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Some paranormed sequence spaces derived from generalized divisor sum function

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We explore the properties of the paranormed sequence spaces $c_0(p,\mathscr{D}^\alpha)$, $c(p,\mathscr{D}^\alpha)$, and $\ell_\infty(p,\mathscr{D}^\alpha)$, which are generated by an infinite matrix $\mathscr{D}^\alpha$ involving a generalized divisor sum function $\sigma^{(\alpha)}$ to classical Maddox spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$, respectively. The matrix $\mathscr{D}^\alpha = (d^\alpha_{m,r})$ is defined such that $d^\alpha_{m,r} = \dfrac{r^\alpha}{\sigma^{(\alpha)}(m)}$ if r is a divisor of m, and 0, otherwise. Our analysis includes the determination of the Schauder basis and the computation of dual spaces (α-, β-, and γ-duals) for these newly defined paranormed spaces. Additionally, we characterize matrix transformations from $\ell_\infty(p,\mathscr{D}^\alpha)$ into several known sequence spaces, and present related matrix characterizations as direct consequences.

Key words and phrases: sequence space, generalized divisor sum function, Schauder basis, α-dual, β-dual, γ-dual, matrix transformation.

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1 Introduction and preliminaries

Throughout the paper $\mathbb{R}$ and $\mathbb{Z}^+$ denote the set of all real numbers and the set of all positive integers, respectively. We define ω as the vector space of all real-valued sequences, represented by $s = (s_m)$, where $s_m \in \mathbb{R}$ for all $m \in \mathbb{Z}^+$. The operations of component-wise addition and scalar multiplication, defined by

$$\begin{aligned}(s_m) + (t_m) &= (s_m + t_m),\\ a(s_m) &= (as_m), \quad a \in \mathbb{R},\end{aligned}$$

respectively, endow ω with a linear structure over $\mathbb{R}$. The subspaces of ω are called sequence spaces. Familiar examples include ℓ_p (the space of absolutely p-summable sequences for $0 < p < \infty$), ℓ_∞ (the space of bounded sequences), c (the space of convergent sequences), and c_0 (the space of null sequences).

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Definition 1.1. *Let Ξ be a subspace of ω, and $g : \Xi \to \mathbb{C}$ be a function. The pair (Ξ, g) defines a paranormed sequence space if, for all $s, t \in \Xi$ and $a \in \mathbb{C}$, the following properties hold:*

(c1) $g(s) \geq 0$, and $g(s) = 0$ if and only if s is the zero element of Ξ;

(c2) $g(-s) = g(s)$;

(c3) $g(s+t) \leq g(s) + g(t)$;

(c4) if $(a_m) \in \omega$ and $(s_m) \in \Xi$ are such that $a_m \to a$ and $g(s_m - s) \to 0$ as $m \to \infty$, then $g(a_m s_m - as) \to 0$ as $m \to \infty$.

The function g is then known as a paranorm on Ξ.

Let us consider a bounded sequence $p = (p_m)$ of positive real numbers satisfying $0 < p_m \leq \sup p_m = P < 1$, with $M = \max\{1, P\}$. Let q_m be the conjugate exponent of p_m, defined by $p_m^{-1} + (q_m)^{-1} = 1$, assuming $1 < p_m$ for all $m \in \mathbb{Z}^+$. Examples of classical paranormed sequence spaces include $c_0(p)$, $c(q)$, and $\ell_\infty(p)$, often referred to as Maddox spaces.

The concept of paranormed sequence spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ was introduced by I.J. Maddox [24] and independently explored by S. Simons [33]. These spaces are given by:

$$\begin{aligned} c_0(p) &= \left\{ s = (s_m) \in \omega : \lim_{m\to\infty} |s_m|^{p_m} = 0 \right\}, \\ c(p) &= \left\{ s = (s_m) \in \omega : \lim_{m\to\infty} |s_m - l|^{p_m} = 0 \ \text{ for some } \ l \in \mathbb{R} \right\}, \\ \ell_\infty(p) &= \left\{ s = (s_m) \in \omega : \sup_{m\in\mathbb{Z}^+} |s_m|^{p_m} < \infty \right\}. \end{aligned}$$

Under the operations of coordinate-wise addition and scalar multiplication, $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ form linear spaces.

Define the function g_1 on the spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ by

$$g_1(s) = \sup_{m\in\mathbb{Z}^+} |s_m|^{p_m/M}.$$

I.J. Maddox [25] demonstrated that if $p \in \ell_\infty$, then $c_0(p)$ and $c(p)$ are complete paranormed spaces with g_1 as their paranorm. Additionally, the inclusion $c_0(p) \subseteq c_0(q)$ is valid if and only if $\liminf q_m/p_m > 0$, as shown in [24]. Furthermore, $\ell_\infty(p)$ becomes a complete paranormed space with respect to g_1 if and only if $\inf p_m > 0$.

For a sequence $s = (s_m) \in \Xi$ and an infinite matrix $\mathfrak{U} = (u_{m,r})$, the sequence

$$t = \mathfrak{U}s = \left(\sum_{r=0}^{\infty} u_{m,r} s_r \right)_{m\in\mathbb{Z}^+}$$

is defined as the $\mathfrak{U}$-transform of s. The domain of the matrix $\mathfrak{U}$ in the sequence space Ξ, denoted $\Xi_{\mathfrak{U}}$, is the set of all sequences s such that their $\mathfrak{U}$-transforms belong to Ξ. Equivalently, $\Xi_{\mathfrak{U}} = \{s \in \omega : \mathfrak{U}s \in \Xi\}$.

Furthermore, given two sequence spaces Ξ and Π, the notation (Ξ, Π) represents the set of matrices $\mathfrak{U} = (u_{m,r})$ that map sequences from Ξ into Π, i. e. $\mathfrak{U}s \in \Pi$ for all $s \in \Xi$.

Paranormed sequence spaces generated by matrix transformations in $\ell(p)$, $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ have been widely studied. Z.U. Ahmad and M. Mursaleen [1], for example, employed the forward difference matrix Δ to create the sequence spaces $\Xi(\Delta, p) = \{s \in \omega : \Delta s \in \Xi(p)\}$, where Ξ is one of the space ℓ_∞, c, or c_0.

B. Altay and F. Başşar [2, 3] contributed Riesz sequence spaces $r^n(p)$, $r_0^n(p)$, $r_c^n(p)$, and $r_\infty^n(p)$, which are the matrix domains of R^n in $\ell(p)$, $c_0(p)$, $c(p)$, and $\ell_\infty(p)$, respectively. Equivalently,

$$r^n(p) = (\ell(p))_{R^n}, \quad r_0^n(p) = (c_0(p))_{R^n}, \quad r_c^n(p) = (c(p))_{R^n}, \quad \text{and} \quad r_\infty^n(p) = (\ell_\infty(p))_{R^n}.$$

E.E. Kara et al. [21] furthered this line of research by introducing paranormed Euler sequence space $e_p^n = (\ell(p))_{E^n}$. Moreover, B. Altay and F. Başşar [4, 5] investigated the paranormed sequence spaces $\ell(u, v; p)$, $c_0(u, v; p)$, $c(u, v; p)$, and $\ell_\infty(u, v; p)$, which are the domains of the weighted mean matrix $W(u, v)$ in $\ell(p)$, $c_0(p)$, $c(p)$, and $\ell_\infty(p)$. M. Yeşilkayagil and F. Başar [37, 38] conducted studies on the paranormed sequence spaces

$$\Xi(N^n, p) = (\Xi(p))_{N^n}, \qquad \Xi \in \{\ell_\infty, c, c_0, \ell_p\},$$

providing detailed analyses of their properties (see also [34]). C. Aydın and F. Başar [6, 7] extended this research by introducing and investigating the spaces $c(A^n, p) = (c(p))_{A^n}$, $c_0(A^n, p) = (c_0(p))_{A^n}$, and $\ell(A^n, p) = (\ell(p))_{A^n}$. Furthermore, E.E. Kara and S. Demiriz [20] examined paranormed sequence spaces $\ell_\infty(\hat{F}, p)$, $c(\hat{F}, p)$, $c_0(\hat{F}, p)$, and $\ell(\hat{F}, p)$ using the Fibonacci difference matrix $\hat{F}$. More recently, S. Erdem et al. [13] introduced and studied properties of paranormed Motzkin sequence spaces $c(\mathfrak{M}, p)$ and $c_0(\mathfrak{M}, p)$ derived from the Motzkin matrix $\mathfrak{M}$. Additionally, M.C. Dağli and T. Yaying [11] explored paranormed sequence spaces $\ell_\infty(\mathfrak{T}, p)$, $c(\mathfrak{T}, p)$, $c_0(\mathfrak{T}, p)$, and $\ell(\mathfrak{T}, p)$ using a regular tribonacci matrix $\mathfrak{T}$. Moreover, M.C. Dağli [10] studied paranormed sequence spaces derived from the Catalan-Motzkin matrix, and studied their algebraical and topological properties. We also recommend the papers [26, 27, 30–32] for deeper understanding about mixed paranormed spaces and other important spaces with intriguing natures. These studies have collectively advanced our understanding of paranormed sequence spaces.

1.1 The spaces f and f_0

This subsection provides a concise overview of *almost convergence*, a generalization of the standard concept of convergence. S. Banach [8] demonstrated the existence of a linear functional φ on the space ℓ_∞ of bounded sequences, satisfying the following properties for all $s, t \in \ell_\infty$ and scalars a, b:

(i) $\varphi(as + bt) = a\varphi(s) + b\varphi(t)$;

(ii) $s_m \geq 0$ implies $\varphi((s_m)_{m=0}^\infty) \geq 0$ for all $m \in \mathbb{Z}^+$;

(iii) $\varphi((s_{m+k})_{m=0}^\infty) = \varphi((s_m)_{m=0}^\infty)$ for all $k \in \mathbb{Z}^+$;

(iv) $\varphi(e) = 1$, where $e = (1, 1, \ldots, 1, \ldots)$.

G.G. Lorentz [23] defined a *Banach limit* as any functional on ℓ_∞ satisfying the properties outlined in conditions (i)–(iv). A sequence $s = (s_m) \in \ell_\infty$ is said to be almost convergent to a generalized limit l, denoted $f\text{-}\lim s_m = l$, if all Banach limits of s are equal to l. The *shift operator* O on ω is defined by $O_m(s) = s_{m+1}$ for every $m \in \mathbb{Z}^+$, and O^r represents the r-fold composition of O. For a sequence $s = (s_m)$, let

$$u_{km}(s) = \frac{1}{k+1} \sum_{r=0}^{k} O_m^r(s), \quad k, m \in \mathbb{Z}^+.$$

G.G. Lorentz [23] showed that $f\text{-}\lim s_m = l$ if and only if $\lim_{k\to\infty} u_{km}(s) = l$ uniformly in m. Based on this, we define the almost null convergent space f_0 and the almost convergent space f as:

$$f_0 = \left\{ s = (s_m) \in \ell_\infty : \lim_{r\to\infty} \frac{1}{r+1} \sum_{m=0}^{r} s_{k+m} = 0 \text{ uniformly in } k \right\},$$
$$f = \left\{ s = (s_m) \in \ell_\infty : \exists a \in \mathbb{C} \text{ such that } \lim_{r\to\infty} \frac{1}{r+1} \sum_{m=0}^{r} s_{k+m} = a \text{ uniformly in } k \right\}.$$

Building upon the work of M. Mursaleen [28], we can summarize the following properties of the almost convergent space f.

(i) Ordinary convergence implies almost convergence to the same limit, but the converse does not always hold.

(ii) The space f is a closed subspace of the non-separable space $(\ell_\infty, \|\cdot\|_\infty)$.

(iii) f is a BK-space under the norm $\|\cdot\|_\infty$.

(iv) f is nowhere dense in ℓ_∞.

(v) Almost convergence is not equivalent to any matrix method.

(vi) From the inclusions $c \subset f \subset \ell_\infty$, we deduce $\ell_1 = \ell_\infty^\lambda \subset f^\lambda \subset c^\lambda = \ell_1$, which implies $f^\lambda = \ell_1$ for $\lambda \in \{\alpha, \beta, \gamma\}$.

1.2 Motivation of the study

The Euler totient matrix $\Phi = (\phi_{m,r})$, defined by

$$\phi_{m,r} = \begin{cases} \dfrac{\varphi(r)}{m}, & \text{if } r \mid m, \\ 0, & \text{if } r \nmid m, \end{cases}$$

for $m, r \in \mathbb{Z}^+$, was introduced by M. İlkhan and E.E. Kara [16]. They proceeded to study the sequence spaces $\ell_\infty(\Phi) = (\ell_\infty)_\Phi$ and $\ell_p(\Phi) = (\ell_p)_\Phi$ (see also [15]). S. Demiriz et al. [12] examined the space $\hat{c}^\Phi = f_\Phi$. M. İlkhan et al. [18] defined the matrix $\Upsilon^t = (\upsilon_{m,r}^t)$ as

$$\upsilon_{m,r}^t = \begin{cases} \dfrac{J_t(r)}{m^t}, & \text{if } r \mid m, \\ 0, & \text{if } r \nmid m, \end{cases}$$

for $m, r \in \mathbb{Z}^+$, and constructed the space $\ell_p(\Upsilon^t) = (\ell_p)_{\Upsilon^t}$. Furthermore, M. İlkhan et al. [17] explored the compactness properties of the sequence spaces $\ell_p(\Upsilon^t) = (\ell_p)_{\Upsilon^t}$, $c_0(\Upsilon^t) = (c_0)_{\Upsilon^t}$, and $\ell_\infty(\Upsilon^t) = (\ell_\infty)_{\Upsilon^t}$. V.A. Khan and U. Tuba [22] investigated paranormed sequence spaces $\ell(\Upsilon^t, q) = (\ell(q))_{\Upsilon^t}$ and $\ell_\infty(\Upsilon^t, q) = (\ell_\infty(q))_{\Upsilon^t}$, and introduced paranormed ideal sequence spaces $c_0^I(\Upsilon^t, q)$, $c^I(\Upsilon^t, q)$, and $\ell_\infty^I(\Upsilon^t, q)$.

T. Yaying and N. Saikia [35] recently introduced an infinite matrix $\mathscr{D} = (d_{m,r})$, defined using the arithmetic divisor sum function $\sigma(m)$ as

$$d_{m,r} = \begin{cases} \dfrac{r}{\sigma(m)}, & \text{if } r \mid m, \\ 0, & \text{if } r \nmid m. \end{cases}$$

Using this matrix $\mathscr{D}$, they defined the sequence spaces $\ell_\infty(\mathscr{D})$, $c(\mathscr{D})$, $c_0(\mathscr{D})$, and $\ell_p(\mathscr{D})$ as

$$\Xi(\mathscr{D}) = \{u \in \omega : \mathscr{D}s \in \Xi\}, \quad \Xi \in \{\ell_\infty, c, c_0, \ell_p\}, \ 0 < p < \infty.$$

Their investigation included the construction of Schauder bases and the determination of dual spaces for these newly defined spaces. Additionally, they explored matrix characterizations and compactness properties.

Extending the investigations from [35], T. Yaying et al. [36] defined new sequence spaces $\Xi(\mathscr{D}^\alpha) = \{s \in \omega : \mathscr{D}^\alpha s \in \Xi\}$, where $\Xi \in \{\ell_p, c_0, c, \ell_\infty\}$ and $0 < p < \infty$. The matrix $\mathscr{D}^\alpha = (d^\alpha_{m,r})$ is defined using a generalized arithmetic function $\sigma^{(\alpha)}(m)$ as

$$d^\alpha_{m,r} = \begin{cases} \dfrac{r^\alpha}{\sigma^{(\alpha)}(m)}, & \text{if } r \mid m, \\ 0, & \text{if } r \mid m. \end{cases}$$

They demonstrated the regularity of $\mathscr{D}^\alpha$. It is observed that $\Xi(\mathscr{D}^\alpha)$ reduces to $\Xi(\mathscr{D})$ when $\alpha = 1$. The study included the determination of Schauder bases and dual spaces, besides characterizing matrix classes related to the spaces $\ell_p(\mathscr{D}^\alpha)$, $c_0(\mathscr{D}^\alpha)$, $c(\mathscr{D}^\alpha)$, and $\ell_\infty(\mathscr{D}^\alpha)$. Finally, compactness of certain matrix operator was investigated by using Hausdorff measure of non-compactness on these spaces.

This current study extends the concept of normed arithmetic sequence spaces $\Xi(\mathscr{D}^\alpha)$ to the more general paranormed arithmetic sequence spaces $\Xi(p, \mathscr{D}^\alpha)$, where $p = (p_m)$ is a bounded sequence of positive real numbers and $\Xi \in \{\ell_\infty, c, c_0\}$. We aim to define the paranormed space $\Xi(p, \mathscr{D}^\alpha)$ as the matrix domain of $\mathscr{D}^\alpha$ within the classical paranormed space $\Xi(p)$. Our goals include determining Schauder bases and dual spaces for the newly defined space. Additionally, we will prove characterization theorems for the matrix class $(\ell_\infty(p, \mathscr{D}^\alpha), \Pi)$, with $\Pi \in \{\ell_\infty, f, f_0, c, c_0\}$, and derive related matrix transformation characterizations as corollaries.

2 Paranormed arithmetic sequence space $\Xi(p, \mathscr{D}^\alpha)$, $\Xi \in \{\ell_\infty, c, c_0\}$

The paranormed sequence spaces $\ell_\infty(p, \mathscr{D}^\alpha)$, $c(p, \mathscr{D}^\alpha)$, and $c_0(p, \mathscr{D}^\alpha)$ are defined as follows:

$$\ell_\infty(p, \mathscr{D}^\alpha) := \left\{ s = (s_m) \in \omega : \sup_{m \in \mathbb{Z}^+} \left| \sum_{r \mid m} \frac{r^\alpha}{\sigma^{(\alpha)}(m)} s_r \right|^{p_m} < \infty \right\},$$

$$c(p, \mathscr{D}^\alpha) := \left\{ s = (s_m) \in \omega : \lim_{m \to \infty} \left| \sum_{r \mid m} \frac{r^\alpha}{\sigma^{(\alpha)}(m)} s_r \right|^{p_m} = l \text{ for some } l \in \mathbb{R} \right\},$$

$$c_0(p, \mathscr{D}^\alpha) := \left\{ s = (s_m) \in \omega : \lim_{m \to \infty} \left| \sum_{r \mid m} \frac{r^\alpha}{\sigma^{(\alpha)}(m)} s_r \right|^{p_m} = 0 \right\}.$$

We define the $\mathscr{D}^\alpha$-transform of a sequence $s=(s_m)\in\omega$ as $t=(t_m)$, where

$$t_m=(\mathscr{D}^\alpha s)_m=\sum_{r|m}\frac{r^\alpha}{\sigma^{(\alpha)}(m)}s_r \tag{1}$$

for every $m\in\mathbb{Z}^+$. This transform allows us to construct new sequence spaces:

$$\begin{aligned}\ell_\infty(p,\mathscr{D}^\alpha)&:=\{s=(s_m)\in\omega: t=\mathscr{D}^\alpha s\in\ell_\infty(p)\},\\ c(p,\mathscr{D}^\alpha)&:=\{s=(s_m)\in\omega: t=\mathscr{D}^\alpha s\in c(p)\},\\ c_0(p,\mathscr{D}^\alpha)&:=\{s=(s_m)\in\omega: t=\mathscr{D}^\alpha s\in c_0(p)\}.\end{aligned}$$

These spaces are equivalently expressed as $(\ell_\infty(p))_{\mathscr{D}^\alpha}$, $(c(p))_{\mathscr{D}^\alpha}$, and $(c_0(p))_{\mathscr{D}^\alpha}$, respectively. In the case where p is the constant sequence e, these spaces reduce to $\ell_\infty(\mathscr{D}^\alpha)$, $c(\mathscr{D}^\alpha)$, and $c_0(\mathscr{D}^\alpha)$, previously studied by T. Yaying et al. [36].

Based on the definition of the $\mathscr{D}^\alpha$-transform in (1), its inverse, $\mathscr{D}^{-\alpha}$-transform, is given by

$$s_m=(\mathscr{D}^{-\alpha}t)_m=\sum_{r|m}\frac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha}t_r \tag{2}$$

for every $m\in\mathbb{Z}^+$. We record here that the sequences s and t in this paper are always connected by these transformations, defined as in equations (1) and (2).

Theorem 2.1. *For any $\Xi\in\{c_0,c,\ell_\infty\}$, $\Xi(p,\mathscr{D}^\alpha)$ is a linear metric space with the paranorm g defined as*

$$g(s)=\sup_{m\in\mathbb{Z}^+}\left|\sum_{r|m}\frac{r^\alpha}{\sigma^{(\alpha)}(m)}s_r\right|^{p_m/P}.$$

Proof. To avoid redundancy, the proof is given only for the space $c_0(p,\mathscr{D}^\alpha)$. Given that the transformation matrix $\mathscr{D}^\alpha$ is a triangle, it follows that $c_0(p,\mathscr{D}^\alpha)$ is a linear space under component-wise addition and scalar multiplication. Furthermore, the following properties are satisfied for all $s\in c_0(p,\mathscr{D}^\alpha)$:

(a) $g(\theta)=0$;

(b) $g(s)\geq 0$;

(c) $g(-s)=g(s)$.

Consider a sequence (s^i) in $c_0(p,\mathscr{D}^\alpha)$ such that $g(s^i-s)\to 0$ as $i\to\infty$. Additionally, let (a_i) be a sequence of scalars such that $a_i\to a$ as $i\to\infty$. Due to the sub-additivity of g, we have $g(s^i)\leq g(s)+g(s^i-s)$. This implies that the sequence $\{g(s^i)\}$ is bounded. We then have

$$g(a_is^i-as)=\sup_{m\in\mathbb{Z}^+}\left|\frac{1}{\sigma^{(\alpha)}(m)}\sum_{r|m}r^\alpha(a_is_r^i-as_r)\right|^{p_m/P}\leq|a_i-a|g(s^i)+|a|g(s^i-s).$$

Taking the limit as $i\to\infty$, we obtain that

$$0\leq\lim_{i\to\infty}g(a_is^i-as)\leq\lim_{i\to\infty}\{|a_i-a|g(s^i)+|a|g(s^i-s)\}=0.$$

This establishes the continuity of scalar multiplication, thus ensuring that g defines a paranorm on $c_0(p,\mathscr{D}^\alpha)$.

Now, we proceed to establish the completeness of the space $c_0(p,\mathscr{D}^\alpha)$. Let (s^i) be a Cauchy sequence in $c_0(p,\mathscr{D}^\alpha)$, where $s^i=\{s_1^{(i)},s_2^{(i)},s_3^{(i)},\ldots\}$. Consequently, for any $\epsilon>0$, there exists $n(\epsilon)>0$ such that $g(s^i-s^j)<\epsilon$ for all $i,j\geq n(\epsilon)$. For a fixed $m\in\mathbb{Z}^+$, the definition of g yields that

$$\left|(\mathscr{D}^\alpha s^i)_m-(\mathscr{D}^\alpha s^j)_m\right|\leq\sup_{m\in\mathbb{Z}^+}\left|(\mathscr{D}^\alpha s^i)_m-(\mathscr{D}^\alpha s^j)_m\right|^{p_m/P}<\epsilon \tag{3}$$

for all $i,j\geq n(\epsilon)$. This shows that $\{(\mathscr{D}^\alpha s^i)_m\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed m. By the completeness of $\mathbb{R}$, the sequence $\{(\mathscr{D}^\alpha s^i)_m\}$ converges, say to some $(\mathscr{D}^\alpha s)_m$ as $i\to\infty$.

By letting $j\to\infty$ in (3), we obtain that

$$\sup_{m\in\mathbb{Z}^+}\left|(\mathscr{D}^\alpha s^i)_m-(\mathscr{D}^\alpha s)_m\right|^{p_m/P}<\epsilon \tag{4}$$

for each $m\in\mathbb{Z}^+$. Given that $|(\mathscr{D}^\alpha s^i)_m|^{p_m/P}<\epsilon$ for all m, it follows from (4) that

$$\left|(\mathscr{D}^\alpha s)_m\right|^{p_m/P}\leq\left|(\mathscr{D}^\alpha s)_m-(\mathscr{D}^\alpha s^i)_m\right|^{p_m/P}+\left|(\mathscr{D}^\alpha s^i)_m\right|^{p_m/P}<2\epsilon.$$

This proves that the sequence $\{(\mathscr{D}^\alpha s)_m\}\in c_0(p,\mathscr{D}^\alpha)$. Since (s^i) is an arbitrary Cauchy sequence, we conclude that $c_0(p,\mathscr{D}^\alpha)$ is complete, as desired.

The proof for other spaces under consideration can be given in a similar fashion. We recommend the papers [2–4, 6, 37, 38] to get a deeper understanding of similar results. □

Remark 2.2. *In the case $p=e$, the unit sequence, the space $\Xi(p,\mathscr{D}^\alpha)$ for $\Xi\in\{c_0,c,\ell_\infty\}$ is a BK-space equipped with the norm*

$$\|s\|=\sup_{m\in\mathbb{Z}^+}\left|\sum_{r|m}\frac{r^\alpha}{\sigma^{(\alpha)}(m)}s_r\right|$$

as discussed in the paper [36].

Theorem 2.3. *The spaces $c_0(p,\mathscr{D}^\alpha)$, $c(p,\mathscr{D}^\alpha)$ and $\ell_\infty(p,\mathscr{D}^\alpha)$ are linearly isomorphic to $c_0(p)$, $c(p)$ and $\ell_\infty(p)$, respectively.*

Proof. It can be observed that the transformation

$$\begin{array}{rccc} h: & \Xi(p,\mathscr{D}^\alpha) & \longrightarrow & \Xi(p) \\ & s & \longmapsto & t=hs=\mathscr{D}^\alpha s \end{array}$$

is a linear bijection that preserves the paranorm, where Ξ represents any of the spaces c_0, c, or ℓ_∞. This observation directly leads to the stated result. □

Remark 2.4. *In the case $p=e$, the spaces $c_0(\mathscr{D}^\alpha)$, $c(\mathscr{D}^\alpha)$ and $\ell_\infty(\mathscr{D}^\alpha)$ are linearly isomorphic to c_0, c and ℓ_∞, respectively, as discussed in the paper* [36].

We now aim to determine the Schauder basis for the spaces $c_0(p,\mathscr{D}^\alpha)$ and $c(p,\mathscr{D}^\alpha)$. First, we recall the definition of a Schauder basis in a paranormed space (Ξ,g).

A sequence $(\mathcal{B}_r)$ in the paranormed space Ξ, with paranorm g, is a Schauder basis for Ξ if, for every $x\in\Xi$, there exists a scalar sequence (a_r) such that

$$\lim_{m\to\infty}g\left(x-\sum_{r=1}^m a_r\mathcal{B}_r\right)=0.$$

It is known that the domain $\Xi_{\mathfrak{U}}$ of a triangular matrix $\mathfrak{U}$ in Ξ has a Schauder basis if and only if Ξ has a Schauder basis (see [19, Theorem 2.3]). Building upon this, we proceed to construct the Schauder basis for $c_0(p,\mathscr{D}^\alpha)$ and $c(p,\mathscr{D}^\alpha)$.

Theorem 2.5. *Let $t_r = (\mathscr{D}^\alpha s)_r$ and $0 < p_r \le P < \infty$ for all $r \in \mathbb{Z}^+$. Define the sequence $\mathcal{B}^r(\alpha) = \{\mathcal{B}_m^{(r)}(\alpha)\}$ for each fixed $r \in \mathbb{Z}^+$ by*

$$\mathcal{B}_m^{(r)}(\alpha) = \begin{cases} \dfrac{\mu\left(\frac{m}{r}\right)\sigma^{(\alpha)}(r)}{m^\alpha}, & r \mid m, \\ 0, & r \nmid m. \end{cases}$$

Then, the following statements hold:

(a) *the sequence $\{\mathcal{B}^r(\alpha)\}_{r\in\mathbb{Z}^+}$ forms a Schauder basis for $c_0(p, \mathscr{D}^\alpha)$, and each $z \in c_0(p, \mathscr{D}^\alpha)$ has a unique representation*
$$z = \sum_{r=1}^{\infty} t_r \mathcal{B}^r(\alpha);$$

(b) *the set $\{e, \mathcal{B}^r(\alpha)\}$ is a Schauder basis for $c(p, \mathscr{D}^\alpha)$, and every $z \in c(p, \mathscr{D}^\alpha)$ can be uniquely written as*
$$z = le + \sum_{r=1}^{\infty} (t_r - l)\mathcal{B}^r(\alpha),$$
where $e = (1, 1, 1, \ldots)$ and $l = \lim_{r\to\infty}(\mathscr{D}^\alpha s)_r$;

(c) *the space $\ell_\infty(p, \mathscr{D}^\alpha)$ does not possess a basis.*

3 α-, β-, and γ-duals of the spaces $c_0(p, \mathscr{D}^\alpha)$, $c(p, \mathscr{D}^\alpha)$ and $\ell_\infty(p, \mathscr{D}^\alpha)$

The α-, β-, and γ-duals of a sequence space Ξ, denoted as Ξ^α, Ξ^β, and Ξ^γ, respectively, are defined as

$$\begin{aligned} \Xi^\alpha &= \{a = (a_r) \in \omega : as = (a_r s_r) \in \ell_1 \text{ for all } s = (s_r) \in \Xi\}, \\ \Xi^\beta &= \{a = (a_r) \in \omega : as = (a_r s_r) \in cs \text{ for all } s = (s_r) \in \Xi\}, \\ \Xi^\gamma &= \{a = (a_r) \in \omega : as = (a_r s_r) \in bs \text{ for all } s = (s_r) \in \Xi\}. \end{aligned}$$

Let $\mathfrak{Z}$ represent the set of all finite subsets of $\mathbb{Z}^+$ and $L, M \in \mathbb{Z}^+$. To aid in our analysis, we utilize the following lemmas.

Lemma 3.1 ([14]). *The following statements hold true for any infinite matrix $\mathfrak{U} = (u_{m,r})$:*

(a) *$\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p), \ell_\infty(q))$ if and only if*
$$\sup_{m\in\mathbb{Z}^+} \left(\sum_{r=1}^{\infty} |u_{m,r}| \, M^{1/p_r} \right)^{q_m} < \infty \quad \text{for all } M > 1; \tag{5}$$

(b) *$\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p), c(q))$ if and only if*
$$\sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} |u_{m,r}| \, M^{1/p_r} < \infty \quad \text{for all } M > 1, \tag{6}$$
$$\forall M \;\; \exists (a_r) \in \mathbb{R} \;\; \lim_{m\to\infty} \left(\sum_{r=1}^{\infty} |u_{m,r} - a_r| \, M^{1/p_r} \right)^{q_m} < \infty; \tag{7}$$

(c) $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p), c_0(q))$ *if and only if*

$$\lim_{m\to\infty} \left(\sum_{r=1}^{\infty} |u_{m,r}| M^{1/p_r} \right)^{q_m} = 0 \quad \text{for all } M > 1; \tag{8}$$

(d) $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p), \ell(q))$ *if and only if* $q_r \geq 1$ *for all* r *and*

$$\sup_{R\in\mathfrak{Z}} \sum_{m=1}^{\infty} \left| \sum_{r\in R} u_{m,r} M^{1/p_r} \right|^{q_m} < \infty \quad \text{for all } M \geq 1. \tag{9}$$

Lemma 3.2 ([14]). *The following statements hold true for any infinite matrix* $\mathfrak{U} = (u_{m,r})$*:*

(a) $\mathfrak{U} = (u_{m,r}) \in (c(p), \ell_\infty(q))$ *if and only if*

$$\exists L > 1 \quad \sup_{m\in\mathbb{Z}^+} \left(\sum_{r=1}^{\infty} |u_{m,r}| L^{-1/p_r} \right)^{q_m} < \infty, \tag{10}$$

$$\sup_{m\in\mathbb{Z}^+} \left| \sum_{r=1}^{\infty} u_{m,r} \right|^{q_m} < \infty; \tag{11}$$

(b) $\mathfrak{U} = (u_{m,r}) \in (c(p), c(q))$ *if and only if*

$$\exists L > 1 \quad \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} |u_{m,r}| L^{-1/p_r} < \infty, \tag{12}$$

$$\forall M \;\; \exists L > 1 \;\; \text{and} \;\; a_r \in \mathbb{R} \quad \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} |u_{m,r} - a_r| M^{1/q_m} L^{-1/p_r} < \infty, \tag{13}$$

$$\exists (a_r) \in \omega \quad \lim_{m\to\infty} |u_{m,r} - a_r|^{q_m} = 0, \tag{14}$$

$$\exists a \in \mathbb{R} \quad \lim_{m\to\infty} \left| \sum_{r=1}^{\infty} u_{m,r} - a \right|^{q_m} = 0; \tag{15}$$

(c) $\mathfrak{U} = (u_{m,r}) \in (c(p), c_0(q))$ *if and only if*

$$\forall M \;\; \exists L > 1 \quad \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} |u_{m,r}| M^{1/q_m} L^{-1/p_r} < \infty, \tag{16}$$

$$\lim_{m\to\infty} |u_{m,r}|^{q_m} = 0 \quad \text{for all } r \in \mathbb{Z}^+, \tag{17}$$

$$\lim_{m\to\infty} \left| \sum_{r=1}^{\infty} u_{m,r} \right|^{q_m} = 0; \tag{18}$$

(d) $\mathfrak{U} = (u_{m,r}) \in (c(p), \ell(q))$ *if and only if*

$$\exists L > 1 \quad \sup_{R\in\mathfrak{Z}} \sum_{m=1}^{\infty} \left| \sum_{r\in R} u_{m,r} L^{-1/p_r} \right|^{q_m} < \infty \quad \text{for any } q_m \geq 1, \tag{19}$$

$$\sum_{m=1}^{\infty} \left| \sum_{r=1}^{\infty} u_{m,r} \right|^{q_m} < \infty \quad \text{for all } q_m \geq 1. \tag{20}$$

Lemma 3.3 ([14]). *The following statements hold true for any infinite matrix* $\mathfrak{U} = (u_{m,r})$*:*

(a) $\mathfrak{U} = (u_{m,r}) \in (c_0(p), \ell_\infty(q))$ *if and only if* (10) *holds;*

(b) $\mathfrak{U} = (u_{m,r}) \in (c_0(p), c(q))$ *if and only if* (12), (13), *and* (14) *hold;*

(c) $\mathfrak{U} = (u_{m,r}) \in (c_0(p), c_0(q))$ *if and only if* (16) *and* (17) *hold;*

(d) $\mathfrak{U} = (u_{m,r}) \in (c_0(p), \ell(q))$ *if and only if* (19) *holds.*

Theorem 3.4. *Define the sets* $A_r(\alpha)$ *for* $r = 1, 2, 3$ *by*

$$\begin{aligned}
A_1(\alpha) &:= \bigcap_{M>1} \left\{ a = (a_r) \in \omega : \sup_{R \in \mathfrak{Z}} \sum_{m=1}^{\infty} \left| \sum_{r \in R,\, r|m} \frac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha} a_m M^{1/p_r} \right| < \infty \right\}, \\
A_2(\alpha) &:= \bigcup_{L>1} \left\{ a = (a_r) \in \omega : \sup_{R \in \mathfrak{Z}} \sum_{m=1}^{\infty} \left| \sum_{r \in R,\, r|m} \frac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha} a_m L^{-1/p_r} \right| < \infty \right\}, \\
A_3(\alpha) &:= \left\{ a = (a_r) \in \omega : \sum_{m=1}^{\infty} \left| \sum_{r=1,\, r|m}^{\infty} \frac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha} a_m \right| < \infty \right\}.
\end{aligned}$$

Then each of the following statements holds true:

(i) $\{\ell_\infty(p, \mathscr{D}^\alpha)\}^\alpha = A_1(\alpha)$;

(ii) $\{c(p, \mathscr{D}^\alpha)\}^\alpha = A_2(\alpha) \cap A_3(\alpha)$;

(iii) $\{c_0(p, \mathscr{D}^\alpha)\}^\alpha = A_2(\alpha)$.

Proof. Based on the definition of the sequence $s = (s_r)$, presented as in (2), it can be shown that

$$a_m s_m = \sum_{r|m} \frac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha} t_r a_m = (\Lambda(\alpha)t)_m \tag{21}$$

for every $m \in \mathbb{Z}^+$. Here, $\Lambda(\alpha) = (\lambda^\alpha_{m,r})$ is an infinite matrix defined by

$$\lambda^\alpha_{m,r} = \begin{cases} \dfrac{\mu(\frac{m}{r})\sigma^{(\alpha)}(r)}{m^\alpha} a_m, & r \mid m, \\ 0, & r \nmid m, \end{cases}$$

for all $r, m \in \mathbb{Z}^+$.

It is observed from (21) that $as = (a_m s_m) \in \ell_1$ whenever $s \in \ell_\infty(p, \mathscr{D}^\alpha)$ if and only if $\Lambda(\alpha)t \in \ell_1$ whenever $t \in \ell_\infty(p)$. It follows that $a = (a_m) \in \{\ell_\infty(p, \mathscr{D}^\alpha)\}^\alpha$ if and only if $\Lambda(\alpha) \in (\ell_\infty(p), \ell_1)$. By applying Lemma 3.1 (d) with $q_m = 1$ for all $m \in \mathbb{Z}^+$, we find that $\{\ell_\infty(p, \mathscr{D}^\alpha)\}^\alpha = A_1(\alpha)$.

The proofs of parts (ii) and (iii) can be shown analogously to part (i) by employing Lemma 3.2 (d) and Lemma 3.3 (d), respectively, instead of Lemma 3.1 (d). Therefore, we do not provide detailed explanations. □

Theorem 3.5. *Define the sets $A_r(\alpha)$ for $r = 4$ to 9 by*

$$A_4(\alpha) := \bigcap_{M>1} \left\{ a = (a_m) \in \omega : \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k \right| M^{1/p_r} < \infty \right\},$$

$$A_5(\alpha) := \bigcap_{M>1} \left\{ a = (a_m) \in \omega : \exists (c_r) \in \mathbb{R} \ni \lim_{m\to\infty} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k - c_r \right| M^{1/p_r} < \infty \right\},$$

$$A_6(\alpha) := \bigcup_{L>1} \left\{ a = (a_m) \in \omega : \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k \right| L^{-1/p_r} < \infty \right\},$$

$$A_7(\alpha) := \bigcup_{L>1} \left\{ a = (a_m) \in \omega : \exists c_r \in \mathbb{R} \ni \sup_{m\in\mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k - c_r \right| L^{-1/p_r} < \infty \right\},$$

$$A_8(\alpha) := \left\{ a = (a_m) \in \omega : \exists (c_r) \in \mathbb{R} \ni \lim_{m\to\infty} \left| \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k - c_r \right| = 0 \right\},$$

$$A_9(\alpha) := \left\{ a = (a_m) \in \omega : \exists c \in \mathbb{R} \ni \lim_{m\to\infty} \left| \sum_{r=1}^{\infty} \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k - c \right| = 0 \right\}.$$

Then each of the following statements holds true:

(a) $\{\ell_\infty(p, \mathscr{D}^\alpha)\}^\beta = A_4(\alpha) \cap A_5(\alpha)$;

(b) $\{c(p, \mathscr{D}^\alpha)\}^\beta = A_6(\alpha) \cap A_7(\alpha) \cap A_8(\alpha) \cap A_9(\alpha)$;

(c) $\{c_0(p, \mathscr{D}^\alpha)\}^\beta = A_6(\alpha) \cap A_7(\alpha) \cap A_8(\alpha)$.

Proof. Consider a sequence $a = (a_m) \in \omega$. By using the relation (2), we can show that

$$\sum_{r=1}^{m} a_r s_r = \sum_{r=1}^{m} \sum_{k|r} \frac{\mu\left(\frac{r}{k}\right)\sigma^{(\alpha)}(k)}{r^\alpha} t_k a_r = \sum_{r=1}^{m} \sum_{\substack{k=r\\ r|k}}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k t_r = (\Omega(\alpha)t)_m$$

for all $m \in \mathbb{Z}^+$, where $\Omega(\alpha) = (\omega^\alpha_{m,r})$ is an infinite matrix given by

$$\omega^\alpha_{m,r} = \begin{cases} \displaystyle\sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} a_k, & 1 \le r \le m, \\ 0, & r > m, \end{cases}$$

for all $m, r \in \mathbb{Z}^+$.

This shows that $as \in cs$ for all $s \in \ell_\infty(p, \mathscr{D}^\alpha)$ if and only if $\Omega(\alpha)t \in c$ for all $t \in \ell_\infty(p)$. Therefore, $a = (a_m) \in \{\ell_\infty(p, \mathscr{D}^\alpha)\}^\beta$ if and only if $\Omega(\alpha) \in (\ell_\infty(p), c)$. By using Lemma 3.1 (b), we find that

$$[\ell_\infty(p, \mathscr{D}^\alpha)]^\beta = A_4(\alpha) \cap A_5(\alpha).$$

Similarly, the β-duals of $c(p, \mathscr{D}^{\alpha})$ and $c_0(p, \mathscr{D}^{\alpha})$ are derived by applying Lemma 3.2 (b) and Lemma 3.3 (b), respectively, with $q_m = 1$ for all $m \in \mathbb{Z}^+$ instead of Lemma 3.1 (b). This completes the proof. □

Theorem 3.6. *Define the set* $A_{10}(\alpha)$ *by*

$$A_{10}(\alpha) := \left\{ a = (a_m) \in \omega : \sup_{m \in \mathbb{Z}^+} \left| \sum_{r=1}^{\infty} \sum_{k=r,\, r|k}^{m} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^{\alpha}} a_k \right| < \infty \right\}.$$

Then, each of the following statements holds true:

(a) $\{\ell_\infty(p, \mathscr{D}^{\alpha})\}^{\gamma} = A_4(\alpha)$;

(b) $\{c(p, \mathscr{D}^{\alpha})\}^{\gamma} = A_6(\alpha) \cap A_1 0(\alpha)$;

(c) $\{c_0(p, \mathscr{D}^{\alpha})\}^{\gamma} = A_6(\alpha)$.

Proof. By following a similar approach as in Theorem 3.5, we find that $as \in bs$ for all $s \in \ell_\infty(p, \mathscr{D}^{\alpha})$ if and only if $\Omega(\alpha)t \in \ell_\infty$ for all $t \in \ell_\infty(p)$. Therefore, $a = (a_m) \in \{\ell_\infty(p, \mathscr{D}^{\alpha})\}^{\gamma}$ if and only if $\Omega(\alpha) \in (\ell_\infty(p), \ell_\infty)$. By applying Lemma 3.1 (a) with $q_m = 1$ for all $m \in \mathbb{Z}^+$, we conclude that

$$\{\ell_\infty(p, \mathscr{D}^{\alpha})\}^{\gamma} = A_4(\alpha).$$

In the same way, the γ-duals of $c(p, \mathscr{D}^{\alpha})$ and $c_0(p, \mathscr{D}^{\alpha})$ are obtained by using Lemma 3.2 (a) and Lemma 3.3 (a), respectively, with $q_m = 1$ for all $m \in \mathbb{Z}^+$ instead of Lemma 3.1 (a). This completes the proof. □

4 Matrix transformations on the space $\ell_\infty(p, \mathscr{D}^{\alpha})$

This section is dedicated to characterizing the class of infinite matrices $(\ell_\infty(p, \mathscr{D}^{\alpha}), \Xi)$, where $\Xi \in \{\ell_\infty, f, f_0, c, c_0\}$. Additionally, we provide characterizations of various other infinite matrix classes by applying a specific theorem.

We define the matrix $\tilde{\mathfrak{U}} = (\tilde{u}_{m,r})$ via an infinite matrix $\mathfrak{U} = (u_{m,r})$ over $\mathbb{R}$ as follows

$$\tilde{u}_{m,r} = \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^{\alpha}} u_{m,k} \quad \text{for all } \; m, r \in \mathbb{Z}^+. \tag{22}$$

Let us introduce the lemma that is crucial for the proofs of the main theorems presented in this section.

Lemma 4.1 ([29, Theorem 5]). *Let* $0 < p_r \le \sup_r p_r < \infty$ *for all* $r \in \mathbb{Z}^+$.

Then $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p), f)$ *if and only if* (5) *holds with* $q_m = 1$ *for all* $m \in \mathbb{Z}^+$, *and*

$$\exists c_r \in \mathbb{R} \;\; \forall r \in \mathbb{Z}^+ \;\; \lim_{l \to \infty} \frac{1}{l+1} \sum_{k=1}^{l} u_{m+k,r} = c_r \text{ uniformly in } m,$$

$$\exists M > 1 \;\; \lim_{l \to \infty} \sum_{r=1}^{\infty} \left| \frac{1}{l+1} \sum_{k=1}^{l} u_{m+k,r} - c_r \right| M^{1/p_r} = 0 \text{ uniformly in } m.$$

Theorem 4.2. *$\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), \ell_\infty)$ if and only if*

$$\mathfrak{U}_m \in \{\ell_\infty(p, \mathscr{D}^\alpha)\}^\beta \quad \textit{for each } m \in \mathbb{Z}^+, \tag{23}$$

$$\sup_{m \in \mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} \right| M^{1/p_r} < \infty \quad \textit{for all } M > 1. \tag{24}$$

Proof. We recall that $\ell_\infty(p, \mathscr{D}^\alpha)$ and ℓ_∞ are linearly paranorm isomorphic.

Let $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), \ell_\infty)$ and $s \in \ell_\infty(p, \mathscr{D}^\alpha)$. Consider the nth partial sum of the series $\sum_{r=1}^{\infty} u_{m,r} s_r$, i.e.

$$\sum_{r=1}^{n} u_{m,r} s_r = \sum_{r=1}^{n} u_{m,r} \left[\sum_{k|r} \frac{\mu\left(\frac{r}{k}\right) \sigma^{(\alpha)}(k)}{r^\alpha} t_k \right] = \sum_{r=1}^{n} \sum_{k=r,\, r|k}^{n} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} t_r, \quad \forall n, m \in \mathbb{Z}^+. \tag{25}$$

Since $\mathfrak{U}s$ exists and belongs to ℓ_∞, the necessity of condition (23) is immediate. Taking the limit as $n \to \infty$ in (25), we obtain that

$$\sum_{r=1}^{\infty} u_{m,r} s_r = \sum_{r=1}^{\infty} \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} t_r, \quad \forall m \in \mathbb{Z}^+. \tag{26}$$

This implies that $\mathfrak{U}s = \tilde{\mathfrak{U}}t$, and therefore $\tilde{\mathfrak{U}}t \in \ell_\infty$. This means that $\tilde{\mathfrak{U}} \in (\ell_\infty(p), \ell_\infty)$. Consequently, $\tilde{\mathfrak{U}}$ satisfies condition (5) with $q_m = 1$ for all $m \in \mathbb{Z}^+$. This shows the necessity of condition (24).

Conversely, suppose that the conditions (23) and (24) hold, and let $s = (s_r) \in \ell_\infty(p, \mathscr{D}^\alpha)$. The condition (23) ensures the existence of $\mathfrak{U}s$. By choosing $M > \max\{1, |t_r|^{p_r}\}$ and by using condition (24) along with equality (26), we obtain that

$$\begin{aligned} \|\mathfrak{U}s\|_{\ell_\infty} = \sup_{m \in \mathbb{Z}^+} \left| \sum_{r=1}^{\infty} u_{m,r} s_r \right| &\le \sup_{m \in \mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} t_r \right| \\ &\le \sup_{m \in \mathbb{Z}^+} \sum_{r=1}^{\infty} \left| \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right) \sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} \right| M^{1/p_r} < \infty. \end{aligned}$$

This implies that $\mathfrak{U}s \in \ell_\infty$. Therefore, $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), \ell_\infty)$, as required. □

We now examine a theorem that characterizes the class of matrices $(\ell_\infty(p, \mathscr{D}^\alpha), f)$.

Theorem 4.3. *$\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), f)$ if and only if each of the conditions* (23) *and* (24) *holds true, and*

$$\exists (\alpha_r) \in \mathbb{R} \quad \textit{such that} \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=r,\, r|j}^{\infty} \frac{\mu\left(\frac{j}{r}\right) \sigma^{(\alpha)}(r)}{j^\alpha} u_{m+k,r} = c_r \tag{27}$$

uniformly in m for all $r \in \mathbb{Z}^+$.

Proof. Let $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), f)$ and $s \in \ell_\infty(p, \mathscr{D}^\alpha)$. Given that $f \subset \ell_\infty$, the necessity of conditions (23) and (24) can be directly deduced from Theorem 4.2. Since $\mathfrak{U}s \in f$ by assumption, we choose $s = \mathcal{B}^r(\alpha)$ for each $r \in \mathbb{Z}^+$ and find that

$$\mathfrak{U}s = \mathfrak{U}\mathcal{B}^r(\alpha) = \left\{ \sum_{k=r,\, r|k}^{\infty} \frac{\mu(\frac{k}{r})\sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} \right\}_{m\in\mathbb{Z}^+} \in f,$$

where $\mathcal{B}^r(\alpha)$ is defined as in Theorem 2.5. This establishes the necessity of condition (27).

Conversely, suppose that the conditions (23), (24), and (27) hold, and take any $s = (s_r) \in \ell_\infty(p, \mathscr{D}^\alpha)$. The condition (23) ensures the existence of $\mathfrak{U}s$. This leads back to the relationship (26). As a result, the matrix $\tilde{\mathfrak{U}}$ fulfills conditions (6) and (7) with $q_m = 1$ for all $m \in \mathbb{Z}^+$. Therefore, $\tilde{\mathfrak{U}}t \in c \subset f$. This, along with (26), establishes that $\mathfrak{U}s \in f$. Hence, $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), f)$. □

Substituting the space f_0 for f in Theorem 4.3 leads to the following result.

Corollary 4.4. $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), f_0)$ *if and only if conditions* (23) *and* (24) *hold, and condition* (27) *holds with* $c_r = 0$ *for all* $r \in \mathbb{Z}^+$.

By replacing the notion of almost convergence with ordinary convegence, Theorem 4.3 is reduced to the following result.

Theorem 4.5. $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), c)$ *if and only if conditions* (23) *and* (24) *hold, and*

$$\exists c_r \in \mathbb{R} \quad \lim_{m\to\infty} \sum_{k=r,\, r|k}^{\infty} \frac{\mu\left(\frac{k}{r}\right)\sigma^{(\alpha)}(r)}{k^\alpha} u_{m,k} = c_r \tag{28}$$

for all $r \in \mathbb{Z}^+$.

Corollary 4.6. $\mathfrak{U} = (u_{m,r}) \in (\ell_\infty(p, \mathscr{D}^\alpha), c_0)$ *if and only if conditions* (23) *and* (24) *hold, and condition* (28) *holds with* $c_r = 0$ *for all* $r \in \mathbb{Z}^+$.

Lemma 4.7 ([9, Lemma 5.3]). *Consider any two sequence spaces* Ξ *and* Π, *an infinite matrix* $\mathfrak{A}$, *and a triangle* $\mathfrak{T}$. *It is established that* $\mathfrak{A} \in (\Xi, \Pi(\mathfrak{T}))$ *if and only if* $\mathfrak{T}\mathfrak{A} \in (\Xi, \Pi)$.

A direct consequence of Lemma 4.7 is the characterization of certain matrix classes from $\Xi \in \{\ell_1, c_0, c, \ell_\infty\}$ to $\ell_\infty(p, \mathscr{D}^\alpha)$.

Theorem 4.8. *Given a space* Ξ *and an infinite matrix* $\mathfrak{U} = (u_{m,r})$, *we define the matrix* $\mathfrak{V} = (v_{m,r})$ *by the relation*

$$v_{m,r} = \sum_{k|m} \frac{k^\alpha}{\sigma^{(\alpha)}(m)} \theta_{k,r}, \quad m, r = 1, 2, 3, \ldots.$$

Accordingly, $\mathfrak{U} \in (\Xi, \Pi(p, \mathscr{D}^\alpha))$ *if and only if* $\mathfrak{V} \in (\Xi, \Pi(p))$.

Proof. It is a straightforward consequence of Lemma 4.7. □

Now, we present the following corollary based on Theorem 4.8 along with Lemmas 3.1, 3.2 and 3.3.

Corollary 4.9. *Given an infinite matrix $\mathfrak{U} = (u_{m,r})$, we have the following results:*

(a) $\mathfrak{U} \in (\ell_\infty(p), \ell_\infty(q, \mathscr{D}^\alpha))$ *if and only if* (5) *holds with* $v_{m,r}$ *instead of* $u_{m,r}$,

(b) $\mathfrak{U} \in (c(p), \ell_\infty(q, \mathscr{D}^\alpha))$ *if and only if* (10) *and* (11) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$,

(c) $\mathfrak{U} \in (c_0(p), \ell_\infty(q, \mathscr{D}^\alpha))$ *if and only if* (10) *holds with* $v_{m,r}$ *instead of* $u_{m,r}$,

(d) $\mathfrak{U} \in (\ell_\infty(p), c(q, \mathscr{D}^\alpha))$ *if and only if* (6) *and* (7) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$,

(e) $\mathfrak{U} \in (c(p), c(q, \mathscr{D}^\alpha))$ *if and only if* (12), (13), (14) *and* (15) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$,

(f) $\mathfrak{U} \in (c_0(p), c(q, \mathscr{D}^\alpha))$ *if and only if* (12), (13), *and* (14) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$,

(g) $\mathfrak{U} \in (\ell_\infty(p), c_0(q, \mathscr{D}^\alpha))$ *if and only if* (8) *holds with* $v_{m,r}$ *instead of* $u_{m,r}$,

(h) $\mathfrak{U} \in (c(p), c_0(q, \mathscr{D}^\alpha))$ *if and only if* (16), (17) *and* (18) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$,

(i) $\mathfrak{U} \in (c_0(p), c_0(q, \mathscr{D}^\alpha))$ *if and only if* (16) *and* (17) *hold with* $v_{m,r}$ *instead of* $u_{m,r}$.

Corollary 4.10. *Consider the matrix $\mathfrak{S} = (s_{m,r})$ defined via an infinite matrix $\mathfrak{U} = (u_{m,r})$ with entries*

$$s_{m,r} = \sum_{k=1}^{m} u_{m,r}$$

for all $m, r \in \mathbb{Z}^+$. Consequently, the following outcomes can be drawn:

(i) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), bs)$ *if and only if the conditions* (23) *and* (24) *hold true with* $s_{m,r}$ *substituted for* $u_{m,r}$,

(ii) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), fs)$ *if and only if the conditions* (23), (24) *and* (27) *hold true, with* $s_{m,r}$ *substituted for* $u_{m,r}$, *where* fs *denotes the space of series whose sequence of partial sums lies in the space* f,

(iii) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), fs_0)$ *if and only if the conditions* (23) *and* (24) *and* (27) *with* $c_r = 0$ *for all* $r \in \mathbb{Z}^+$ *hold true and with* $s_{m,r}$ *substituted for* $u_{m,r}$, *where* fs_0 *refers to the space of series that almost converge to zero,*

(iv) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), cs)$ *if and only if the conditions* (23), (24) *and* (28) *hold true with* $s_{m,r}$ *substituted for* $u_{m,r}$,

(v) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), cs_0)$ *if and only if the conditions* (23) *and* (24) *and* (28) *hold true with* $c_r = 0$ *forall* $r \in \mathbb{Z}^+$ *and* $s_{m,r}$ *substituted for* $u_{m,r}$, *where* cs_0 *refers to the space of the series converging to zero,*

(vi) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), \ell_\infty(\mathscr{D}^\alpha))$ *if and only if the conditions* (23) *and* (24) *hold true with* $v_{m,r}$ *substituted for* $u_{m,r}$,

(vii) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), c(\mathscr{D}^\alpha))$ *if and only if the conditions* (23), (24) *and* (28) *hold true with* $v_{m,r}$ *substituted for* $u_{m,r}$,

(viii) $\mathfrak{U} \in (\ell_\infty(p, \mathscr{D}^\alpha), c_0(\mathscr{D}^\alpha))$ *if and only if the conditions* (23), (24) *and* (28) *with* $c_r = 0$ *for all* $r \in \mathbb{Z}^+$ *hold true and* $v_{m,r}$ *substituted for* $u_{m,r}$.

5 Conclusion

The present work focusses on the application of a generalized arithmetic operator $\mathscr{D}^{\alpha}$ in defining novel paranormed sequence spaces. More specifically, we developed paranormed sequence spaces $\Xi(p,\mathscr{D}^{\alpha})$, $\Xi \in \{\ell_\infty, c, c_0\}$, which are derived as the domain of the arithmetic operator $\mathscr{D}^{\alpha}$ in the paranormed space $\Xi(p)$. Our work involved determining Schauder bases and exploring the α-, β-, and γ-duals for these newly defined spaces. Furthermore, we characterized the matrix class $(\ell_\infty(p,\mathscr{D}^{\alpha}),\Pi)$, where Π is any of the spaces ℓ_∞, f, f_0, c, or c_0. By utilizing a well known result due F. Başar and B. Altay [9], we derived several insightful corollaries. To maintain conciseness, we have omitted the analysis of the matrix classes $(c(p,\mathscr{D}^{\alpha}),\Pi)$ and $(c_0(p,\mathscr{D}^{\alpha}),\Pi)$. Future research could address these matrix classes and examine the relationships of inclusion between the new sequence spaces.

Furthermore, it would be valuable to explore sequence spaces defined by the generalized arithmetic operator $\mathscr{D}^{\alpha}$ in the spaces f of almost convergent sequences and f_0 of almost null sequences. More specifically, we propose studying the following spaces:

$$f(\mathscr{D}^{\alpha}) = \{s=(s_k)\in\omega : \mathscr{D}^{\alpha}s \in f\},$$
$$f_0(\mathscr{D}^{\alpha}) = \{s=(s_k)\in\omega : \mathscr{D}^{\alpha}s \in f_0\}.$$

Future investigations could focus on determining Schauder bases, obtaining important duals, and characterizing matrix classes related to $f(\mathscr{D}^{\alpha})$ and $f_0(\mathscr{D}^{\alpha})$, in the line of the works of S. Demiriz et al. [12].

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Яїнг Т., Діксіт М.М. *Деякі паранормовані простори послідовностей, породжені узагальненою функцією суми дільників* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 289–306.

У роботі досліджуються властивості паранормованих просторів послідовностей $c_0(p, \mathscr{D}^\alpha)$, $c(p, \mathscr{D}^\alpha)$ та $\ell_\infty(p, \mathscr{D}^\alpha)$, які породжуються нескінченною матрицею $\mathscr{D}^\alpha$, що містить узагальнену функцію суми дільників $\sigma^{(\alpha)}$, і є аналогами класичних просторів Меддокса $c_0(p)$, $c(p)$ та $\ell_\infty(p)$ відповідно. Матриця $\mathscr{D}^\alpha = (d^\alpha_{m,r})$ визначається таким чином: $d^\alpha_{m,r} = \dfrac{r^\alpha}{\sigma^{(\alpha)}(m)}$, якщо r є дільником числа m, і 0 — в іншому випадку. Проведений аналіз включає визначення базису Шаудера та обчислення спряжених просторів (α-, β- та γ-спряжень) для цих нововизначених паранормованих просторів. Крім того, охарактеризовано матричні перетворення з простору $\ell_\infty(p, \mathscr{D}^\alpha)$ у кілька відомих просторів послідовностей, а також наведено відповідні матричні характеризації як безпосередні наслідки отриманих результатів.

Ключові слова і фрази: простір послідовностей, узагальнена функція суми дільників, базис Шаудера, α-спряження, β-спряження, γ-спряження, матричне перетворення.