called the *index of nilpotency* of $\mathcal{A}$.

Definition 2. *An n-dimensional algebra $\mathcal{A}$ is called null-filiform if* $\dim \mathcal{A}^i = (n+1) - i$ *for* $1 \leq i \leq n+1$.

All null-filiform associative algebras were described in [25, Proposition 5.3].

Theorem 1 ([25]). *An arbitrary n-dimensional null-filiform associative algebra is isomorphic to the algebra*

$$\mu_0^n : \ e_i \cdot e_j = e_{i+j}, \quad 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra μ_0^n and the omitted products vanish.

Theorem 2 ([4]). *A linear map $\varphi : \mu_0^n \to \mu_0^n$ is an automorphism of the algebra μ_0^n if and only if the map φ has the following form*

$$\varphi(e_1) = \sum_{i=1}^{n} A_i e_i, \quad \varphi(e_i) = \sum_{j=i}^{n} \sum_{k_1+k_2+\cdots+k_i=j} A_{k_1} A_{k_2} \ldots A_{k_i} e_j, \quad 2 \leq i \leq n,$$

where $A_1 \neq 0$.

3 Transposed Poisson algebra structures on null-filiform associative algebras

Theorem 3. *Let $(\mu_0^n, \cdot, [-,-])$ be a transposed Poisson algebra defined on the associative algebra $(\mu_0^n, \cdot)$. Then the multiplication of $(\mu_0^n, \cdot, [-,-])$ has the following form*

$$\mathbf{TP}(\alpha_2, \ldots, \alpha_n) : \begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_i, e_j] = (j-i) \sum\limits_{t=i+j-1}^{n} \alpha_{t-i-j+3} e_t, & 3 \leq i+j \leq n+1. \end{cases}$$

Proof. Let $(\mu_0^n, \cdot, [-,-])$ be a transposed Poisson algebra. To establish the table of multiplications, we set

$$[e_1, e_2] = \sum_{t=1}^{n} \alpha_t e_t.$$

Now, we consider the condition (2) for the triple $\{e_1, e_1, e_2\}$, namely

$$2e_1 \cdot [e_1, e_2] = [e_1 \cdot e_1, e_2] + [e_1, e_1 \cdot e_2],$$

$$2e_1 \cdot \sum_{t=1}^{n} \alpha_t e_t = [e_2, e_2] + [e_1, e_3].$$

From this, we derive the following product $[e_1, e_3] = 2\sum_{t=2}^{n} \alpha_{t-1} e_t$. By induction on i, we can show that

$$[e_1, e_i] = (i-1) \sum_{t=i-1}^{n} \alpha_{t-i+2} e_t.$$

The equality holds for $i = 2$. Assuming it holds for i, we prove the equality for $i+1$. We consider the identity (2) for the triple $\{e_1, e_1, e_i\}$, i.e.

$$2e_1 \cdot [e_1, e_i] = [e_1 \cdot e_1, e_i] + [e_1, e_1 \cdot e_i],$$

$$2e_1 \cdot (i-1) \sum_{t=i-1}^{n} \alpha_{t-i+2} e_t = [e_2, e_i] + [e_1, e_{i+1}].$$

Thus we obtain

$$[e_2, e_i] + [e_1, e_{i+1}] = 2(i-1) \sum_{t=i}^{n} \alpha_{t-i+1} e_t. \tag{3}$$

Similarly, considering (2) for the triple $\{e_{i-1}, e_1, e_2\}$, we get

$$2e_{i-1} \cdot [e_1, e_2] = [e_{i-1} \cdot e_1, e_2] + [e_1, e_{i-1} \cdot e_2],$$

$$2e_{i-1} \cdot \sum_{t=1}^{n} \alpha_t e_t = [e_i, e_2] + [e_1, e_{i+1}],$$

so we have

$$[e_i, e_2] + [e_1, e_{i+1}] = 2 \sum_{t=i}^{n} \alpha_{t-i+1} e_t. \tag{4}$$

From equations (3) and (4), we deduce

$$[e_1, e_{i+1}] = i \sum_{t=i}^{n} \alpha_{t-i+1} e_t \qquad \text{and} \qquad [e_i, e_2] = (2-i) \sum_{t=i}^{n} \alpha_{t-i+1} e_t.$$

Applying induction and the identity (2) for $2 \leq i \leq n$, we establish

$$[e_i, e_j] = (j-i) \sum_{t=i+j-2}^{n} \alpha_{t-i-j+3} e_t. \tag{5}$$

We can write

$$2e_i \cdot [e_1, e_j] = [e_i \cdot e_1, e_j] + [e_1, e_i \cdot e_j],$$

$$2e_i \cdot (j-1) \sum_{t=j-1}^{n} \alpha_{t-j+2} e_t = [e_{i+1}, e_j] + [e_1, e_{i+j}],$$

$$2(j-1) \sum_{t=i+j-1}^{n} \alpha_{t-i-j+2} e_t = [e_{i+1}, e_j] + (i+j-1) \sum_{t=i+j-1}^{n} \alpha_{t-i-j+2} e_t.$$

Thus, we derive

$$[e_{i+1}, e_j] = (j-i-1) \sum_{t=i+j-1}^{n} \alpha_{t-i-j+2} e_t.$$

Next, applying the Jacobi identity to $\{e_1, e_3, e_n\}$, we get

$$\begin{aligned}
0 &= [[e_1, e_3], e_n] + [[e_3, e_n], e_1] + [[e_n, e_1], e_3] \\
&= [2\sum_{t=2}^{n} \alpha_{t-1} e_t, e_n] + [(1-n)\sum_{t=n-1}^{n} \alpha_{t-n+2} e_t, e_3] \\
&= 2(n-2)\alpha_1 \sum_{k=n}^{n} \alpha_{k-n+1} e_k + (1-n)(4-n)\alpha_1 \sum_{k=n}^{n} \alpha_{k-n+1} e_k \\
&= (2(n-2) + (1-n)(4-n))\alpha_1^2 e_n \\
&= (n^2 - 3n)\alpha_1^2 e_n.
\end{aligned}$$

From here, we conclude that $\alpha_1 = 0$ for $n > 3$.

Now, we consider two and three-dimensional cases separately.

For the case $n = 2$, we analyze the identity (2) for the triple $\{e_1, e_1, e_2\}$, namely

$$0 = [e_1 \cdot e_1, e_2] + [e_1, e_1 \cdot e_2] = 2e_1 \cdot [e_1, e_2] = 2e_1 \cdot (\alpha_1 e_1 + \alpha_2 e_2) = 2\alpha_1 e_2.$$

From this, we have $\alpha_1 = 0$.

As for the case $n = 3$, we analyze the identity (2) for the triple $\{e_2, e_1, e_2\}$, i.e.

$$[e_3, e_2] = [e_2 \cdot e_1, e_2] + [e_1, e_2 \cdot e_2] = 2e_2 \cdot [e_1, e_2] = 2e_2 \cdot (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = 2\alpha_1 e_3.$$

On the other hand, from equality (5), we have $[e_2, e_3] = \alpha_1 e_3$. Thus, we obtain $\alpha_1 = 0$.

Hence, we obtain the transposed Poisson algebras $\mathbf{TP}(\alpha_2, \ldots, \alpha_n)$ given by the following multiplications

$$[e_i, e_j] = (j-i)\sum_{t=i+j-1}^{n} \alpha_{t-i-j+3} e_t, \quad 3 \le i+j \le n+1.$$

□

The following theorem establishes a necessary and sufficient condition for two algebras in the family $\mathbf{TP}(\alpha_2, \ldots, \alpha_n)$ to be isomorphic.

Definition 3. *Two transposed Poisson algebras $(\mathfrak{L}_1, \cdot_1, [\cdot, \cdot]_1)$ and $(\mathfrak{L}_2, \cdot_2, [\cdot, \cdot]_2)$ are said to be isomorphic if there exists a bijective linear map $\varphi : \mathfrak{L}_1 \to \mathfrak{L}_2$ satisfying*

$$\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y), \qquad \varphi([x, y]_1) = [\varphi(x), \varphi(y)]_2$$

for all $x, y \in \mathfrak{L}_1$.

Theorem 4. *Let $\mathbf{TP}(\alpha_2, \ldots, \alpha_n)$ and $\mathbf{TP}'(\alpha_2', \ldots, \alpha_n')$ are isomorphic algebras. Then there exists an automorphism φ between these algebras such that the following relation*

$$\sum_{i=2}^{t} \sum_{k_1+\cdots+k_i=t} A_{k_1} \ldots A_{k_i} \alpha_i' = \sum_{j=2}^{t} \sum_{i=1}^{t-j+1} \sum_{k_1+k_2=t-i-j+3} (t-2i-j+3) A_i A_{k_1} A_{k_2} \alpha_j \tag{6}$$

holds for $2 \le t \le n$.

Proof. Using the description of automorphisms of the algebra μ_0^n (see, Theorem 2), we introduce the following notations $e'_i = \varphi(e_i)$, $1 \le i \le n$.

Thus, we consider

$$[e'_1, e'_2] = \sum_{i=2}^{n} \alpha'_i e'_i = \sum_{i=2}^{n} \alpha'_i \sum_{j=i}^{n} \sum_{k_1+\cdots+k_i=j} A_{k_1} \dots A_{k_i} e_j$$
$$= \sum_{i=2}^{n} \sum_{j=i}^{n} \sum_{k_1+\cdots+k_i=j} \alpha'_i A_{k_1} \dots A_{k_i} e_j = \sum_{t=2}^{n} \sum_{i=2}^{t} \sum_{k_1+\cdots+k_i=t} \alpha'_i A_{k_1} \dots A_{k_i} e_t.$$

On the other hand, we have

$$[e'_1, e'_2] = [\sum_{i=1}^{n} A_i e_i, \sum_{j=2}^{n} \sum_{k_1+k_2=j} A_{k_1} A_{k_2} e_j] = \sum_{i=1}^{n} \sum_{j=2}^{n} \sum_{k_1+k_2=j} A_i A_{k_1} A_{k_2} [e_i, e_j]$$
$$= \sum_{i=1}^{n} \sum_{j=2}^{n} \sum_{k_1+k_2=j} A_i A_{k_1} A_{k_2} (j-i) \sum_{t=i+j-1}^{n} \alpha_{t-i-j+3} e_t$$
$$= \sum_{i=1}^{n} \sum_{j=2}^{n} \sum_{t=i+j-1}^{n} \sum_{k_1+k_2=j} (j-i) A_i A_{k_1} A_{k_2} \alpha_{t-i-j+3} e_t$$
$$= \sum_{i=1}^{n} \sum_{t=i+1}^{n} \sum_{j=2}^{t-i+1} \sum_{k_1+k_2=j} (j-i) A_i A_{k_1} A_{k_2} \alpha_{t-i-j+3} e_t$$
$$= \sum_{t=2}^{n} \sum_{i=1}^{t-1} \sum_{j=2}^{t-i+1} \sum_{k_1+k_2=j} (j-i) A_i A_{k_1} A_{k_2} \alpha_{t-i-j+3} e_t.$$

Comparing the coefficients of the obtained expressions for the basis elements for $2 \le t \le n$, we get the following restrictions

$$\sum_{i=2}^{t} \sum_{k_1+\cdots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{i=1}^{t-1} \sum_{j=2}^{t-i+1} \sum_{k_1+k_2=j} (j-i) A_i A_{k_1} A_{k_2} \alpha_{t-i-j+3}.$$

Now we transform the right-hand side of the above equation. Setting $m := t-i-j+3$, we obtain $2 \le m \le t-i+1$, so we replace j with $t-i-m+3$. This yields

$$\sum_{i=1}^{t-1} \sum_{j=2}^{t-i+1} \sum_{k_1+k_2=j} (j-i) A_i A_{k_1} A_{k_2} \alpha_{t-i-j+3} = \sum_{i=1}^{t-1} \sum_{m=2}^{t-i+1} \sum_{k_1+k_2=t-i-m+3} (t-2i-m+3) A_i A_{k_1} A_{k_2} \alpha_m.$$

Now, interchanging the order of the first and second summations, we obtain

$$\sum_{i=1}^{t-1} \sum_{m=2}^{t-i+1} \sum_{k_1+k_2=t-i-m+3} (t-2i-m+3) A_i A_{k_1} A_{k_2} \alpha_m$$
$$= \sum_{m=2}^{t} \sum_{i=1}^{t-m+1} \sum_{k_1+k_2=t-i-m+3} (t-2i-m+3) A_i A_{k_1} A_{k_2} \alpha_m.$$

Finally, replacing m with j, we get the following equality

$$\sum_{i=2}^{t} \sum_{k_1+\cdots+k_i=t} A_{k_1} \dots A_{k_i} \alpha'_i = \sum_{j=2}^{t} \sum_{i=1}^{t-j+1} \sum_{k_1+k_2=t-i-j+3} (t-2i-j+3) A_i A_{k_1} A_{k_2} \alpha_j$$

for $2 \le t \le n$. □

Lemma 1. *Let* $\mathbf{TP}(0,\dots,0,\alpha_s,\dots,\alpha_n)$*, with* $\alpha_s \neq 0$*, be a transposed Poisson algebra defined above. Then there exists* $A \in \mathbb{C}$ *such that the relation* $\alpha'_s = \alpha_s A^{3-s}$ *holds for any* $2 \leq s \leq n$.

Proof. Let $\mathbf{TP}(0,\dots,0,\alpha_s,\dots,\alpha_n)$ be a transposed Poisson algebra. Consider a change of basis. Then for some natural number s the restriction (6) has the form

$$\sum_{i=2}^{s}\sum_{k_1+\dots+k_i=s} A_{k_1}\dots A_{k_i}\alpha'_i = \sum_{j=2}^{s}\sum_{i=1}^{s-j+1}\sum_{k_1+k_2=s-i-j+3}(s-2i-j+3)A_iA_{k_1}A_{k_2}\alpha_j.$$

Now, considering that $\alpha_i = 0$ for $2 \leq i < s$, this reduces to

$$\sum_{k_1+\dots+k_s=s} A_{k_1}\dots A_{k_s}\alpha'_s = \sum_{i=1}^{s-s+1}\sum_{k_1+k_2=s-i-s+3}(s-2i-s+3)A_iA_{k_1}A_{k_2}\alpha_s,$$

which further simplifies to

$$\sum_{k_1+\dots+k_s=s} A_{k_1}\dots A_{k_s}\alpha'_s = \sum_{k_1+k_2=2} A_1A_{k_1}A_{k_2}\alpha_s.$$

From this equality, we obtain the relation $\alpha'_s = \alpha_s A_1^{3-s}$. □

Lemma 2. *If* $\alpha_2 \neq 0$*, then* $\mathbf{TP}(\alpha_2,\dots,\alpha_n)$ *is isomorphic to the algebra* $\mathbf{TP}(1,0,\dots,0)$.

Proof. Let $\alpha_2 \neq 0$. Consider the change of basis, defined by $e'_i = \alpha_2^{-i}e_i$, $1 \leq i \leq n$. Applying this transformation to the product $[e_1,e_2] = \sum_{t=2}^{n}\alpha_t e_t$, we can assume that $\alpha_2 = 1$.

Next, consider another change of basis, i.e. $e'_1 = e_1 - \frac{\alpha_3}{2}e_2$. From the relations $e_i \cdot e_j = e_{i+j}$ for $2 \leq i+j \leq n$, we obtain

$$e'_i = \sum_{t=0}^{i}\binom{i}{t}\Big(-\frac{\alpha_3}{2}\Big)^t e_{i+t}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$. Considering $[e_1,e_2] = e_2 + \sum_{t=3}^{n}\alpha_t e_t$ we conclude that $\alpha_3 = 0$.

Now, we prove by induction that it is possible to set $\alpha_j = 0$ for all $3 \leq j \leq n$. The base case $j = 3$ is already established. Assuming the claim holds for some j, we show it also holds for $j+1$. Consider the change of basis, defined by

$$e'_1 = e_1 - \frac{\alpha_{j+1}}{j}e_j.$$

Using $e_ie_j = e_{i+j}$ for $2 \leq i+j \leq n$, we derive

$$e'_i = \sum_{t=0}^{i}\binom{i}{t}\Big(-\frac{\alpha_{j+1}}{j}\Big)^t e_{i+t(j-1)}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$ in this sum. Considering $[e_1,e_2] = e_2 + \sum_{t=j+1}^{n}\alpha_t e_t$ we conclude that $\alpha_{j+1} = 0$.

By induction, we have $\alpha_j = 0$ for all $3 \leq j \leq n$, proving that the algebra $\mathbf{TP}(\alpha_2,\dots,\alpha_n)$ is isomorphic to $\mathbf{TP}(1,0,\dots,0)$ with the following multimplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_i,e_j] = (j-i)e_{i+j-1}, & 3 \leq i+j \leq n+1. \end{cases}$$

□

Lemma 3. *If $\alpha_3 \neq 0$, then $\mathbf{TP}(0,\alpha_3,\dots,\alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0,\alpha,0,\dots,0)$.*

Proof. Let $\alpha_3 \neq 0$, and consider the change of basis, defined by $e'_1 = e_1 - \frac{\alpha_4}{\alpha_3}e_2$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \leq i+j \leq n$, we obtain

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} \Big(-\frac{\alpha_4}{\alpha_3} \Big)^t e_{i+t}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = \alpha_3 e_3 + \sum_{t=4}^{n} \alpha_t e_t$, we conclude that $\alpha_4 = 0$.

Now, we prove by induction that we can eliminate α_j for all $4 \leq j \leq n$. The base case $j = 4$ follows from the above step. Assuming that $\alpha_j = 0$ holds for some $j \geq 4$, we prove it for $j+1$. Consider the change of basis, defined by $e'_1 = e_1 - \frac{\alpha_{j+1}}{(j-2)\alpha_3}e_{j-1}$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \leq i+j \leq n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} \Big(-\frac{\alpha_{j+1}}{(j-2)\alpha_3} \Big)^t e_{i+t(j-2)}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_2 + \sum_{t=j+1}^{n} \alpha_t e_t$, we conclude that $\alpha_{j+1} = 0$.

Thus, we have shown that $\alpha_j = 0$ for $4 \leq j \leq n$, and the algebra $\mathbf{TP}(0,\alpha_3,\dots,\alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0,\alpha,0,\dots,0)$ with the following multimplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \leq i+j \leq n, \\ [e_i, e_j] = (j-i)\alpha e_{i+j}, & 3 \leq i+j \leq n. \end{cases}$$

□

Lemma 4. *If $\alpha_4 \neq 0$, then $\mathbf{TP}(0,0,\alpha_4,\dots,\alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0,0,1,\alpha,0,\dots,0)$.*

Proof. From the equality (6), we obtain the following equality in general substitution:

$$\alpha'_4 = \frac{\alpha_4}{A_1}, \qquad \alpha'_5 = \frac{\alpha_5}{A_1^2}.$$

Since $\alpha_4 \neq 0$, we consider the change of basis $e'_i = \alpha_4^i e_i$ for $1 \leq i \leq n$. Considering the product $[e_1, e_2] = \sum_{t=4}^{n} \alpha_t e_t$ we can assume that $\alpha_4 = 1$.

Now, consider the change of basis, defined by $e'_1 = e_1 - \alpha_6 e_3$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \leq i+j \leq n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} (-\alpha_6)^t e_{i+2t}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_4 + \sum_{t=5}^{n} \alpha_t e_t$, we obtain that $\alpha_6 = 0$.

Now we prove by induction that it is possible to set $\alpha_j = 0$ for $6 \leq j \leq n$. The base case $j = 6$ holds by the above argument. Now, assuming it holds for some j, we prove it for $j+1$. Consider basis change $e'_1 = e_1 - \frac{\alpha_{j+1}}{j-4}e_{j-2}$. From $e_i \cdot e_j = e_{i+j}$ for $2 \leq i+j \leq n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} \Big(-\frac{\alpha_{j+1}}{j-4} \Big)^t e_{i+t(j-3)}, \quad 1 \leq i \leq n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_2 + \sum_{t=j+1}^{n} \alpha_t e_t$, we obtain that $\alpha_{j+1} = 0$.

Thus, we have shown that $\alpha_j = 0$ for $6 \le j \le n$, and the algebra $\mathbf{TP}(0, 0, \alpha_4, \dots, \alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0, 0, 1, \alpha, 0, \dots, 0)$ with the following multimplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \le i + j \le n, \\ [e_i, e_j] = (j - i)(e_{i+j+1} + \alpha e_{i+j+2}), & 3 \le i + j \le n - 1. \end{cases}$$

□

Lemma 5. *If $\alpha_s \neq 0$ for some $s \ge 5$, then $\mathbf{TP}(0, \dots, 0, \alpha_s, \dots, \alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0, \dots, 0, 1_s, 0, \dots, 0, \alpha_{2s-3}, 0, \dots, 0)$.*

Proof. Let $\alpha_s \neq 0$ and consider the change of basis, defined by $e'_i = \alpha_s^{\frac{i}{s-3}} e_i$, $1 \le i \le n$. Considering the product $[e_1, e_2] = \sum_{t=s}^{n} \alpha_t e_t$ we can assume that $\alpha_s = 1$.

Now consider the change of basis, defined by $e'_1 = e_1 + \frac{\alpha_{s+1}}{s-4} e_2$. From $e_i \cdot e_j = e_{i+j}$ for $2 \le i + j \le n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} \Big(\frac{\alpha_{s+1}}{s-4}\Big)^t e_{i+t}, \quad 1 \le i \le n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_s + \sum_{t=s+1}^{n} \alpha_t e_t$ we can derive that $\alpha_{s+1} = 0$.

Now we prove by induction using successive changes of the basis elements that it is possible to make $\alpha_j = 0$ for $s + 1 \le j \le 2s - 4$. If $j = s + 1$, the relationship is fulfilled according to the above equalities. Now, that it holds for some j, we prove it for $j + 1$. We consider the change of the basis element e_1 as $e'_1 = e_1 + \frac{\alpha_{j+1}}{2s-4-j} e_{j-s+2}$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \le i + j \le n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} \Big(\frac{\alpha_{j+1}}{2s-4-j}\Big)^t e_{i+tj}, \quad 1 \le i \le n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_s + \sum_{t=j+1}^{n} \alpha_t e_t$ we can derive that $\alpha_{j+1} = 0$. Then we have

$$[e_1, e_2] = e_s + \sum_{t=2s-3}^{n} \alpha_t e_t.$$

From the equality (6), we obtain the following equality in general substitution

$$\alpha'_{2s-3} = \alpha_{2s-3}.$$

Now, consider the change of basis elements, defined by $e'_1 = e_1 - \alpha_{2s-2} e_{s-1}$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \le i + j \le n$, we derive

$$e'_i = \sum_{t=0}^{i} \binom{i}{t} (-\alpha_{2s-2})^t e_{i+t(s-2)}, \quad 1 \le i \le n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_s + \sum_{t=s+1}^{n} \alpha_t e_t$, we can derive that $\alpha_{2s-2} = 0$.

Now we prove by induction that it is possible to set $\alpha_j = 0$ for $2s-2 \le j \le n$. If $j = 2s-2$, the relationship is fulfilled according to the above equalities. Now, that it holds for j, we prove it for $j+1$. Consider the change of basis element e_1 as $e_1' = e_1 - \frac{\alpha_{j+1}}{2s-4-j} e_{j-s+2}$. From the products $e_i \cdot e_j = e_{i+j}$ for $2 \le i+j \le n$, we derive

$$e_i' = \sum_{t=0}^{i} \binom{i}{t} \Big(- \frac{\alpha_{j+1}}{2s-4-j} \Big)^t e_{i+t(j-s+1)}, \quad 1 \le i \le n,$$

where we assume that $e_t = 0$ for $t > n$. Considering the product $[e_1, e_2] = e_2 + \sum_{t=j+1}^{n} \alpha_t e_t$, we can derive that $\alpha_{j+1} = 0$.

Thus, we have shown that $\alpha_j = 0$ for $2s-2 \le j \le n$, and the algebra $\mathbf{TP}(0, \dots, 0, \alpha_s, \dots, \alpha_n)$ is isomorphic to the algebra $\mathbf{TP}(0, \dots, 0, 1_s, 0, \dots, 0, \alpha_{2s-3}, 0, \dots, 0)$ with the following multimplication rules:

$$\begin{cases} e_i \cdot e_j = e_{i+j}, & 2 \le i+j \le n, \\ [e_i, e_j] = (j-i)(e_{s+i+j-3} + \alpha_{2s-3} e_{2s+i+j-6}), & 3 \le i+j \le n-s+3. \end{cases}$$

□

Theorem 5. *Let $(\mu_0^n, \cdot, [-,-])$ be a transposed Poisson algebra and $n \ge 5$. Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}(1, 0, \dots, 0), \quad \mathbf{TP}(0, \alpha, 0, \dots, 0),$$

$$\mathbf{TP}(0, \dots, 0, 1_s, 0, \dots, 0, \alpha_{2s-3}, 0, \dots, 0), \quad 4 \le s \le n, \ \alpha \in \mathbb{C}.$$

Proof. Let $(\mu_0^n, \cdot, [-,-])$ be a transposed Poisson algebra. Then Theorem 3 implies that it is isomorphic to the algebra $\mathbf{TP}(\alpha_2, \dots, \alpha_n)$. Moreover,

- if $\alpha_2 \neq 0$, then from Lemma 2 we have $\mathbf{TP}(1, 0, \dots, 0)$;
- if $\alpha_2 = 0$, $\alpha_3 \neq 0$, then from Lemma 3 we have $\mathbf{TP}(0, \alpha, 0, \dots, 0)$, where $\alpha \neq 0$;
- if $\alpha_2 = \alpha_3 = 0$, $\alpha_4 \neq 0$, then from Lemma 4 we have $\mathbf{TP}(0, 0, 1, \alpha, 0, \dots, 0)$;
- if $\alpha_2 = \dots = \alpha_{s-1} = 0$, $\alpha_s \neq 0$, $s \ge 5$, then from Lemma 5 we have

$$\mathbf{TP}(0, \dots, 0, 1_s, 0, \dots, 0, \alpha_{2s-3}, 0, \dots, 0), \quad 5 \le s \le n;$$

- if $\alpha_i = 0, 2 \le i \le n$, then we have $\mathbf{TP}(0, \dots, 0)$.

□

In the following theorems, we consider 2, 3 and 4-dimensional cases. In [5], it is proved that any 2-dimensional complex transposed Poisson algebra on the 2-dimensional associative algebra, whose product is given by $e_1 \cdot e_1 = e_2$, is isomorphic to one of the following transposed Poisson algebras:

$$\mathbf{TP}(0): \ e_1 \cdot e_1 = e_2; \quad \mathbf{TP}(1): \ e_1 \cdot e_1 = e_2, \ [e_1, e_2] = e_2.$$

Theorem 6. *Let $(\mu_0^3, \cdot, [-,-])$ be a transposed Poisson algebra. Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}(1,0), \quad \mathbf{TP}(0,\alpha), \quad \alpha \in \mathbb{C}.$$

Proof. Let $(\mu_0^3, \cdot, [-,-])$ be a transposed Poisson algebra defined on a null-filiform associative algebra. Then according to Theorem 3 it is isomorphic to the algebra $\mathbf{TP}(\alpha_2, \alpha_3)$. Applying a change of basis and using equality (6), we get the following relations:

$$\alpha_2' = A_1\alpha_2, \qquad \alpha_3' = \frac{A_1\alpha_3 + 2A_2\alpha_2}{A_1}.$$

Let $\alpha_2 \neq 0$. Then, by selecting

$$A_1 = \frac{1}{\alpha_2}, \qquad A_2 = -\frac{\alpha_3}{2\alpha_2^2},$$

we get the algebra $\mathbf{TP}(1,0)$.

Let $\alpha_2 = 0$. Then we derive the algebra $\mathbf{TP}(0,\alpha)$. □

Theorem 7. *Let $(\mu_0^4, \cdot, [-,-])$ be a transposed Poisson algebra. Then this algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathbf{TP}(1,0,0), \quad \mathbf{TP}(0,\alpha,0), \quad \mathbf{TP}(0,0,1), \quad \alpha \in \mathbb{C}.$$

Proof. Let $(\mu_0^n, \cdot, [-,-])$ be a transposed Poisson algebra. According to Theorem 3, this algebra is isomorphic to the algebra $\mathbf{TP}(\alpha_2, \alpha_3, \alpha_4)$. Applying a change of basis and using the equality (6), we get the following relations:

$$\alpha_2' = A_1\alpha_2, \quad \alpha_3' = \frac{A_1\alpha_3 + 2A_2\alpha_2}{A_1}, \quad \alpha_4' = \frac{A_1^2\alpha_4 + A_1A_2\alpha_3 + (3A_1A_3 - 2A_2^2)\alpha_2}{A_1^3}.$$

Let $\alpha_2 \neq 0$. Then by selecting

$$A_1 = \frac{1}{\alpha_2}, \quad A_2 = -\frac{\alpha_3}{2\alpha_2^2}, \quad A_3 = \frac{\alpha_3^2 - \alpha_2\alpha_4}{3\alpha_2^3},$$

we obtain the algebra $\mathbf{TP}(1,0,0)$.

Let $\alpha_2 = 0$ and $\alpha_3 \neq 0$. Choosing $A_1 = -\alpha_3$, $A_2 = \alpha_4$, we derive the algebra $\mathbf{TP}(0,\alpha,0)$, where $\alpha \neq 0$.

Let $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 \neq 0$. Then, by setting $A_1 = \alpha_1$, we obtain the algebra $\mathbf{TP}(0,0,1)$.

If $\alpha_2 = \alpha_3 = \alpha_4 = 0$, then the algebra simplifies to $\mathbf{TP}(0,0,0)$. □

Remark 1. *Thus, from the following table and the obtained results, it can be seen that the classifications of the transposed Poisson algebra structures defined on the algebra μ_0^n leads to the following pairwise non-isomorphic algebras:*

- *algebras $\mathbf{TP}(0,\alpha,0,\dots,0)$ and $\mathbf{TP}(0,\dots,0,1_s,0,\dots,0,\alpha_{2s-3},0,\dots,0)$ for $4 \leq s \leq n$, $\alpha \in \mathbb{C}$, are nilpotent;*
- *algebra $\mathbf{TP}(1,0,\dots,0)$ is solvable.*

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Received 09.03.2025
Revised 13.06.2026

Адашев Ж., Бердалова Х., Тоштемірова Ф. *Транспоновані структури Пуассона на нуль-філіформних асоціативних алгебрах* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 504–516.

У статті описано всі транспоновані структури Пуассона на асоціативних нуль-філіформних алгебрах. Розглянуто єдину n-вимірну нуль-філіформну асоціативну алгебру та знайдено всі дужки Лі, які разом із заданим асоціативним множенням задовольняють тотожність сумісності транспонованих алгебр Пуассона. Спочатку отримано загальний вигляд усіх структур транспонованої алгебри Пуассона на нуль-філіформних асоціативних алгебрах. Далі, на основі дослідження групи автоморфізмів відповідної асоціативної алгебри, встановлено явні критерії ізоморфності та класифіковано ці структури з точністю до ізоморфізму. Як наслідок, отримано повний список попарно неізоморфних транспонованих алгебр Пуассона довільної розмірності. Окремо проаналізовано випадки малих розмірностей, що дало повні класифікації в розмірностях два, три і чотири.

Ключові слова і фрази: алгебра многочленів, алгебра Лі, алгебра Пуассона, транспонована алгебра Пуассона.