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On the maximal term of series in systems of functions in a disk

Sheremeta M.M.[1]**, Dobushovskyy M.S.**[2]

For an entire transcendental function f and for a sequence (λ_n) of positive numbers increasing to $+\infty$, a series $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ is said to be regularly convergent in $\{z : |z| < R[A] < +\infty\}$, if $\mathfrak{M}(r, A) = \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty$ for all $r \in [0, R[A])$, where $R[A]$ is the radius of a regular convergence of $A(z)$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$.

We have found conditions on (λ_n) and f, under which $\ln \mathfrak{M}(r, A) \sim \ln \mu(r, A)$ as $r \to R[A]$, where $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) : n \geq 1\}$ is the maximal term of the series.

At the end of the paper, an unresolved problem is stated.

Key words and phrases: series over a system of functions, regularly converging series, maximal term.

[1] Ivan Franko Lviv National University, 1 Universytetska str., 79000, Lviv, Ukraine
[2] Carpathian Branch of Subbotin Institute of Geophysics, 3b Naukova str., 79060, Lviv, Ukraine
E-mail: m.m.sheremeta@gmail.com (Sheremeta M.M.), mdobush19@gmail.com (Dobushovskyy M.S.)

Introduction

For an entire function $\phi(z) = \sum_{k=0}^{\infty} \phi_k z^k$, let $M_\phi(r) = \max\{|\phi(z)| : |z| = r\}$ and let $\mu_\phi(r) = \max\{|\phi_k| r^k : k \geq 0\}$ be the maximal term. It is known (see, for example, [1, p. 31]) that if ϕ has a finite order of growth, then $\ln M_\phi(r) \sim \ln \mu_\phi(r)$ as $r \to +\infty$.

A direct generalization of the power expansion of an entire function is the Dirichlet series $F(s) = \sum_{k=0}^{\infty} \varphi_k \exp\{s\lambda_k\}$, which is absolutely convergent for all $s = \sigma + it \in \mathbb{C}$, where $0 \leq \lambda_k \uparrow +\infty$. For such a function, we put $M_F(\sigma) = \sup\{|F(s)| : |t| < +\infty\}$ and $\mu_F(\sigma) = \max\{|\varphi_k| \exp\{\sigma\lambda_k\} : k \geq 0\}$. Then $\ln M_F(\sigma) \sim \ln \mu(\sigma)$ as $\sigma \to +\infty$ (see [2]), provided $|\varphi_k| \leq \exp\{-\lambda_k(h(\lambda_k)\}$ for $k \geq k_0$ and $\ln k = O(h(\lambda_k))$ as $k \to \infty$, where h is a positive continuous function on $[0, +\infty)$ increasing to $+\infty$. In [3,4], the conditions were studied under which $\psi(\ln M_F(\sigma)) \sim \psi(\ln \mu(\sigma))$ as $\sigma \to +\infty$, where ψ is a positive continuous function on $[0, +\infty)$ increasing to $+\infty$.

The condition $|\varphi_k| \leq \exp\{-\lambda_k(h(\lambda_k)\}$ for $k \geq k_0$ in the case of entire Dirichlet series is actually a condition for the growth of the maximal term from above. On the other hand, $\ln \sigma = O(\ln \mu_F(\sigma))$ as $\sigma \to +\infty$. If the Dirichlet series has a zero abscissa of absolute convergence, then the maximal term can be bounded and in order that $\mu_F(\sigma) \to +\infty$ as $\sigma \uparrow 0$, it is necessary and sufficient that $\overline{\lim\limits_{k\to\infty}} |\varphi_k| = +\infty$. The same situation occurs for analytic functions in the unit disk. This indicates that in addition to restrictions on the growth of the function F (or ϕ) from above, restrictions from below are required. For example, in [5] it is proven that if an analytic in the disk $\{z : |z| < 1\}$ function ϕ has a finite order, i.e. $\varrho = \overline{\lim\limits_{r\uparrow 1}} \dfrac{\ln^+ \ln M_\phi(r)}{-\ln(1-r)} < +\infty$

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and $\lambda = \varliminf\limits_{r\uparrow 1} \dfrac{\ln^+ M_\phi(r)}{-\ln(1-r)} = +\infty$, then $\ln M_\phi(r) \sim \ln \mu_\phi(r)$ as $r \uparrow 1$. This statement is derived from the following proposition.

Proposition 1 ([5]). *Let the function F is represented as a Dirichlet series with a zero abscissa of absolute convergence and the functions Φ and L are positive, continuous and increasing to $+\infty$ on $[0, +\infty)$ and $\varlimsup\limits_{x\to+\infty} \dfrac{L(\lambda x)}{L(x)} = c(\lambda) < +\infty$ for each $\lambda \in (0, +\infty)$. If*

$$\varlimsup_{\sigma\uparrow 0} \frac{\ln M_F(\sigma)}{|\sigma|\Phi(1/|\sigma|)} < +\infty, \qquad \varliminf_{\sigma\uparrow 0} \frac{\ln M_F(\sigma)}{|\sigma| L(1/|\sigma|)} = +\infty, \qquad \varlimsup_{t\to+\infty} \frac{\Phi^{-1}(t)\ln n(t)}{L(\Phi^{-1}(t))} < +\infty,$$

then $\ln M_F(\sigma) \sim \ln \mu_F(\sigma)$ as $\sigma \uparrow 0$.

Using modifications of the Wiman-Valiron method, in a number of works (see, for example, [6–9]) the equivalence of the logarithms of the maximum modulus and the maximum term outside some exceptional sets has been studied.

Now, let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire transcendental function and

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

be a series with respect to the system $f(\lambda_n z)$, where $\Lambda = (\lambda_n)$ is a sequence of positive numbers increasing to $+\infty$. Let $R[A]$ be the radius of a regular convergence of series (2), i.e.

$$\mathfrak{M}(r, A) = \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty \tag{3}$$

holds for $r < R[A]$ and does not hold for $r > R[A]$. If (2) holds for all $r \geq 0$, then we put $R[A] = +\infty$. Denote

$$\Gamma_f(r) = \frac{d \ln M_f(r)}{d \ln r}.$$

Note that in points, where the derivative does not exist, under $\dfrac{d\ln M_f(r)}{d\ln r}$ we mean right-hand derivative. Since the function $\ln M_f(r)$ is logarithmically convex, we have $\Gamma_f(r) \nearrow +\infty$ as $r \to +\infty$.

Let $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) : n \geq 1\}$ be the maximal term of series (3) and let $\nu(r, A) = \max\{n \geq 1 : |a_n| M_f(r\lambda_n) = \mu(r, A)\}$ be its central index.

It is known (see [10]) that if $\Gamma_f(cr) \asymp \Gamma_f(r)$ as $r \to +\infty$ for each $c \in (0, +\infty)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$, then

$$R[A] = \varliminf_{n\to\infty} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right).$$

If $R(A) = +\infty$, then let h be a positive continuous function on $[0, +\infty)$ increasing to $+\infty$ and $\mathbf{S}_h(f, \Lambda)$ be a class of functions (2) such that $|a_n| M_f(\lambda_n h(\lambda_n)) \to 0$ as $n \to +\infty$. By E we denote a class of entire functions (1) such that $r = O(\ln M_f(r))$ and $\ln M_f(r) = O(\Gamma_f(r))$ as $r \to +\infty$. Then for $f \in E$ we have $\ln M_f(r)) = o(\ln \mathfrak{M}(r, A))$ as $r \to +\infty$ (see [11]).

In [11], the following theorem is proved.

Proposition 2 ([11]). *If $R = +\infty$ and $\ln n = O(h(\lambda_n))$ as $n \to \infty$, then $\ln \mathfrak{M}(r, A) \sim \ln \mu(r, A)$ as $r \to +\infty$ for every function $A \in \mathbf{S}_h(f, \Lambda)$ and every $f \in E$.*

Let us note, that condition $\ln n = O(h(\lambda_n))$ as $n \to \infty$ from Proposition 2 cannot be weakened.

In the current paper, we will consider the case when $0 < R[A] < +\infty$.

Main results

It is clear that $\mu(r, A) \le \mathfrak{M}(r, A)$. On the other hand, if $R[A] < +\infty$, then the function $\mu(r, A)$ can be bounded on $[0, R[A])$, therefore the following statement holds.

Proposition 3. *In order that $\mu(r, A) \to +\infty$ as $r \to R[A]$, it is necessary and sufficient that*

$$\varlimsup_{n\to\infty} |a_n| M_f(R[A]\lambda_n) = +\infty.$$

Proof. If $\varlimsup\limits_{n\to\infty} |a_n| M_f(R[A]\lambda_n) < +\infty$, then there exists $K < +\infty$ such that $|a_n| M_f(R[A]\lambda_n) \le K$ for all $n \ge 1$ and, therefore, $|a_n| M_f(r\lambda_n) \le K$ for all $n \ge 1$ and $r \in [0, R[A])$, i.e. $\mu(r, A) \le K$ for all $r \in [0, R[A])$.

On the contrary, if $\mu(r, A) \le K$ for all $r \in [0, R[A])$, then $|a_n| M_f(r\lambda_n) \le K$ for all $n \ge 1$ and $r \in [0, R[A])$. Fixing n and letting $r \to R[A]$, we obtain $|a_n| M_f(R[A]\lambda_n) \le K$. □

In what follows, we will assume that the condition $\varlimsup\limits_{n\to\infty} |a_n| M_f(R[A]\lambda_n) = +\infty$ is satisfied. Moreover, let us assume that for all $r \in [0, R[A])$ we have

$$\Psi\left(\frac{1}{R[A] - r}\right) \le \ln \mu(r, A) \le \Phi\left(\frac{1}{R[A] - r}\right), \tag{4}$$

where functions Ψ and Φ are positive continuous and increasing to $+\infty$ on $[1/(R[A], +\infty)$. Then the following theorem is true.

Theorem 1. *Suppose that the function Φ^{-1} is slowly increasing and for some $\eta > 1$ we have*

$$x\Phi^{-1}(x) = o\left(\Gamma_f\left(\left(R[A] - \frac{\eta}{\Phi^{-1}(x)}\right)x\right)\right), \quad x \to +\infty, \tag{5}$$

and

$$\ln n_c(x) = o\left(\min\left\{\frac{\Gamma_f\left(\left(R[A] - \frac{\eta}{\Phi^{-1}(x)}\right)x\right)}{\Phi^{-1}(x)}, \Psi(\Phi^{-1}(x))\right\}\right), \quad x \to \infty, \tag{6}$$

where $n_c(t) = \sum\limits_{\lambda_n \le t} 1$ is the counting function of the sequence (λ_n). Then

$$\ln \mathfrak{M}(r, A) \sim \ln \mu(r, A) \quad \text{as} \quad r \uparrow R[A].$$

Proof. From (4) it follows that $\ln |a_n| \le \Phi(1/(R[A] - r)) - \ln M_f(r\lambda_n)$ for all r and n. Choosing $r = r_n = R[A] - 1/\Phi^{-1}(\lambda_n)$, we get $\ln |a_n| \le \lambda_n - \ln M_f((R[A] - 1/\Phi^{-1}(\lambda_n))\lambda_n)$, whence for all n we have

$$|a_n| \le \frac{e^{\lambda_n}}{M_f\left(\left(R[A] - \frac{1}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}. \tag{7}$$

Let $n_0(r)=\min\{n:\lambda_n\ge\Phi(\eta/(R[A]-r))\}$. Then (7) implies

$$\begin{aligned}\mathfrak{M}(r,A)&\le\sum_{n=1}^{n_0(r)-1}|a_n|M_f(r\lambda_n)+\sum_{n=n_0(r)}^{\infty}|a_n|M_f(r\lambda_n)\\&\le n_0(r)\mu(r,A)+\sum_{n=n_0(r)}^{\infty}\frac{e^{\lambda_n}M_f(r\lambda_n)}{M_f\left(\left(R[A]-\frac{1}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}.\end{aligned}\tag{8}$$

Since $r\le R[A]-\eta/\Phi^{-1}(\lambda_n)$ for $n\ge n_0(r)$, we have

$$\begin{aligned}\frac{M_f(r\lambda_n)}{M_f\left(\left(R[A]-\frac{1}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}&\le\exp\left\{-\left(\ln M_f\left(\left(R[A]-\frac{1}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)\right.\right.\\&\qquad\left.\left.-\ln M_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)\right)\right\}\\&=\exp\left\{-\int_{(R[A]-\eta/\Phi^{-1}(\lambda_n)\lambda_n)}^{(R[A]-1/\Phi^{-1}(\lambda_n)\lambda_n)}\Gamma_f(x)\,d\ln x\right\}\\&\le\exp\left\{-\Gamma_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)\ln\frac{R[A]-1/\Phi^{-1}(\lambda_n)}{R[A]-\eta/\Phi^{-1}(\lambda_n)}\right\}\\&=\exp\left\{-\Gamma_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)\frac{(1+o(1))(\eta-1)}{R[A]\Phi^{-1}(\lambda_n)}\right\}\end{aligned}$$

as $n\to\infty$. In view of (5), we get

$$\begin{aligned}\frac{M_f(r\lambda_n)e^{\lambda_n}}{M_f\left(\left(R[A]-\frac{1}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}&\le\exp\left\{-\frac{(1+o(1))(\eta-1)}{R[A]}\frac{\Gamma_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}{\Phi^{-1}(\lambda_n)}+\lambda_n\right\}\\&=\exp\left\{-\frac{(1+o(1))(\eta-1)}{R[A]}\frac{\Gamma_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right)}{\Phi^{-1}(\lambda_n)}\right\}\end{aligned}\tag{9}$$

as $n\to\infty$. From (6) it follows that

$$\ln n\le\frac{\eta-1}{2R[A]\Phi^{-1}(\lambda_n)}\Gamma_f\left(\left(R[A]-\frac{\eta}{\Phi^{-1}(\lambda_n)}\right)\lambda_n\right),\quad n\to\infty,$$

and, thus, from (8) and (9) we obtain the inequality $\mathfrak{M}(r,A)\le n_0(r)\mu(r,A)+\text{const}$, whence $\ln\mathfrak{M}(r,A)\le\ln\mu(r,A)+\ln n_0(r)+o(1)$ as $r\uparrow R[A]$.

On the other hand, the equality $n_0(r)=\min\{n:\lambda_n\ge\Phi(\eta/(R[A]-r))\}$ implies the inequality $n_0(r)\le n_c(\Phi(\eta/(R[A]-r))$ and, thus, in view of (6), we obtain

$$\begin{aligned}\varlimsup_{r\uparrow R[A]}\frac{\ln n_0(r)}{\ln\mu(r,A)}&\le\varlimsup_{r\uparrow R[A]}\frac{\ln n_c(\Phi(\eta/(R[A]-r))}{\Psi(1/(R[A]-r))}\\&=\varlimsup_{r\uparrow R[A]}\frac{\ln n_c(\Phi(\eta/(R[A]-r))}{\Psi(\eta/(R[A]-r))}=\varlimsup_{x\to+\infty}\frac{\ln n_c(\Phi(x))}{\Psi(x)}=0,\end{aligned}$$

i.e. $\ln\mathfrak{M}(r,A)\sim\ln\mu(r,A)$ as $r\uparrow R[A]$. Theorem 1 is proved. $\square$

An estimate of $\mathfrak{M}(r, A)$ by $\mu(r, A)$ can be obtained using a slightly different method. First, we prove the following statement.

Proposition 4. *Let γ be a positive continuous function on $[0, +\infty)$ increasing to $[+\infty)$ and*

$$A_\gamma(z) = \sum_{n=1}^{\infty} a_n \gamma(\lambda_n) f(\lambda_n z).$$

If $\Gamma_f(cr) \asymp \Gamma_f(r)$ as $r \to +\infty$ for each $c \in (0, +\infty)$, $\ln \gamma(x) = o(\Gamma_f(x))$ as $x \to +\infty$ and $\overline{\lim\limits_{n\to\infty}} \ln n/\gamma(\lambda_n) < 1$, then

$$\mathfrak{M}(r, A) = O(\mu(r, A_\gamma) \quad as \quad r \to +\infty.$$

Proof. Since $\Gamma_f(cr) \asymp \Gamma_f(r)$ as $r \to +\infty$, we have

$$\underline{\lim_{n\to\infty}} \frac{1}{\lambda_n} M_f^{-1}\left(\frac{1}{|a_n|}\right) = R[A] > 0,$$

i.e. $M_f^{-1}\left(\frac{1}{|a_n|}\right) \geq R_2\lambda_n$ for every $R_2 \in (0, R[A])$ and all sufficiently large n. Therefore,

$$\frac{1}{|a_n|\gamma(\lambda_n)} \geq \frac{M_f(R_2\lambda_n)}{\gamma(\lambda_n)}.$$

Let $0 < R_1 < R_2$. Then

$$\ln M_f(R_2\lambda_n) - \ln M_f(R_1\lambda_n) = \int_{R_1\lambda_n}^{R_2\lambda_n} \Gamma_f(x) d\ln x \geq \Gamma_f(R_1\lambda_n) \ln(R_2/R_1)$$

and, thus,

$$\frac{1}{|a_n|\gamma(\lambda_n)} \geq M_f(R_1\lambda_n) \exp\{\Gamma_f(R_1\lambda_n) \ln(R_2/R_1) - \ln \gamma(\lambda_n)\}.$$

Since $\ln \gamma(x) = o(\Gamma_f(x))$ as $x \to +\infty$, we have $\ln \gamma(\lambda_n) = o(\Gamma_f(R_1\lambda_n))$ as $n \to \infty$ and $\Gamma_f(R_1\lambda_n) \ln(R_2/R_1) - \ln \gamma(\lambda_n) \geq h > 0$ for all n enough large. Therefore,

$$M_f^{-1}\left(\frac{1}{|a_n|\gamma(\lambda_n)}\right) \geq M_f^{-1}(M_f(R_1\lambda_n)e^h) \sim R_1, \quad n \to \infty,$$

because the function M_f^{-1} is slowly increasing, it follows that $R[A_\gamma] \geq R_1$. Since $\gamma(\lambda_n) \geq 1$, we have $R[A_\gamma] \leq R[A]$. Therefore, in view of the arbitrariness of R_1 we get $R[A_\gamma] = R[A]$.

We remark also that the condition $\overline{\lim\limits_{n\to\infty}} \ln n/\gamma(\lambda_n) < 1$ implies $\ln n \leq b\gamma(\lambda_n)$ for some $b < 1$ and all $n \geq n_0$. Therefore,

$$\begin{aligned}
\mathfrak{M}(r, A) &\leq \sum_{n=1}^{\infty} \frac{|a_n|\gamma(\lambda_n) M_f(r\lambda_n)}{\gamma(\lambda_n)} \\
&\leq \max\left\{|a_n|\gamma(\lambda_n) M_f(r\lambda_n) : n \geq 1\right\} \sum_{n=1}^{\infty} \frac{1}{\gamma(\lambda_n)} \\
&\leq \mu(r, A_\gamma)\left(\sum_{n=1}^{n_0} \frac{1}{\gamma(\lambda_n)} + \sum_{n=n_0}^{\infty} \exp\{-(\ln n)/b\}\right) \\
&= K\mu(r, A_\gamma), \qquad K = \text{const}.
\end{aligned}$$

□

Using Proposition 4, let us prove the following theorem.

Theorem 2. *Let the conditions of Proposition 4 hold. If* $\Gamma_f(r) \geq hr$ *for some* $h > 0$ *and all* $r > 0$*, condition* (4) *holds with* $\ln\, \mu(r, A_\gamma)$ *instead of* $\ln\, \mu(r, A)$ *and*

$$\ln\, \gamma(2x\Phi(2x)/h) = o(\Psi(x)), \quad x \to +\infty, \tag{10}$$

then

$$\ln \mathfrak{M}(r, A) \sim \ln\, \mu(r, A) \quad as \quad r \uparrow R[A].$$

Proof. Let $\nu(r, A) = \max\{n \geq 1 : |a_n| M_f(r\lambda_n) = \mu(r, A)\}$ is the central index of series (1). Then the function $\nu(r, A)$ is increasing [12] and

$$\ln\, \mu(r, A) - \ln\, \mu(0, A) = \int_0^r \frac{\Gamma_f(t\lambda_{\nu(t,A)})}{t}\, dt, \quad 0 \leq r_0 \leq r < +\infty.$$

Since $\Gamma_f(r) \geq hr$, it was obtained that

$$\ln\, \mu\left(\frac{R[A]+r}{2}, A_\gamma\right) \geq h \int_r^{(R[A]+r)/2} \lambda_{\nu(t,A_\gamma)}\, dt \geq h\lambda_{\nu(r,A_\gamma)} \frac{R[A]-r}{2},$$

i.e.

$$\lambda_{\nu(r,A_\gamma)} \leq \frac{2}{h(R[A]-r)} \ln\, \mu\left(\frac{R[A]+r}{2}, A_\gamma\right).$$

On the other hand, we have

$$\ln\, \mu(r, A_\gamma) = \ln\left(\gamma(\lambda_{\nu(r,A_\gamma)})|a_{\nu(r,A_\gamma)}| M_f(r\lambda_{\nu(r,A_\gamma)}\right) \leq \ln\, \mu(r, A) + \ln\, \gamma(\lambda_{\nu(r,A_\gamma)}),$$

i.e.

$$\begin{aligned} \ln\, \mu(r, A) &\geq \ln\, \mu(r, A_\gamma)\left(1 - \frac{\ln\, \gamma(\lambda_{\nu(r,A_\gamma)})}{\ln\, \mu(r, A_\gamma)}\right) \\ &\geq \ln\, \mu(r, A_\gamma)\left(1 - \frac{\ln\, \gamma\left(\frac{2}{h(R[A]-r)} \ln\, \mu\left(\frac{R[A]+r}{2}, A_\gamma\right)\right)}{\ln\, \mu(r, A_\gamma)}\right). \end{aligned} \tag{11}$$

In view of (4) with $\ln\, \mu(r, A_\gamma)$ instead of $\ln\, \mu(r, A)$ and (10) we obtain

$$\begin{aligned} \frac{\ln\, \gamma\left(\frac{2}{h(R[A]-r)} \ln\, \mu\left(\frac{R[A]+r}{2}, A_\gamma\right)\right)}{\ln\, \mu(r, A_\gamma)} &\leq \frac{\ln\, \gamma\left(\frac{2}{h(R[A]-r)} \Phi\left(\frac{2}{R[A]-r}\right)\right)}{\Psi\left(\frac{1}{R[A]-r}\right)} \\ &= \frac{\ln\, \gamma\left(\frac{2x\Phi(2x)}{h}\right)}{\Psi(x)} \to 0 \end{aligned}$$

as $x = \dfrac{1}{R[A]-r} \to +\infty$.

Therefore, $\ln\, \mu(r, A) \geq (1 + o(1)) \ln\, \mu(r, A_\gamma)$ as $r \uparrow R[A]$ and by Proposition 4 we get

$$\ln\, \mu(r, A) \leq \ln \mathfrak{M}(r, A) \leq \ln\, \mu(r, A_\gamma) + \ln K \leq (1 + o(1)) \ln\, \mu(r, A), \quad r \uparrow R[A].$$

Thus, $\ln \mathfrak{M}(r, A) \sim \ln\, \mu(r, A)$ as $r \uparrow R[A]$. Theorem 2 is proved. $\square$

Remark 1. *In both Theorem 1 and Theorem 2, the conditions under which* $\ln \mathfrak{M}(r, A) \sim \ln \mu(r, A)$ *as* $r \uparrow R[A]$ *appeared as a result of the methods used. Apparently they can be weakened. For example, in the proof of Theorem 1, to estimate the coefficients, use the inequality*

$$\ln |a_n| \le \min\{\Phi(1/(R[A]-r)) - \ln M_f(r\lambda_n) : r \in [0, R[A]\}.$$

This minimum is achieved at the point $r = r_n$*, which is a solution to the very difficult equation*

$$\Phi'(1/(R[A]-r))(R[A]-r)^{-2} = \Gamma_f(r\lambda_n)/r,$$

and it is not clear what to do next.

There is another option. Instead of (4) *we impose a condition*

$$\ln \Psi(r) \le \ln \mu(r, A) \le \ln \Phi(r),$$

where the function $\ln \Phi(r)$ *is convex on* $[0, R[A])$.

Then $\min\{\ln \Phi(r) - \ln M_f(r\lambda_n) : r \in [0, R[A]\}$ *is achieved at the point* $r = r_n$*, which is a solution to the equation* $\Gamma_\Phi(r) = \Gamma_f(r\lambda_n)$. *Considering the function* $\omega(r) = \Gamma_f^{-1}(\Gamma_\Phi(r))/r$ *to be increasing, we obtain* $r_n = \omega^{-1}(\lambda_n)$ *and, thus,*

$$\ln |a_n| \le \ln \Phi(\omega^{-1}(\lambda_n)) - \lambda_n \omega^{-1}(\lambda_n).$$

Perhaps, we will get the desired result in this direction.

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Шеремета М.М., Добушовський М.С. *Про максимальний член рядів за системами фукнцій в крузі* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 11–18.

Для цілої трансцендентної функції f і зростаючої до $+\infty$ послідовності (λ_n) додатних чисел ряд $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$ за системою $f(\lambda_n z)$ називають регулярно збіжним у множині $\{z : |z| < R[A]\}$, якщо $\mathfrak{M}(r, A) = \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty$ для всіх $r \in [0, R[A])$, де $R[A]$ — це радіус регулярної збіжності ряду $A(z)$ і $M_f(r) = \max\{|f(z)| : |z| = r\}$.

Знайдено умови на послідовність (λ_n) та функцію f, за яких $\ln \mathfrak{M}(r, A) \sim \ln \mu(r, A)$ при $r \to R[A]$, де $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) : n \geq 1\}$ — максимальний член ряду.

Наприкінці статті сформульовано нерозв'язану проблему.

Ключові слова і фрази: ряд за системою функцій, регулярно збіжний ряд, максимальний член.