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On embedding semigroups into trioids

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J.-L. Loday and M.O. Ronco introduced the concepts of a trialgebra and a trioid, and defined the constructions of a free trialgebra and a free monogenic trioid. Trialgebras are related to the operads associated with chain modules of simplices and Stasheff polytopes. A trioid is the basis of a trialgebra and it is defined as a set with three binary associative operations satisfying the same axioms as a trialgebra, so trialgebras are linear analogs of trioids. If the operations of a trioid coincide, it becomes a semigroup. In this paper, we study the natural relationships between arbitrary semigroups and trioids defined by these semigroups. We present new classes of trioids constructed from various semigroups and show that any semigroup can be embedded into a suitable non-trivial trioid as a subtrioid in which all operations coincide.

Key words and phrases: trioid, dimonoid, semigroup, monomorphism.

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1 Introduction

The notion of a trioid first appeared in the paper of J.-L. Loday and M.O. Ronco [11] at the study of some properties of ternary planar trees. Recall that an algebraic system $(T, \dashv, \vdash, \perp)$ with three arbitrary binary associative operations $\dashv$, $\vdash$, and $\perp$ is called *a trioid* if for all $x, y, z \in T$ the following conditions hold:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \qquad (T_1)$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \qquad (T_2)$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \qquad (T_3)$$
$$(x \dashv y) \dashv z = x \dashv (y \perp z), \qquad (T_4)$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \qquad (T_5)$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \qquad (T_6)$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \qquad (T_7)$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z). \qquad (T_8)$$

It is known that trioids are a basis of the notion of a trialgebra [11] that is a non-commutative version of Poisson algebras; in addition, trioids naturally generalize semigroups and

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dimonoids [10]. A nonempty set T equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying axioms (T_1)–(T_3) is called a *dimonoid*. Triods have also different relationships with such structures as doppelsemigroups [21,34], g-dimonoids [14,26] and (generalised) digroups [16,35,36], doppelalgebras and dialgebras [13,15], trigroups [3], Rota-Baxter algebras and dendriform algebras [8,19], Hopf algebras and tridendriform algebras [5–7], Poisson structures and Leibniz algebras [1,2,9], graphs and hypergraphs [4,12], and other structures (see, e.g., [17,18,25,27]). One of the first results about trioids is the description of the construction of a free monogenic trioid [11]. Simpler and more convenient model of the free trioid of rank 1 was presented in [31], where the endomorphism semigroup of the free monogenic trioid was also investigated (in particular, see [32]). The construction of the free trioid of an arbitrary rank has a similar structure and it was given independently in [24,30]. Further, the main attention at the study of trioids was devoted to the construction of relatively free trioids and the description of respected least congruences on the free trioid, the investigation of different properties of relatively free trioids. For example, the structure of free abelian trioids was described in [28], certain congruences on free trioids were found in [23]. The new models for the free commutative monogenic trioid and its endomorphism monoid were obtained in [33]. The latest results on relatively free trioids and verbal congruences on free trioids were presented in [20,22]. Besides, ordered trioids were considered in [29], where in particular it was proved that every ordered trioid is isomorphic to some ordered trioid of binary relations and representations of ordered trioids by reflexive and transitive binary relations were described. In this paper, we study the natural relationships between arbitrary semigroups and trioids constructed from these semigroups.

The paper is organised as follows. In Section 2, we give examples of trioids defined by additive semigroups of natural, integer, rational, real and complex numbers. In Section 3, we present three new classes of trioids that are based on arbitrary semigroups and study some properties of these trioids. In addition, we show that every semigroup can be embedded into a suitable non-trivial trioid as a subtrioid in which all operations coincide.

2 New examples of trioids

Let $\mathcal{N} = (N, +)$ be the additive semigroup of all natural numbers. By $T(N)$ we denote the union of N and all possible direct products $N^2 = N \times N$, $N^3 = N \times N \times N$, $\ldots$, that is, $T(N) = \bigcup_{i \in N} N^i$. For all $x_1 \in N$ we identify the expression (x_1) with the corresponding x_1. For every $x = (x_1, x_2, \ldots, x_n) \in T(N)$ we put $x^+ = x_1 + x_2 + \cdots + x_n$. Further, we extend the semigroup operation on $\mathcal{N}$ to three binary operations $\dashv, \vdash$, and $\perp$ on $T(N)$ as follows:

$$(a_1, a_2, \ldots, a_n) \dashv (b_1, b_2, \ldots, b_m) = (a_1, \ldots, a_{n-1}, a_n + b^+ + m - 1),$$
$$(a_1, a_2, \ldots, a_n) \vdash (b_1, b_2, \ldots, b_m) = (a^+ + b_1 + n - 1, b_2, \ldots, b_m),$$
$$(a_1, a_2, \ldots, a_n) \perp (b_1, b_2, \ldots, b_m) = (a_1, \ldots, a_{n-1}, a_n + b_1, b_2, \ldots, b_m)$$

for all $a_i, b_j \in N$, where $1 \leq i \leq n, 1 \leq j \leq m$.

Proposition 1. *The algebra $\mathcal{T}(\mathcal{N}) = (T(N), \dashv, \vdash, \perp)$ is a trioid.*

Proof. Firstly, we show that all operations $\dashv, \vdash$, and $\perp$ of $\mathcal{T}(\mathcal{N})$ are associative. For all $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$, $c = (c_1, c_2, \ldots, c_k) \in T(N)$, we have

$$\begin{aligned} a \dashv (b \dashv c) &= a \dashv (b_1, \dots, b_{m-1}, b_m + c^+ + k - 1) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + c^+ + m + k - 2) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + m - 1) \dashv c = (a \dashv b) \dashv c, \end{aligned}$$

$$\begin{aligned} a \vdash (b \vdash c) &= a \vdash (b^+ + c_1 + m - 1, c_2, \dots, c_k) \\ &= (a^+ + b^+ + c_1 + n + m - 2, c_2, \dots, c_k) \\ &= (a^+ + b_1 + n - 1, b_2, \dots, b_m) \vdash c = (a \vdash b) \vdash c, \end{aligned}$$

$$\begin{aligned} a \perp (b \perp c) &= a \perp (b_1, \dots, b_{m-1}, b_m + c_1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b_1, \dots, b_{m-1}, b_m + c_1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b_1, b_2, \dots, b_m) \perp c = (a \perp b) \perp c. \end{aligned}$$

Therefore, $(T(N), *)$, $* \in \{\dashv, \vdash, \perp\}$, are semigroups. Taking into account the equalities

$$\begin{aligned} a \dashv (b \vdash c) &= a \dashv (b^+ + c_1 + m - 1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + c^+ + m + k - 2) = (a \dashv b) \dashv c, \end{aligned}$$

$$\begin{aligned} a \vdash (b \vdash c) &= (a^+ + b^+ + c_1 + n + m - 2, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + m - 1) \vdash c = (a \dashv b) \vdash c, \end{aligned}$$

$$\begin{aligned} a \dashv (b \perp c) &= a \dashv (b_1, \dots, b_{m-1}, b_m + c_1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + c^+ + m + k - 2) = (a \dashv b) \dashv c, \end{aligned}$$

$$\begin{aligned} a \vdash (b \vdash c) &= (a^+ + b^+ + c_1 + n + m - 2, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b_1, b_2, \dots, b_m) \vdash c = (a \perp b) \vdash c, \end{aligned}$$

we conclude that the axioms (T_1), (T_3), (T_4), and (T_8) hold.

Now we check axioms (T_5)–(T_7):

$$\begin{aligned} a \perp (b \dashv c) &= a \perp (b_1, \dots, b_{m-1}, b_m + c^+ + k - 1) \\ &= (a_1, \dots, a_{n-1}, a_n + b_1, b_2, \dots, b_{m-1}, b_m + c^+ + k - 1) \\ &= (a_1, \dots, a_{n-1}, a_n + b_1, b_2, \dots, b_m) \dashv c = (a \perp b) \dashv c, \end{aligned}$$

$$\begin{aligned} a \perp (b \vdash c) &= a \perp (b^+ + c_1 + m - 1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + m - 1 + c_1, c_2, \dots, c_k) \\ &= (a_1, \dots, a_{n-1}, a_n + b^+ + m - 1) \perp c = (a \dashv b) \perp c, \end{aligned}$$

$$\begin{aligned} a \vdash (b \perp c) &= a \vdash (b_1, \dots, b_{m-1}, b_m + c_1, c_2, \dots, c_k) \\ &= (a^+ + b_1 + n - 1, b_2, \dots, b_{m-1}, b_m + c_1, c_2, \dots, c_k) \\ &= (a^+ + b_1 + n - 1, b_2, \dots, b_m) \perp c = (a \vdash b) \perp c. \end{aligned}$$

Finally, the axiom (T_2) also holds, since

$$\begin{aligned} a \vdash (b \dashv c) &= a \vdash (b_1, \dots, b_{m-1}, b_m + c^+ + k - 1) \\ &= (a^+ + b_1 + n - 1, b_2, \dots, b_{m-1}, b_m + c^+ + k - 1) \\ &= (a^+ + b_1 + n - 1, b_2, \dots, b_m) \dashv c = (a \vdash b) \dashv c. \end{aligned}$$

□

Remark 1. *A triod is called commutative (see, e.g., [22]) if all trioid operations are commutative. Note,* $\mathcal{T}(\mathcal{N}) = (T(N), \dashv, \vdash, \perp)$ *is not a commutative trioid (all operations of* $\mathcal{T}(\mathcal{N})$ *are not commutative) while* $\mathcal{N} = (N, +)$ *is a commutative semigroup. In addition, the semigroups* $(T(N), \dashv)$ *and* $(T(N), \vdash)$ *of the trioid* $\mathcal{T}(\mathcal{N})$ *are anti-isomorphic.*

Proposition 2. *Let* $\mathcal{S} = (S, +)$ *be one of such semigroups as the additive semigroup* $\mathcal{Z}$ *of all integers, the additive semigroup* $\mathcal{Q}$ *of all rational numbers, the additive semigroup* $\mathcal{R}$ *of all real numbers or the additive semigroup* $\mathcal{C}$ *of all complex numbers. Then the algebraic system* $\mathcal{T}(\mathcal{S}) = (T(S), \dashv, \vdash, \perp)$*, defined analogously to* $\mathcal{T}(\mathcal{N})$ *in Proposition 1, is a trioid.*

Proof. It is analogous to the proof of Proposition 1. □

Remark 2. *For a semigroup* $\mathcal{S} = (S, +)$ *consisting of some numbers, the algebraic system* $\mathcal{T}(\mathcal{S}) = (T(S), \dashv, \vdash, \perp)$*, defined exactly as in Proposition 1, is a trioid if and only if* $S + N \subseteq S$*. For example, the additive semigroups of the form* $S_n = \{x \in N : x \geq n\}$*, where* $n \in N$*, satisfy the condition* $S_n + N = S_{n+1} \subset S_n$*, so they define corresponding trioids. However, e.g., for the semigroup* $2\mathcal{Z} = (2Z, +)$*, the algebra* $\mathcal{T}(2\mathcal{Z}) = (T(2Z), \dashv, \vdash, \perp)$*, defined analogously to* $\mathcal{T}(\mathcal{N})$ *in Proposition 1, does not form a trioid.*

Let $\mathcal{S} = (S, +)$ be a semigroup consisting of some numbers and satisfying the condition $S + N \subseteq S$. By Remark 2, the algebra $\mathcal{T}(\mathcal{S})$ defined as in Proposition 1, is a trioid. We call $\mathcal{T}(\mathcal{S})$ *the trioid defined by the additive semigroup* $\mathcal{S}$. Thus, we obtain a class of trioids that are based on additive semigroups of given numbers. In particular, the algebras from Propositions 1 and 2 are trioids defined by the additive semigroups of all natural, integer, rational, real and complex numbers respectively.

3 Main results

A trioid we call *trivial* if at least some two of its three operations coincide, and *non-trivial* otherwise. Thus, a non-trivial trioid has three pairwise different operations. If $\mathcal{S} = (S, *)$ is a semigroup, sometimes we refer to $\mathcal{S}$ as a trivial trioid $(S, *, *, *)$.

By [36, Theorem 4], for an arbitrary group H there exists a non-trivial digroup (binary operations of such digroup are distinct) such that the group part of this digroup coincides with H. A similar statement holds for dimonoids: for an arbitrary semigroup there exists a non-trivial dimonoid containing this semigroup as a subdimonoid in which operations coincide (see [36, Theorem 1]). In connection with this, it is natural to consider the following question: is there for an arbitrary semigroup a non-trivial trioid containing it as a subtriod, in which all operations coincide? The answer to this question is positive.

Let $\mathcal{S} = (S, *)$ be an arbitrary semigroup. We extend the operation of $\mathcal{S}$ to the binary operations $\dashv$, $\vdash$, and $\perp$ on $Tr(S) = S \cup (S \times S)$ by

$$a \dashv (b, c) = a * b * c, \quad (b, c) \dashv a = (b, c * a), \quad (a, b) \dashv (c, d) = (a, b * c * d),$$
$$a \vdash (b, c) = (a * b, c), \quad (b, c) \vdash a = b * c * a, \quad (a, b) \vdash (c, d) = (a * b * c, d),$$
$$a \perp (b, c) = a * b * c, \quad (b, c) \perp a = b * c * a, \quad (a, b) \perp (c, d) = a * b * c * d$$

for all elements $a, b, c, d \in S$.

One of the main results of this section is the following statement.

Theorem 1. *For an arbitrary semigroup $\mathcal{S} = (S, *)$, the algebra $\mathcal{T}r(\mathcal{S}) = (Tr(S), \dashv, \vdash, \perp)$ is a non-trivial trioid containing $\mathcal{S}$ as a subtrioid, in which all operations coincide.*

Proof. By [36, Theorem 2], $(Tr(S), \dashv, \vdash)$ is a dimonoid, so operations $\dashv$ and $\vdash$ are associative and (T_1)–(T_3) hold. It is clear that $\perp$ is an associative operation too. Further we will check the last five trioid axioms for $\mathcal{T}r(\mathcal{S})$.

Let $a, b, c \in Tr(S)$. The case $a, b, c \in S$ is trivial. If $a \in S \times S$, $a = (a_1, a_2)$, and $b, c \in S$, then

$$\begin{aligned}(a \dashv b) \dashv c &= (a_1, a_2 * b) \dashv c = (a_1, a_2 * b * c) \\ &= (a_1, a_2) \dashv (b * c) = a \dashv (b \perp c), \\ (a \perp b) \dashv c &= (a_1 * a_2 * b) \dashv c = a_1 * a_2 * b * c, \\ (a \dashv b) \perp c &= (a_1, a_2 * b) \perp c = a_1 * a_2 * b * c, \\ (a \vdash b) \perp c &= (a_1 * a_2 * b) \perp c = a_1 * a_2 * b * c, \\ (a \perp b) \vdash c &= (a_1 * a_2 * b) \vdash c = a_1 * a_2 * b * c\end{aligned}$$

and on the other hand,

$$\begin{aligned}a \perp (b \dashv c) &= a \perp (b \vdash c) = a \vdash (b \perp c) \\ &= a \vdash (b \vdash c) = a_1 * a_2 * b * c.\end{aligned}$$

The case $a, b \in S$ and $c = (c_1, c_2) \in S \times S$ can be proved similarly to the case above.

Let $a, c \in S$ and $b = (b_1, b_2) \in S \times S$. Taking into account $a \dashv b = a \perp b = a * b_1 * b_2$, we obtain

$$\begin{aligned}(a \dashv b) \dashv c &= (a \perp b) \dashv c = (a \dashv b) \perp c \\ &= (a \perp b) \vdash c = a * b_1 * b_2 * c, \\ a \dashv (b \perp c) &= a \dashv (b_1 * b_2 * c) = a * b_1 * b_2 * c, \\ a \perp (b \dashv c) &= a \perp (b_1, b_2 * c) = a * b_1 * b_2 * c, \\ a \perp (b \vdash c) &= a \perp (b_1 * b_2 * c) = a * b_1 * b_2 * c, \\ a \vdash (b \vdash c) &= a \vdash (b_1 * b_2 * c) = a * b_1 * b_2 * c, \\ (a \vdash b) \perp c &= (a * b_1, b_2) \perp c = a * b_1 * b_2 * c \\ &= a \vdash (b_1 * b_2 * c) = a \vdash (b \perp c).\end{aligned}$$

Assume $a = (a_1, a_2)$, $b = (b_1, b_2) \in S \times S$ and $c \in S$. In this case, we obtain

$$\begin{aligned}(a \dashv b) \dashv c &= (a_1, a_2 * b_1 * b_2) \dashv c = (a_1, a_2 * b_1 * b_2 * c) \\ &= (a_1, a_2) \dashv (b_1 * b_2 * c) = a \dashv (b \perp c), \\ (a \perp b) \dashv c &= (a_1 * a_2 * b_1 * b_2) \dashv c = a_1 * a_2 * b_1 * b_2 * c \\ &= (a_1, a_2) \perp (b_1, b_2 * c) = a \perp (b \dashv c), \\ (a \dashv b) \perp c &= (a_1, a_2 * b_1 * b_2) \perp c = a_1 * a_2 * b_1 * b_2 * c \\ &= (a_1, a_2) \perp (b_1 * b_2 * c) = a \perp (b \vdash c), \\ (a \vdash b) \perp c &= (a_1 * a_2 * b_1, b_2) \perp c = a_1 * a_2 * b_1 * b_2 * c \\ &= (a_1, a_2) \vdash (b_1 * b_2 * c) = a \vdash (b \perp c), \\ (a \perp b) \vdash c &= (a_1 * a_2 * b_1 * b_2) \vdash c = a_1 * a_2 * b_1 * b_2 * c \\ &= (a_1, a_2) \vdash (b_1 * b_2 * c) = a \vdash (b \vdash c).\end{aligned}$$

For the cases $a \in S$, $b = (b_1, b_2), c = (c_1, c_2) \in S \times S$, or $a = (a_1, a_2), c = (c_1, c_2) \in S \times S$ and $b \in S$, the proofs are similar to the case above.

Finally, for $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2) \in S \times S$, we get

$$\begin{aligned}
(a \dashv b) \dashv c &= (a_1, a_2 * b_1 * b_2) \dashv c = (a_1, a_2 * b_1 * b_2 * c_1 * c_2) \\
&= (a_1, a_2) \dashv (b_1 * b_2 * c_1 * c_2) = a \dashv (b \perp c), \\
(a \perp b) \dashv c &= (a_1 * a_2 * b_1 * b_2) \dashv c = a_1 * a_2 * b_1 * b_2 * c_1 * c_2 \\
&= (a_1, a_2) \perp (b_1, b_2 * c_1 * c_2) = a \perp (b \dashv c), \\
(a \dashv b) \perp c &= (a_1, a_2 * b_1 * b_2) \perp c = a_1 * a_2 * b_1 * b_2 * c_1 * c_2 \\
&= (a_1, a_2) \perp (b_1 * b_2 * c_1, c_2) = a \perp (b \vdash c), \\
(a \vdash b) \perp c &= (a_1 * a_2 * b_1, b_2) \perp c = a_1 * a_2 * b_1 * b_2 * c_1 * c_2 \\
&= (a_1, a_2) \vdash (b_1 * b_2 * c_1 * c_2) = a \vdash (b \perp c), \\
(a \perp b) \vdash c &= (a_1 * a_2 * b_1 * b_2) \vdash c = (a_1 * a_2 * b_1 * b_2 * c_1, c_2) \\
&= (a_1, a_2) \vdash (b_1 * b_2 * c_1, c_2) = a \vdash (b \vdash c).
\end{aligned}$$

Therefore, the axioms (T_4)–(T_8) hold for $\mathcal{T}r(\mathcal{S})$ too and so, it is a trioid. By construction, $\mathcal{T}r(\mathcal{S})$ is non-trivial and contains $\mathcal{S}$ as a trivial subtrioid. $\square$

Some classes of trioids that are defined using arbitrary commutative semigroups can be found, for example, in [28]. Now we will construct two more classes of non-trivial trioids that are also defined by arbitrary semigroups.

For a semigroup $\mathcal{S} = (S, *)$, we denote by $TR(S)$ the union of S and all possible direct products $S^2 = S \times S$, $S^3 = S \times S \times S$, ..., that is, $TR(S) = \bigcup_{i \in N} S^i$. We identify the expression (x_1) with the corresponding $x_1 \in S$. For every $x = (x_1, x_2, \dots, x_n) \in TR(S)$ we put $x^* = x_1 * x_2 * \cdots * x_n$. Further, we extend the semigroup operation on $\mathcal{S}$ to four binary operations $\dashv, \vdash$, $\perp$, and $\perp'$ on $TR(S)$ as follows:

$$\begin{aligned}
(a_1, a_2, \dots, a_n) \dashv (b_1, b_2, \dots, b_m) &= (a_1, \dots, a_{n-1}, a_n * b^*), \\
(a_1, a_2, \dots, a_n) \vdash (b_1, b_2, \dots, b_m) &= (a^* * b_1, b_2, \dots, b_m), \\
(a_1, a_2, \dots, a_n) \perp (b_1, b_2, \dots, b_m) &= (a_1, \dots, a_{n-1}, a_n * b_1, b_2, \dots, b_m), \\
(a_1, a_2, \dots, a_n) \perp' (b_1, b_2, \dots, b_m) &= a^* * b^*
\end{aligned}$$

for all $a_i, b_j \in S$, where $1 \le i \le n, 1 \le j \le m$.

Theorem 2. *For any semigroup $\mathcal{S} = (S, *)$, each of the algebras $\mathcal{TR}(\mathcal{S}) = (TR(S), \dashv, \vdash, \perp)$ and $\mathcal{TR}'(\mathcal{S}) = (TR(S), \dashv, \vdash, \perp')$ is a non-trivial trioid containing $\mathcal{S}$ as a subtrioid, in which all operations coincide.*

Proof. We show first that operations $\dashv$, $\vdash$, and $\perp$ of $\mathcal{TR}(\mathcal{S})$ are associative. For all elements $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_m),\ c = (c_1, c_2, \dots, c_k) \in TR(S)$, we get

$$\begin{aligned}
a \dashv (b \dashv c) &= a \dashv (b_1, \dots, b_{m-1}, b_m * c^*) = (a_1, \dots, a_{n-1}, a_n * b^* * c^*) \\
&= (a_1, \dots, a_{n-1}, a_n * b^*) \dashv c = (a \dashv b) \dashv c, \\
a \vdash (b \vdash c) &= a \vdash (b^* * c_1, c_2, \dots, c_k) = (a^* * b^* * c_1, c_2, \dots, c_k) \\
&= (a^* * b_1, b_2, \dots, b_m) \vdash c = (a \vdash b) \vdash c
\end{aligned}$$

and

$$\begin{aligned} a \perp (b \perp c) &= a \perp (b_1, \ldots, b_{m-1}, b_m * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b_1, \ldots, b_{m-1}, b_m * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b_1, b_2, \ldots, b_m) \perp c = (a \perp b) \perp c. \end{aligned}$$

Thus, $(TR(S), \circ)$ is a semigroup for any $\circ \in \{\dashv, \vdash, \perp\}$. Since

$$\begin{aligned} a \dashv (b \vdash c) &= a \dashv (b^* * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^* * c^*) = (a \dashv b) \dashv c, \\ a \vdash (b \vdash c) &= (a^* * b^* * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^*) \vdash c = (a \dashv b) \vdash c, \\ a \dashv (b \perp c) &= a \dashv (b_1, \ldots, b_{m-1}, b_m * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^* * c^*) = (a \dashv b) \dashv c, \\ a \vdash (b \vdash c) &= (a^* * b^* * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b_1, b_2, \ldots, b_m) \vdash c = (a \perp b) \vdash c, \end{aligned}$$

the axioms (T_1), (T_3), (T_4), and (T_8) hold.

The axiom (T_2) also holds, since

$$\begin{aligned} a \vdash (b \dashv c) = a \vdash (b_1, \ldots, b_{m-1}, b_m * c^*) &= (a^* * b_1, b_2, \ldots, b_{m-1}, b_m * c^*) \\ = (a^* * b_1, b_2, \ldots, b_m) \dashv c &= (a \vdash b) \dashv c. \end{aligned}$$

Further we check axioms (T_5)–(T_7):

$$\begin{aligned} a \perp (b \dashv c) &= a \perp (b_1, \ldots, b_{m-1}, b_m * c^*) \\ &= (a_1, \ldots, a_{n-1}, a_n * b_1, b_2, \ldots, b_{m-1}, b_m * c^*) \\ &= (a_1, \ldots, a_{n-1}, a_n * b_1, b_2, \ldots, b_m) \dashv c = (a \perp b) \dashv c, \\ a \perp (b \vdash c) &= a \perp (b^* * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^* * c_1, c_2, \ldots, c_k) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^*) \perp c = (a \dashv b) \perp c, \\ a \vdash (b \perp c) &= a \vdash (b_1, \ldots, b_{m-1}, b_m * c_1, c_2, \ldots, c_k) \\ &= (a^* * b_1, b_2, \ldots, b_m * c_1, c_2, \ldots, c_k) \\ &= (a^* * b_1, b_2, \ldots, b_m) \perp c = (a \vdash b) \perp c. \end{aligned}$$

Obviously, $\perp'$ is associative on $TR(S)$. To prove that $\mathcal{TR}'(\mathcal{S}) = (TR(S), \dashv, \vdash, \perp')$ is a trioid it is enough to check the trioid axioms (T_4)–(T_8). For all $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$, $c = (c_1, c_2, \ldots, c_k) \in TR(S)$, we have

$$\begin{aligned} a \dashv \left(b \perp' c\right) &= a \dashv (b^* * c^*) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^* * c^*) = (a \dashv b) \dashv c, \\ a \perp' (b \dashv c) &= a \perp' (b_1, \ldots, b_{m-1}, b_m * c^*) = a^* * (b^* * c^*) \\ &= (a^* * b^*) \dashv c = \left(a \perp' b\right) \dashv c, \\ a \perp' (b \vdash c) &= a \perp' (b^* * c_1, c_2, \ldots, c_k) = a^* * (b^* * c^*) \\ &= (a_1, \ldots, a_{n-1}, a_n * b^*) \perp' c = (a \dashv b) \perp' c, \end{aligned}$$

and

$$\begin{aligned} a \vdash \big(b \perp' c\big) &= a \vdash (b^* * c^*) = a^* * (b^* * c^*) \\ &= (a^* * b_1, b_2, \ldots, b_m) \perp' c = (a \vdash b) \perp' c, \\ a \vdash (b \vdash c) &= (a^* * b^* * c_1, c_2, \ldots, c_k) \\ &= (a^* * b^*) \vdash c = (a \perp' b) \vdash c. \end{aligned}$$

Consequently, $\mathcal{TR}(\mathcal{S})$ and $\mathcal{TR}'(\mathcal{S})$ are trioids that are obviously non-trivial, and their operations $\dashv, \vdash, \perp$, and $\perp'$ on S coincide. This means that each of these trioids contains $\mathcal{S}$ as a subtrioid whose operations all coincide. □

We call the obtained trioids $\mathcal{T}r(\mathcal{S})$, $\mathcal{TR}(\mathcal{S})$, and $\mathcal{TR}'(\mathcal{S})$ from the theorems above *the trioid extensions of the semigroup $\mathcal{S}$*.

Theorems 1 and 2 immediately imply the following statement.

Corollary 1. *Any semigroup can be embedded into suitable non-trivial trioids as a subtrioid, in which all operations coincide.*

Remark 3. *Theorems 1, 2 give new classes of trioids that are defined by arbitrary semigroups. We note that all operations on the trioid $\mathcal{TR}(\mathcal{S})$ are not commutative, while $\mathcal{S} = (S, *)$ is an arbitrary semigroup; the operation $\perp$ (respectively, $\perp'$) of the trioid $\mathcal{T}r(\mathcal{S})$ (respectively, $\mathcal{TR}'(\mathcal{S})$) is commutative if $\mathcal{S} = (S, *)$ is commutative.*

Let $\mathcal{T} = (T, \prec, \succ, \dagger)$ and $\mathcal{T}' = (T', \prec', \succ', \dagger')$ be arbitrary trioids. Recall that a mapping $\phi : T \to T'$ is called a *homomorphism* of $\mathcal{T}$ into $\mathcal{T}'$ if $(x * y)\phi = x\phi *' y\phi$ for all $x, y \in T$ and any $* \in \{\prec, \succ, \dagger\}$.

Finally, we consider some properties of trioids constructed in Theorems 1 and 2.

Proposition 3. *Let $\mathcal{S} = (S, *)$ be an arbitrary semigroup. Then*

(i) $\mathcal{S}$ as a trivial trioid is an epimorphic image of each of the trioids $\mathcal{T}r(\mathcal{S})$, $\mathcal{TR}(\mathcal{S})$ and $\mathcal{TR}'(\mathcal{S})$;

(ii) the semigroups $(Tr(S), \dashv)$ and $(Tr(S), \vdash)$ of the trioid $\mathcal{T}r(\mathcal{S})$ are anti-isomorphic if $\mathcal{S}$ is commutative;

(iii) the semigroups $(TR(S), \dashv)$ and $(TR(S), \vdash)$ of the trioid $\mathcal{TR}(\mathcal{S})$ are anti-isomorphic if $\mathcal{S}$ is commutative;

(iv) there exists a monomorphism of $\mathcal{T}r(\mathcal{S})$ into $\mathcal{TR}'(\mathcal{S})$.

Proof. *(i)* Define a mapping φ from the trioid $\mathcal{TR}(\mathcal{S})$ onto the trivial trioid $(S, *, *, *)$ by

$$\varphi : a = (a_1, a_2, \ldots, a_n) \mapsto a^* = a_1 * a_2 * \cdots * a_n.$$

It is clear that $S\varphi = S$, therefore φ is a surjection. Moreover, for all $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m) \in TR(S)$ we have

$$\begin{aligned} ((a_1, a_2, \ldots, a_n) \dashv (b_1, b_2, \ldots, b_m))\varphi &= (a_1, \ldots, a_{n-1}, a_n * b^*)\varphi \\ &= a_1 * a_2 * \cdots * a_n * b^* = a^* * b^* = a\varphi * b\varphi, \end{aligned}$$

$$\begin{aligned}((a_1,a_2,\ldots,a_n)\vdash(b_1,b_2,\ldots,b_m))\varphi &= (a^* * b_1,b_2,\ldots,b_m)\varphi\\ &= a^* * b_1 * b_2 * \cdots * b_m = a^* * b^* = a\varphi * b\varphi,\\ ((a_1,a_2,\ldots,a_n)\perp(b_1,b_2,\ldots,b_m))\varphi &= (a_1,\ldots,a_{n-1},a_n * b_1,b_2,\ldots,b_m)\varphi\\ &= a_1 * \cdots * a_{n-1} * a_n * b_1 * b_2 * \cdots * b_m = a\varphi * b\varphi.\end{aligned}$$

Analogously, $\mathcal{S}$ as a trivial trioid is an epimorphic image of $\mathcal{TR}'(\mathcal{S})$. In particular, a mapping $\psi : Tr(S) \to S$ such that $(a,b)\psi = a * b$ and $c\psi = c$ for all $a,b,c \in S$ is an epimorphism of $\mathcal{T}r(\mathcal{S})$ onto $\mathcal{S}$.

(ii) Let $\mathcal{S} = (S,*)$ be a commutative semigroup. Define a transformation η of $Tr(S)$ by $(a,b)\eta = (b,a)$ and $c\eta = c$ for all $a,b,c \in S$. It is obvious that η is a bijection and $\eta|_S$ is an anti-isomorphism of $\mathcal{S}$. Besides,

$$\begin{aligned}((a,b)\dashv(c,d))\eta &= (a,b*c*d)\eta = (b*c*d,a)\\ &= (d*c*b,a) = (d,c)\vdash(b,a) = (c,d)\eta\vdash(a,b)\eta,\\ ((a,b)\dashv c)\eta &= (a,b*c)\eta = (b*c,a)\\ &= (c*b,a) = c\vdash(b,a) = c\eta\vdash(a,b)\eta,\\ (a\dashv(b,c))\eta &= (a*b*c)\eta = a*b*c\\ &= c*b*a = (c,b)\vdash a = (b,c)\eta\vdash a\eta\end{aligned}$$

for all $a,b,c,d \in S$, so η is an anti-isomorphism of $(Tr(S),\dashv)$ into $(Tr(S),\vdash)$. The statement *(iii)* can be proved in a similar way.

(iv) It is not difficult to check that $\mathcal{T}r(\mathcal{S})$ is identically embedded into $\mathcal{TR}'(\mathcal{S})$. ☐

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Жучок Ю.В. *Про вкладення напівгруп у тріоїди* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 516–526.

Ж.-Л. Лодей та М.О. Ронко ввели поняття триалгебри і тріоїда, а також визначили конструкції вільної триалгебри і вільного моногенного тріоїда. Триалгебри пов'язані з операдами, асоційованими з ланцюговими модулями сімплексів і політопами Сташефа. Тріоїд є основою триалгебри і визначається як множина з трьома бінарними асоціативними операціями, що задовольняють тим же аксіомам, що і триалгебра, тому триалгебри є лінійними аналогами тріоїдів. Якщо операції тріоїда збігаються, то він стає напівгрупою. У цій статті вивчаються природні зв'язки між довільними напівгрупами та тріоїдами, що визначаються цими напівгрупами. Представлено нові класи тріоїдів, побудованих з різних напівгруп, і показано, що будь-яка напівгрупа може бути вкладена у деякий нетривіальний тріоїд як підтріоїд, в якому всі операції збігаються.

Ключові слова і фрази: тріоїд, дімоноїд, напівгрупа, мономорфізм.