



Adequacy of nonsingular matrices over commutative principal ideal domains

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The notion of adequacy for commutative domains was introduced by O. Helmer in [Bull. Amer. Math. Soc. 1943, 49 (4), 225–236]. In the present paper, we extend the concept of adequacy to non-commutative Bézout rings. We show that the set of nonsingular 2×2 matrices over a commutative principal ideal domain is adequate.

Key words and phrases: adequate ring, principal ideal domain, divisor of a matrix.

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1 Introduction and results

Let $U(R)$ be the group of units of an associative, commutative ring R with $1 \neq 0$. The elements $a, b \in R$ are called *strongly associated* if there exists $e \in U(R)$ such that $a = be$ (see [1, Definition 2.1, p. 441] and [6]). The set of all non strongly associate elements of the ring R is denoted by R^* . Of course, we always assume $1 \in R^*$. The matrix $\text{diag}(d_1, \dots, d_n)$ means a matrix having $d_1, \dots, d_n \in R$ on the main diagonal and zeros elsewhere (by the main diagonal we mean the one beginning at the upper left corner). The set of all matrices of size $n \times m$ over a ring R is denoted by $R^{n \times m}$.

A commutative ring R is called an *elementary divisor ring* [13, p. 465] if, for each matrix $A \in R^{n \times m}$, there exist invertible matrices P_A and Q_A such that

$$P_A A Q_A = \text{diag}(\alpha_1, \dots, \alpha_s) \in R^{n \times m}, \quad (1)$$

where $s := \min(n, m)$ and each α_i divides α_{i+1} for $i = 1, \dots, s - 1$. The diagonal matrix $\text{diag}(\alpha_1, \dots, \alpha_s)$ is called a *Smith form* of A (unique up to strong associates of its diagonal elements). Accordingly, we can always choose $\alpha_1, \dots, \alpha_s \in R^*$ so that the matrix $\text{diag}(\alpha_1, \dots, \alpha_s)$ is uniquely defined; it is called the *Smith normal form* of the matrix A and is denoted by $\text{SNF}(A)$. The matrices P_A and Q_A (see (1)) are called the *left* and *right transforming* matrices of A , respectively. The sets of all left and right transforming matrices of $A \in R^{n \times n}$ with the Smith normal form $\Phi := \text{diag}(\alpha_1, \dots, \alpha_n)$ have the form of right and left cosets $\mathbf{G}_\Phi P_A$ and $Q_A \mathbf{G}_\Phi^T$ by the subgroups $\mathbf{G}_\Phi, \mathbf{G}_\Phi^T < \text{GL}_n(R)$, respectively. Here \mathbf{G}_Φ is the Zelisko group [2, 3, 18] of the matrix Φ , defined as

$$\mathbf{G}_\Phi := \{H \in \text{GL}_n(R) : \exists S \in \text{GL}_n(R) \text{ such that } H\Phi = \Phi S\}$$

УДК 512.55

2020 *Mathematics Subject Classification:* 15A23, 19D10, 16U30, 15A24.

The research was supported by UAEU UPAR grants G00002160 and G00003658.

and $\mathbf{G}_\Phi^T := \{H^T : H \in \mathbf{G}_\Phi\}$.

The *greatest common divisor* and the *least common multiple* of $a, b \in R$, which are unique up to strong associates, are denoted by (a, b) and $[a, b]$, respectively; and $a \mid b$ means that a is a divisor of b .

Let R be a commutative domain with $1 \neq 0$ in which every finitely generated ideal is principal (Bézout domain). Let $a, b \in R$, and $b \neq 0$. According to O. Helmer [12, p. 225], the *relatively prime part* of b with respect to a , denoted $RP(a, b)$, refers to a factor t of b such that, if $b = st$, then

- (i) $(t, a) = 1$;
- (ii) $(s', a) \neq 1$ for any non-unit factor s' of s .

The element s (if it exists) is called an *adequate part* of b with respect to a . A ring R is called *adequate* if $RP(a, b)$ exists for all $a, b \in R$ with $b \neq 0$. This concept is essentially a formalization of properties of entire analytic functions rings. Each commutative principal ideal domain (PID) is adequate, but the converse is not true, in particular, the ascending chain condition on ideals may not be satisfied. Each adequate ring is an elementary divisor ring [12, Theorem 3, p. 234]. The ring of all continuous real-valued functions defined on a completely regular (Hausdorff) space X is an example of an adequate ring, which is regular and every prime ideal is maximal [11, Corollaries 3.6, 3.8 p. 386]. Each local ring as well as each commutative von Neumann regular ring is adequate [10, Theorem 11, p. 365]. Adequate rings with zero-divisors in their Jacobson radical were investigated by Kaplansky [13, Theorem 5.3, p. 473]. Note that not every elementary divisor ring is adequate [11, Corollary 6.7, p. 386] and in an adequate domain each nonzero prime ideal is contained in a unique maximal ideal [11, Corollary 6.6, p. 386]. Bézout rings in which each regular element is adequate were investigated in [17]. Moreover, generalized adequate rings were introduced in [16], forming a new class of elementary divisor rings that includes adequate rings as a subclass.

A.I. Gatalevych [9] was the first to attempt applying the notion of adequacy to noncommutative rings. He introduced a new concept of adequacy for noncommutative rings and proved that a generalized right adequate (in the sense of Gatalevych) duo Bézout domain is an elementary divisor domain [9, Theorem 2, p. 117]. In the present article, we propose an alternative definition of adequate rings, which differs from the one introduced by A.I. Gatalevych [9, Definition 1, p. 116]. By means of the example in the last section, we illustrate certain advantages of our definition. Note that some other generalizations of the notion of adequacy and their applications have been considered in [4, 5, 7–9, 16].

Our definition of the adequacy of a ring is the following.

Let K be a Bézout (not necessarily commutative) ring with $1 \neq 0$. An element $0 \neq b \in K$ is called *left adequate* to $a \in K$ if either $aK + bK = K$ or, if $aK + bK \neq K$ then there exists $s \in K$ such that $b = st$ and the following conditions hold:

- (i) $s'K + aK \neq K$ for each $s' \in K$ such that $sK \subset s'K \neq K$;
- (ii) for each $t' \in K$ such that $tK \subset t'K \neq K$ there exists a decomposition $st' = pq$ such that $pK + aK = K$.

The element s is called a *left adequate part* of b with respect to a . The right adequate part of b with respect to a is defined analogously.

A subset $A \subseteq K$ is called *left* (respectively, *right*) *adequate* if each of its nonzero elements is left (respectively, right) adequate to all elements of A . If every nonzero element of A is both left and right adequate to all elements of A , then the set A is called *adequate*.

Remark 1. *At first sight, Helmer's definition and ours differ substantially, since in the former the emphasis is placed on the coprime part of the elements, while in the latter it is on their adequate part. However, in the case of a commutative PID R , these definitions coincide. Let us explain this by the following example.*

Let $b := \alpha^3 \beta^2 \gamma^7$, $a := \alpha^2 \beta^4 \delta^5$, where $\alpha, \beta, \gamma, \delta \in R \setminus U(R)$. Then $b = s \cdot t$, in which $s := \alpha^3 \beta^2$, $t := \gamma^7 = RP(a, b)$, is a decomposition of the element b according to Helmer's definition.

Note that the element s is in some sense "maximal" since its "enlargement" would lead to considering elements of the form $\alpha^3 \beta^2 \gamma^i$ ($1 \leq i \leq 7$). Since these contain factors of the form γ^i that are coprime to a , this contradicts condition (i). Therefore $\alpha^3 \beta^2$ is the adequate part of b with respect to a . Hence, the decomposition of b respect to a according to our definition, coincides with that given by the Helmer's definition.

In our definition, the element t is also coprime to a , since if $(t, a) = \mu \neq 1$, then $b = (s\mu) \frac{t}{\mu}$, where each divisor of $s\mu$ has a common divisor with a , which contradicts condition (ii).

A shift of emphasis from the coprime part to the adequate part necessitates further clarification. The point is that for any $b \neq 0$ and a their coprime part always exists (at least it is 1); however, in the case where b and a are coprime, their adequate part does not exist. Therefore, this case is treated separately in our definition.

Our first main result is the following.

Theorem 1. *Let R be a commutative PID such that $1 \neq 0$. The set of nonsingular 2×2 matrices over R is an adequate set.*

Let R be a commutative PID with $1 \neq 0$. A subset of $R^* \setminus \{1\}$ consisting of all indecomposable divisors of an element $a \in R$ is called the *spectrum* of a and is denoted by $\Sigma(a)$. The spectrum of a nonsingular matrix $A \in R^{2 \times 2}$ is the set $\Sigma(A) := \Sigma(\alpha_2)$ (see (1)). Matrices $M, N \in R^{2 \times 2}$ are called *strongly right associated* if there is a matrix $U \in GL_2(R)$ such that $M = NU$.

Let $A, B, C, D, A_1, B_1 \in R^{2 \times 2}$. If $A = BC$, then A is called a *right multiple* of B . If $A = DA_1$ and $B = DB_1$, then D is called a *left common divisor* of A and B . In addition, if D is a right multiple of each left common divisor of A and B , then D is called a *left greatest common divisor* of A and B , which we denoted by $D := (A, B)_l$. The left greatest common divisor $(A, B)_l$ is unique up to right strongly associates [14, Theorem 1.12, p. 39].

Let $A \in R^{2 \times 2}$. In view of equation (1), we use the following presentation

$$A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1}, \quad (2)$$

in which $\text{diag}(\alpha_1, \alpha_2) = \text{SNF}(A)$, and P_A, Q_A are the left and right transforming matrices of the matrix A .

Our next main result is the following.

Theorem 2. *Let R be a commutative PID such that $1 \neq 0$ and let*

$$A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1} \quad \text{and} \quad S := P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1}$$

be nonsingular matrices of the form (2). Each left divisor of the matrix S has a nontrivial left common divisor with the matrix A if and only if $\Sigma(\sigma_i) \subseteq \Sigma(\alpha_i)$ for $i = 1, 2$ and one of the following conditions holds:

(i) $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1)$;

(ii)

$$P_S = \begin{bmatrix} m_{11} & m_{12} \\ q_1 \cdots q_k m_{21} & m_{22} \end{bmatrix} P_A, \quad m_{ij} \in R,$$

where $\{q_1, \dots, q_k\} = \Sigma(\sigma_2) \setminus \Sigma(\alpha_1)$.

2 Preliminaries, lemmas and proofs

For each 2×2 nonsingular matrices A, B of the form (2) we define the matrix $[\tau_{ij}] := P_B P_A^{-1}$ and the set

$$\mathbf{L}_{\alpha_1, \beta_2} := \left\{ \begin{bmatrix} l_{11} & l_{12} \\ \frac{\beta_2}{(\beta_2, \alpha_1)} l_{12} & l_{22} \end{bmatrix} \in \text{GL}_2(R) : l_{ij} \in R \right\}. \quad (3)$$

In the sequel, we will use the following facts.

Fact 1 ([15, Theorem 1, p. 851]). *Let R be a commutative elementary divisor ring and let $A, B \in R^{2 \times 2}$ of the form (2). Then*

(i) $\text{SNF}((A, B)_l) = \text{diag}((\alpha_1, \beta_1), (\alpha_2, \beta_2, [\alpha_1, \beta_1] \tau_{21}))$;

(ii) A, B are left relatively prime (i.e. $(A, B)_l = I$) if and only if

$$(\alpha_2, \beta_2, [\alpha_1, \beta_1] \tau_{21}) = 1.$$

Fact 2 ([14, Theorem 4.3, p. 127]). *Let R be a commutative elementary divisor ring and let $A, B \in R^{2 \times 2}$ of the form (2). The matrix B is a left divisor of A (i.e. $A = BC$) if and only if $\beta_i | \alpha_i$ for $i = 1, 2$ and $P_B = LP_A$, in which $L \in \mathbf{L}_{\alpha_1, \beta_2}$ (see (3)).*

Fact 3 ([14, Theorem 4.4, p. 128]). *Let R be a commutative elementary divisor ring and let $A \in R^{2 \times 2}$ of the form (2). Let $\beta_1, \beta_2 \in R$ such that $\beta_1 | \beta_2$ and $\beta_i | \alpha_i$ for $i = 1, 2$. The set of all left divisors of A with the Smith form $\text{diag}(\beta_1, \beta_2)$ has the form*

$$(\mathbf{L}_{\alpha_1, \beta_2} P_A)^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot \text{GL}_2(R).$$

Lemma 1. *Let R be a commutative Bézout domain and let $A, B \in R^{n \times n}$ (with $n \geq 2$) be nonsingular matrices. If $\det(B)$ is indecomposable in R and $(A, B)_l \neq I$ then $A = BC$.*

Proof. Let $D := (A, B)_l \neq I$. Clearly, $B = DB_1$ and $\det(D) | \det(B)$. Thus $\det(D)$ and $\det(B)$ are strong associates in R , i.e. $\det(B) = \det(D)e$ for some $e \in U(R)$. Consequently, $\det(B_1) = e$, so $B_1 \in \text{GL}_n(R)$ and $D = BB_1^{-1}$. Since $A = DA_1$, we have $A = BB_1^{-1}A_1 = BC$, where $C = B_1^{-1}A_1$. □

Proof of Theorem 2. Necessity. Let $\omega \in \Sigma(\sigma_1)$. Thus $\sigma_1 = \omega\sigma'_1$ and $\sigma_2 = \omega\sigma'_2$ for some $\sigma'_1, \sigma'_2 \in R$. If $M := P^{-1} \cdot \text{diag}(1, \omega) \cdot Q^{-1}$ and

$$M_1 := (Q \cdot \text{diag}(\omega, 1) \cdot P) (P_S^{-1} \cdot \text{diag}(\sigma'_1, \sigma'_2) \cdot Q_S^{-1}),$$

in which P, Q are arbitrary invertible matrices, then

$$S = P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1} = M \cdot M_1$$

and $(A, M)_l \neq I$. Taking into account that $\det(M)$ is indecomposable in R , we obtain that $A = MA_1$ by Lemma 1. Consequently, all matrices L with $\text{SNF}(L) = \text{diag}(1, \omega)$ are left divisors of A . In accordance with [14, Theorem 5.3 p. 152 and Property 4.11 p. 147] we have $\omega | \alpha_1$ and $\Sigma(\sigma_1) \subseteq \Sigma(\alpha_1)$.

Case 1. Suppose that $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1)$. Reasoning similarly as before, we obtain that every matrix with Smith normal form $\text{diag}(\sigma_1, \sigma_2)$ has a nontrivial left common divisor with A .

Case 2. Let $\mu \in \Sigma(\sigma_2) \setminus \Sigma(\alpha_2)$. Thus $\sigma_2 = \mu \cdot \mu_1$ and $(\mu, \alpha_2) = 1$. If $C := P_S^{-1} \cdot \text{diag}(1, \mu)$ and $C_1 := \text{diag}(\sigma_1, \mu_1) \cdot Q_S^{-1}$, then $S = CC_1$. Since $(\det(C), \det(A)) = 1$, we have $(A, C)_l = I$, a contradiction. Consequently, we get $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_2)$.

Let $\Sigma(\sigma_2) \setminus \Sigma(\alpha_1) = \{q_1, \dots, q_k\}$ for $k \geq 1$ and let $i \in \{1, \dots, k\}$. Thus $\sigma_2 = q_i \delta_i$ for some $\delta_i \in R$. If $D := P_S^{-1} \cdot \text{diag}(1, q_i)$ and $D_1 := \text{diag}(\sigma_1, \delta_i) \cdot Q_S^{-1}$, then

$$S = P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1} = D \cdot D_1.$$

All left divisors L of S (including D) such that $\text{SNF}(L) = \text{diag}(1, q_i)$ belong to the set

$$\begin{aligned} \mathbf{W} &= \left\{ (\mathbf{L}_{\sigma_1, q_i} P_S)^{-1} \cdot \text{diag}(1, q_i) \cdot \text{GL}_2(R) \right\} && \text{by Fact 3} \\ &= \left\{ \left(\begin{bmatrix} l_{11} & l_{12} \\ q_i l_{12} & l_{22} \end{bmatrix} P_S \right)^{-1} \cdot \text{diag}(1, q_i) \cdot \text{GL}_2(R) : l_{ij} \in R \right\}, && \text{since } (q_i, \sigma_1) = 1. \end{aligned}$$

Let us fix $M := P_M^{-1} \cdot \text{diag}(1, q_i) \cdot Q_M^{-1} \in \mathbf{W}$, in which $P_M := \begin{bmatrix} h_{11} & h_{12} \\ q_i h_{21} & h_{22} \end{bmatrix} P_S$ for some $h_{pl} \in R$ and $Q_M \in \text{GL}_2(R)$ is fixed. The matrix M is a left divisor of S , so $(A, M)_l \neq I$. Hence, $d_i := (\alpha_2, q_i, \alpha_1 \tau_{21}^{(i)}) \neq 1$ (see Fact 1 (ii)), where

$$[\tau_{mn}^{(i)}] := P_M P_A^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ q_i h_{21} & h_{22} \end{bmatrix} (P_S P_A^{-1}). \quad (4)$$

Since $d_i | q_i$ and both d_i and q_i are indecomposable elements of R , it follows that they are strongly associated. Taking into account that $d_i, q_i \in R^*$, we obtain

$$d_i = q_i = (\alpha_2, q_i, \alpha_1 \tau_{21}^{(i)}) = (\alpha_2, (q_i, \alpha_1 \tau_{21}^{(i)})) = (\alpha_2, q_i, \tau_{21}^{(i)}),$$

so $q_i | \tau_{21}^{(i)}$, i.e. $\tau_{21}^{(i)} = q_i n_i$ for some $n_i \in R$. It is obvious (see (4)) that

$$P_S P_A^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ q_i h_{21} & h_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tau_{11}^{(i)} & \tau_{12}^{(i)} \\ q_i n_i & \tau_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ q_i p_{21} & p_{22} \end{bmatrix}, \quad p_{mn} \in R. \quad (5)$$

We now show that (5) holds independently of the choices of P_S and P_A . Let P'_S and P'_A be arbitrary left transforming matrices of S and A , respectively. By [14, Property 2.2 p. 61]),

$$P'_S = FP_S \quad \text{and} \quad P'_A = TP_A,$$

where

$$F := \begin{bmatrix} f_{11} & f_{12} \\ \frac{\sigma_2}{\sigma_1}f_{21} & f_{22} \end{bmatrix}, \quad T^{-1} := \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1}t_{21} & t_{22} \end{bmatrix} \in \text{GL}_2(R), \quad f_{mn}, t_{mn} \in R.$$

Thus

$$\begin{aligned} P'_S(P'_A)^{-1} &= F(P_S P_A^{-1})T^{-1} \\ &= \begin{bmatrix} f_{11} & f_{12} \\ \frac{\sigma_2}{\sigma_1}f_{21} & f_{22} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} \\ q_i p_{21} & p_{22} \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1}t_{21} & t_{22} \end{bmatrix}. \end{aligned}$$

Since $q_i \in \Sigma(\sigma_2) \setminus \Sigma(\alpha_1)$ and $\Sigma(\sigma_1) \subset \Sigma(\alpha_1)$, then $q_i \in \Sigma(\sigma_2) \setminus \Sigma(\sigma_1)$. Hence, $q_i \in \Sigma\left(\frac{\sigma_2}{\sigma_1}\right)$. As $q_i \notin \Sigma(\alpha_1)$ therefore $q_i \in \Sigma\left(\frac{\alpha_2}{\alpha_1}\right)$ and $q_i \in \Sigma\left(\frac{\sigma_2}{\sigma_1}\right) \cap \Sigma\left(\frac{\alpha_2}{\alpha_1}\right)$. Therefore

$$P'_S(P'_A)^{-1} = \begin{bmatrix} p'_{11} & p'_{12} \\ q_i p'_{21} & p'_{22} \end{bmatrix}, \quad p'_{mn} \in R.$$

Consequently, (5) holds regardless of the choice of P_S and P_A .

We now proceed in the same way with the remaining elements of the set $\{q_1, \dots, q_k\}$. As a result, the matrix P_S takes the form described in Theorem 2 (ii).

Sufficiency. Let $S = LM$, in which the nontrivial divisor $L := P_L^{-1} \cdot \text{diag}(\lambda_1, \lambda_2) \cdot Q_L^{-1}$ has the form (2).

Case 1. If $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1)$, then $\Sigma(\lambda_2) \subseteq \Sigma(\alpha_1)$ by Fact 2. This yields $(\alpha_2, \lambda_2, \alpha_1) \neq 1$, so $(A, L)_l \neq I$ by Fact 1 (ii) for arbitrary $P_S \in \text{GL}_2(R)$.

Case 2. Let $\Sigma(\sigma_2) \subseteq \Sigma(\alpha_1) \cup \{q_1, \dots, q_k\}$ for $k \geq 1$ and $q_i \notin \Sigma(\alpha_1)$ for each $i = 1, \dots, k$. If $1 \neq \gamma \in \Sigma(\lambda_2) \cap \Sigma(\alpha_1)$, then $L = L_1 L_2$, where

$$L_1 := P_L^{-1} \cdot \text{diag}(1, \gamma) \quad \text{and} \quad L_2 := \text{diag}\left(\lambda_1, \frac{\lambda_2}{\gamma}\right) \cdot Q_L^{-1}.$$

According to the above considerations, L_1 is a left divisor of A .

Using $(\alpha_2, \gamma, \alpha_1) \neq 1$ and Fact 1 (ii) we have $(A, L_1)_l \neq I$. The element $\det(L_1)$ is indecomposable in R , so $A = L_1 A_1$ by Lemma 1 and $(A, L)_l \neq I$.

Suppose $\delta \in \{q_1, \dots, q_k\} \cap \Sigma(\lambda_2)$. It is easy to see that $L = F_1 F_2$, where

$$F_1 := P_L^{-1} \cdot \text{diag}(1, \delta) \quad \text{and} \quad F_2 := \text{diag}\left(\lambda_1, \frac{\lambda_2}{\delta}\right) \cdot Q_L^{-1}.$$

The set of all left divisors of S with Smith normal form $\text{diag}(1, \delta)$ (see Fact 2) is given by

$$\mathbf{W} := \left\{ (\mathbf{L}_{\sigma_1, \delta} P_S)^{-1} \cdot \text{diag}(1, \delta) \cdot \text{GL}_2(R) \right\}.$$

Since $(\delta, \sigma_1) = 1$, any matrix $D \in \mathbf{W}$ can be written in the form $D = P_D^{-1} \cdot \text{diag}(1, \delta) \cdot Q_D^{-1}$, where $P_D = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix} P_S$, and $Q_D \in \text{GL}_2(R)$. Consequently, we have

$$P_D P_A^{-1} = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix} P_S P_A^{-1} = \begin{bmatrix} l_{11} & l_{12} \\ \delta l_{12} & l_{22} \end{bmatrix} \cdot \begin{bmatrix} m_{11} & m_{12} \\ q_1 \cdots q_k m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} l'_{11} & l'_{12} \\ \delta l'_{12} & l'_{22} \end{bmatrix},$$

so $P_D = \begin{pmatrix} l'_{11} & l'_{12} \\ \delta l'_{12} & l'_{22} \end{pmatrix} P_A$. Therefore $A = D A_2$ by Fact 2. It follows that each left divisor D of S with $\text{SNF}(D) = \text{diag}(1, q_i)$ for $i = 1, \dots, k$ (including L_1) is a left divisor of the matrix A too. Consequently, $F_1 := P_L^{-1} \cdot \text{diag}(1, \delta)$ is a left divisor of A . It means that $(A, L)_l \neq I$. \square

Let A and B be nonsingular matrices. We study the properties and structure of the left divisors of B that have a nontrivial left common divisor with A .

Lemma 2. *Let R be a commutative PID and let A, S, T be nonsingular matrices in $R^{2 \times 2}$. If all left divisors of S have a common left divisor with A , then*

$$\Sigma(S) \subseteq \Sigma((A, ST)_l).$$

Proof. Let $ST := P_{ST}^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_{ST}^{-1}$ and $S = P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1}$ have form (2). Let $\mu \in \Sigma(S)$. Thus $\sigma_2 = \mu \sigma'_2$ and $S = S_1 S_2$, where

$$S_1 := P_S^{-1} \cdot \text{diag}(1, \mu) \quad \text{and} \quad S_2 := \text{diag}(\sigma_1, \sigma'_2) \cdot Q_S^{-1}.$$

By assumption, $(A, S_1)_l \neq I$. Since $\det(S_1)$ is an indecomposable element of R , it follows from Lemma 1 that S_1 is a left divisor of A . Hence, S_1 is a left common divisor of the matrices A and ST , and thus a left divisor of $(A, ST)_l$. Consequently, $\mu = \Sigma(S_1) \subseteq \Sigma((A, ST)_l)$, and therefore $\Sigma(S) \subseteq \Sigma((A, ST)_l)$. \square

Lemma 3. *Let R be a commutative PID and let $A, B, S \in R^{2 \times 2}$ be nonsingular matrices of the form (2), namely*

$$\begin{aligned} A &:= P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1}, & B &:= P_B^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_B^{-1}, \\ S &:= P_S^{-1} \cdot \text{diag}(\sigma_1, \sigma_2) \cdot Q_S^{-1}, & \text{and } [\tau_{ij}] &:= P_B P_A^{-1}. \end{aligned}$$

Each left divisor of the matrix S has a nontrivial left common divisor with A and $B = ST$ if and only if S satisfies the conditions of Theorem 2 and

$$\left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) \mid \tau_{21}, \quad (6)$$

where $\sigma_2 = q_1^{r_1} \cdots q_k^{r_k} d_2$ for $q_1, \dots, q_k \in \Sigma(\sigma_2) \setminus \Sigma(\alpha_1)$, $r_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, k$, and $\Sigma(d_2) \subseteq \Sigma(\alpha_1)$.

Proof. Necessity. Since S is a left divisor of B , $\Sigma(\sigma_i) \subseteq \Sigma(\beta_i)$ for $i = 1, 2$ and $P_S = L P_B$, where

$$L := \begin{bmatrix} l_{11} & l_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} l_{21} & l_{22} \end{bmatrix}, \quad l_{ij} \in R,$$

by Fact 2. Each left divisor of S has a left common divisor with A , so S satisfies the conditions of Theorem 2. Hence $P_S = NP_A$, where

$$N := \begin{bmatrix} n_{11} & n_{12} \\ q_1 \cdots q_k n_{21} & n_{22} \end{bmatrix}, \quad n_{ij} \in R.$$

Consequently, $P_S = NP_A = LP_B$. It follows that

$$\begin{aligned} [\tau_{ij}] &= P_B P_A^{-1} = L^{-1} N = \underbrace{\begin{bmatrix} l'_{11} & l'_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} l'_{21} & l'_{22} \end{bmatrix}}_{L^{-1}} \begin{bmatrix} n_{11} & n_{12} \\ q_1 \cdots q_k n_{21} & n_{22} \end{bmatrix} \\ &= \begin{bmatrix} m_{11} & m_{12} \\ \left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) m_{21} & m_{22} \end{bmatrix}, \quad l'_{ij}, m_{ij} \in R. \end{aligned}$$

Therefore, the condition (6) is fulfilled.

Sufficiency. There exist invertible matrices (see [14, Lemma 5.10, p. 193])

$$C^{-1} := \begin{bmatrix} c_{11} & c_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} c_{21} & c_{22} \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} d_{11} & d_{12} \\ q_1 \cdots q_k d_{21} & d_{22} \end{bmatrix},$$

such that $P_B P_A^{-1} = C^{-1} D$. The matrix $S := (C P_B)^{-1} \cdot \text{diag}(\sigma_1, \sigma_2)$ is a left divisor of B by Fact 2. Moreover, each left divisor of $S = (D P_A)^{-1} \cdot \text{diag}(\sigma_1, \sigma_2)$ has a nontrivial left common divisor with A by Theorem 2.

Let us show that (6) holds independently of the choices of $P_B, P_A \in \text{GL}_2(R)$. Indeed, if we choose a different ordered pair $(P'_B, P'_A) \neq (P_B, P_A)$, then $P'_B = H P_B$ and $P'_A = T P_A$ by [14, Property 2.2, p. 61], where

$$H := \begin{bmatrix} h_{11} & h_{12} \\ \frac{\beta_2}{\beta_1} h_{21} & h_{22} \end{bmatrix} \quad \text{and} \quad T^{-1} := \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1} t_{21} & t_{22} \end{bmatrix}, \quad h_{ij}, t_{ij} \in R.$$

Thus

$$\begin{aligned} [\tau'_{ij}] &:= P'_B (P'_A)^{-1} = H P_B P_A^{-1} T^{-1} = H [\tau_{ij}] T^{-1} \\ &= \begin{bmatrix} h_{11} & h_{12} \\ \frac{\beta_2}{\beta_1} h_{21} & h_{22} \end{bmatrix} \cdot \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ \frac{\alpha_2}{\alpha_1} t_{21} & t_{22} \end{bmatrix}. \end{aligned}$$

Hence,

$$\tau'_{21} = \tau_{21} (h_{22} t_{11}) + \frac{\beta_2}{\beta_1} (h_{21} \tau_{11} t_{11} + \frac{\alpha_2}{\alpha_1} h_{21} \tau_{12} t_{21}) + \frac{\alpha_2}{\alpha_1} (h_{22} \tau_{22} t_{21}).$$

Obviously, $\frac{\beta_2(\sigma_2, \beta_1)}{\beta_1 \sigma_2} = \frac{(\beta_2 \sigma_2, \beta_2 \beta_1)}{\beta_1 \sigma_2} \in R$, so $\frac{\sigma_2}{(\sigma_2, \beta_1)} \mid \frac{\beta_2}{\beta_1}$. Taking into account that $q_1, \dots, q_k \in \Sigma(\alpha_2)$ and $(q_1 \cdots q_k, \alpha_1) = 1$, we obtain that $(q_1 \cdots q_k) \mid \frac{\alpha_2}{\alpha_1}$, and $\left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) \mid \left(\frac{\beta_2}{\beta_1}, \frac{\alpha_2}{\alpha_1} \right)$. Consequently, $\left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) \mid \tau'_{21}$. \square

Proof of Theorem 1. If $(A, B)_I = I$, then B is adequate to A .

Let $(A, B)_I \neq I$, where $A := P_A^{-1} \cdot \text{diag}(\alpha_1, \alpha_2) \cdot Q_A^{-1}$ and $B = P_B^{-1} \cdot \text{diag}(\beta_1, \beta_2) \cdot Q_B^{-1}$ have the form (2). Set $\text{SNF}((A, B)_I) := \text{diag}(\omega_1, \omega_2)$ and $[\tau_{ij}] := P_B P_A^{-1}$.

Due to Lemma 2, if D is a left divisor of B and none of its left divisors is relatively prime to A , then $\Sigma(D) \subseteq \Sigma((A, B)_l)$. By Fact 1 (i), we have $\Sigma(\omega_i) \subseteq \Sigma(\alpha_i)$ for $i = 1, 2$. Set

$$\begin{aligned}\Sigma(\omega_1) &= \Sigma((\alpha_1, \beta_1)) := \{p_1, \dots, p_m\}; \\ \Sigma(\omega_2) &:= \{p_1, \dots, p_n\} \cup \{q_1, \dots, q_l\} \cup \{q_{l+1}, \dots, q_k\},\end{aligned}$$

where

$$\begin{aligned}\{p_1, \dots, p_n\} &\subseteq \Sigma(\alpha_1), \quad n \geq m, \quad q_i \notin \Sigma(\alpha_1), \quad i = 1, \dots, k, \\ \{q_1, \dots, q_l\} &\subseteq \Sigma(\tau_{21}), \quad \{q_{l+1}, \dots, q_k\} \cap \Sigma(\tau_{21}) = \emptyset.\end{aligned}$$

By Fact 1 (i), we have $\omega_i \mid \beta_i$ for $i = 1, 2$, so we can write

$$\begin{aligned}\beta_1 &= \underbrace{(p_1^{r_1} \cdots p_m^{r_m})}_{\sigma_1} \cdot (q_1^{u_1} \cdots q_l^{u_l}) \cdot (q_{l+1}^{u_{l+1}} \cdots q_k^{u_k}) \cdot d = \sigma_1 \cdot \beta'_1, \quad u_i \in \mathbb{N} \cup \{0\}, \\ \beta_2 &= \underbrace{\left((p_1^{r'_1} \cdots p_m^{r'_m}) \cdot (p_{m+1}^{r_{m+1}} \cdots p_n^{r_n}) \cdot (q_1^{u'_1} \cdots q_l^{u'_l}) \cdot (q_{l+1}^{u_{l+1}} \cdots q_k^{u_k}) \right)}_{\sigma_2} \beta'_2 = \sigma_2 \cdot \beta'_2,\end{aligned}$$

where $(d, \alpha_2) = 1$, $(\beta'_2, p_1 \cdots p_n \cdot q_1 \cdots q_l) = 1$, $r'_i \geq r_i$, for $i = 1, \dots, m$ and $u'_j \geq u_j \geq 0$ for $j = 1, \dots, l$. It follows that

$$\frac{\sigma_2}{(\sigma_2, \beta_1)} = (p_1^{r'_1 - r_1} \cdots p_m^{r'_m - r_m}) \cdot (p_{m+1}^{r_{m+1}} \cdots p_n^{r_n}) \cdot (q_1^{u'_1 - u_1} \cdots q_l^{u'_l - u_l}).$$

Since $q_1, \dots, q_l \in \Sigma(\tau_{21})$, we get

$$\left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 \cdots q_k \right) = (q_1^{u'_1 - u_1} \cdots q_l^{u'_l - u_l}, q_1 \cdots q_k) \mid \tau_{21}.$$

According to [14, Lemma 5.10, p. 193], we can write

$$P_B P_A^{-1} = \begin{bmatrix} f_{11} & f_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} \\ q_1 \cdots q_k l_{21} & l_{22} \end{bmatrix}, \quad f_{ij}, l_{ij} \in R. \quad (7)$$

Let us consider the matrix

$$S := \left(\begin{bmatrix} f_{11} & f_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} f_{21} & f_{22} \end{bmatrix}^{-1} P_B \right)^{-1} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}. \quad (8)$$

Using Fact 2, S is the left divisor of B , i.e. $B = ST$ for some $T \in R^{2 \times 2}$. From (7), we have

$$\begin{bmatrix} f_{11} & f_{12} \\ \frac{\sigma_2}{(\sigma_2, \beta_1)} f_{21} & f_{22} \end{bmatrix}^{-1} P_B = \begin{bmatrix} l_{11} & l_{12} \\ q_1 \cdots q_k l_{21} & l_{22} \end{bmatrix} P_A.$$

It follows that the matrix S can also be written in the following form

$$S = \left(\begin{bmatrix} l_{11} & l_{12} \\ q_1 \cdots q_k l_{21} & l_{22} \end{bmatrix} P_A \right)^{-1} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}.$$

Consequently, each left divisor of S has a nontrivial common left divisor with A by Theorem 2. Therefore, S satisfies part (i) of the definition of an adequate part of B with respect to A .

Assume that $T = T_1 T_2$ is a decomposition of T into a product of two of its nontrivial divisors. Let us consider the following two cases.

Case 1. Let $\Sigma(ST_1) \not\subseteq \Sigma((A, B)_l) \neq \emptyset$. Hence, there exists $t \in \Sigma(ST_1) \setminus \Sigma((A, B)_l)$. It means that ST_1 has a left divisor L with $\text{SNF}(L) = \text{diag}(1, t)$, such that $(A, L)_l = I$ (by the same trick as the one used in the proof of Lemma 2).

Case 2. Let $\Sigma(ST_1) \subseteq \Sigma((A, B)_l)$ and $\text{SNF}(ST_1) = \text{diag}(\mu_1, \mu_2)$. Based on the construction of the elements σ_1 and σ_2 , it follows that $\det(ST_1)$ has the divisor $q_i^{u_i+1}$ in which $l + 1 \leq i \leq k$.

Case 2a. Let $q_i | \mu_1$. Any matrix with the Smith normal form $\text{diag}(q_i, q_i)$ is a left divisor of ST_1 by [14, Theorem 5.3 p. 152 and Property 4.11 p. 147]. Consider the matrix $M := P_M^{-1} \text{diag}(q_i, q_i)$, where $P_M := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P_A$. It is obvious that $M = M_1 M_2$, where $M_1 := P_M^{-1} \text{diag}(1, q_i)$ and $M_2 := \text{diag}(q_i, 1)$. Since $(\alpha_1, q_i) = 1$ and $P_M P_A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $(A, M_1)_l = I$ by Fact 1 (ii). Thus $ST_1 = MN = M_1(M_2 N)$ for some N .

Case 2b. Let $(q_i, \mu_1) = 1$. Clearly $q_i^{u_i+1} | \mu_2$. The matrix $K := P_{ST_1}^{-1} \text{diag}(1, q_i)$ is a left divisor of ST_1 , and therefore also a left divisor of the matrix B . Since $\frac{q_i^{u_i+1}}{(q_i^{u_i+1}, \beta_1)} = q_i$, we have

$P_{ST_1} := \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix} P_B$ by Fact 2. Thus

$$\begin{aligned} [\tau'_{ij}] &:= P_{ST_1} P_A^{-1} = \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix} (P_B P_A^{-1}) = \begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ q_i k_{21} \tau_{11} + k_{22} \tau_{21} & * \end{bmatrix}. \end{aligned}$$

The matrix $\begin{bmatrix} k_{11} & k_{12} \\ q_i k_{21} & k_{22} \end{bmatrix}$ is invertible. Hence, $(q_i, k_{22}) = 1$. By assumption, $(q_i, \tau_{21}) = 1$, so $(q_i, \tau'_{21}) = 1$. Since $q_i \notin \Sigma(\alpha_1)$, we have $(q_i, \alpha_1) = 1$, so $(\alpha_2, q_i^{u_i+1}, \alpha_1 \tau'_{21}) = 1$. Consequently, $(A, K)_l = I$ by Fact 1 (ii). Thus S satisfies condition (ii) in the definition of an adequate part of B with respect to A , and therefore the set of nonsingular 2×2 matrices over R is a left adequate. Applying the transpose operator, we obtain that this set is also a right adequate set. Hence, the set of all nonsingular 2×2 matrices over R is an adequate set. \square

We illustrate with an example the calculation of the adequate part.

Example 1. Let R be a PID and let $a, b, c, f, m, n \in R \setminus \{U(R) \cup \{0\}\}$ be pairwise relatively prime indecomposable elements. Let

$$\begin{aligned} A &:= \text{diag}(ab, ab^2cfm), & B &:= \begin{bmatrix} 1 & 0 \\ -f & 1 \end{bmatrix} \text{diag}(b^2c, ab^3c^2fn), \\ P_A &= I, & P_B &= \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}, & [\tau_{ij}] &:= P_B P_A^{-1} = P_B = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}. \end{aligned}$$

Clearly, $\Sigma(A) = \{a, b, c, f, m\}$, $\Sigma(B) = \{a, b, c, f, n\}$ and $\text{SNF}((A, B)_l) = \text{diag}(b, ab^2cf)$ by Fact 1 (i). Using the notation of Theorem 1, we have that $q_1 q_2 = cf$. An adequate part of B with

respect to A (see Theorem 1) has the Smith normal form $\text{diag}(b^2, ab^3cf) := \text{diag}(\sigma_1, \sigma_2)$. Note that

$$\left(\frac{\sigma_2}{(\sigma_2, \beta_1)}, q_1 q_2 \right) = (abf, cf) = f | \tau_{21}.$$

It is easy to check that

$$P_B P_A^{-1} = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ abfy & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ cfx & 1 \end{bmatrix},$$

in which $cx + aby = 1$. It follows that

$$\begin{bmatrix} 1 & 0 \\ -abfy & 1 \end{bmatrix} P_B = \begin{bmatrix} 1 & 0 \\ cfx & 1 \end{bmatrix} P_A.$$

Consequently, an adequate part of B with respect to A has the following form

$$\begin{aligned} S &:= \left(\begin{bmatrix} 1 & 0 \\ -abfy & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3cf \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ f(aby - 1) & 1 \end{bmatrix} \begin{bmatrix} b^2 & 0 \\ 0 & ab^3cf \end{bmatrix} = \begin{bmatrix} b^2 & 0 \\ b^2 f(aby - 1) & ab^3cf \end{bmatrix} \end{aligned}$$

by Theorem 1. In this case $B = ST$, where $T = \begin{bmatrix} c & 0 \\ -y & cn \end{bmatrix}$.

3 Adequate part not defined up to strong association

As noted in the Introduction, every commutative PID R is adequate in the sense of Helmer. It is straightforward to verify that the adequate and relatively prime parts of an element $0 \neq b \in R$ with respect to $a \in R$ are defined up to strong association. However, this statement does not hold in the case of the ring $R^{2 \times 2}$, as shown in the following example.

Example 2. Let $R = \mathbb{Z}$ be the ring of integers. Let

$$A := \text{diag}(\alpha_1, \alpha_2) = \text{diag}(2, 2 \cdot 3 \cdot 5 \cdot 7), \quad B := \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \cdot \text{diag}(2 \cdot 3^2 \cdot 5^2, 2^2 \cdot 3^3 \cdot 5^4).$$

Then

$$P_A = I, \quad P_B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad [\tau_{ij}] := P_B P_A^{-1} = P_B,$$

$$\Sigma(\alpha_1) = \{2\}, \quad \Sigma(\omega_2) = \{2, 3, 5\}, \quad \Sigma(\tau_{21}) = \{1, 3\}, \quad \{5\} \cap \Sigma(\tau_{21}) = \emptyset.$$

According to Fact 1 (i), $\text{SNF}((A, B)_l) = \text{diag}(\omega_1, \omega_2) = \text{diag}(2, 2 \cdot 3 \cdot 5)$. The left adequate part of B with respect to A has the following Smith normal form $\Phi := \text{diag}(2, 2^2 \cdot 3^3 \cdot 5^2)$ (see the proof of Theorem 1). The matrices

$$S := \begin{bmatrix} 1 & 0 \\ -3 \cdot 5 & 1 \end{bmatrix} \cdot \Phi \quad \text{and} \quad S_1 := \begin{bmatrix} 1 & 0 \\ 3 \cdot 5 & 1 \end{bmatrix} \cdot \Phi$$

are left divisors of the matrix

$$B = S \begin{bmatrix} 3^2 \cdot 5^2 & 0 \\ 2 & 5^2 \end{bmatrix} = S_1 \begin{bmatrix} 3^2 \cdot 5^2 & 0 \\ -3 & 5^2 \end{bmatrix},$$

and are also adequate parts of B with respect to A by Theorem 2. However (see [14, Theorem 4.5, p. 128]) the matrices S and S_1 are not right strong associates.

Let S be an adequate part of B with respect to A with the presentation (8). Example 2 shows that if S' is another adequate part of B with respect to A , then S and S' are not necessarily right associated. Based on this example, we put forward the following.

Hypothesis. *The adequate part of B with respect to A is defined up to equivalence.*

4 Adequate rings in the sense of Gatalevych

A.I. Gatalevych [9, Definition 1, p. 116] proposed the following definition for noncommutative Bézout rings, which was already indicated in the Introduction.

Let K be a Bézout ring and let $a \in K$. An element $b \in K$ is called *left adequate in the sense of Gatalevych* to $a \in K$ if the following conditions hold:

- (i) there exist elements $s, t \in K$ such that $b = st$ and $tK + aK = K$;
- (ii) $s'K + aK \neq K$ for each $s' \in K \setminus U(K)$ such that $sK \subset s'K \neq K$.

The shortcomings of definition are demonstrated by the following example.

Example 3. *Let R be a commutative PID, and let $a, d, c \in R \setminus \{U(R) \cup \{0\}\}$ be pairwise relatively prime indecomposable elements. Let*

$$A := \text{diag}(a, a^2dc), \quad B := \begin{bmatrix} 1 & 0 \\ d & d^3c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & d^3c^2 \end{bmatrix}, \quad P_B = \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix},$$

$$A_1 := \text{diag}(a, a^2c), \quad P_A = I, \quad T := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \text{diag}(1, d^2c^2), \quad S := \text{diag}(1, d).$$

It is easy to check that $A = SA_1$ and $B = ST$. Since $(A, T)_l = I$ (see Fact 1 (ii)), the decomposition $B = ST$ satisfies the definition of Gatalevych.

On the other hand,

$$B = S_1T_1 = (P_{S_1}^{-1} \cdot \text{diag}(1, d^3) \cdot Q_{S_1}^{-1}) \cdot (P_{T_1}^{-1} \cdot \text{diag}(1, c^2) \cdot Q_{T_1}^{-1}),$$

where

$$S_1 = \begin{bmatrix} 1 & 0 \\ d^3 + d & d^3 \end{bmatrix}, \quad P_{S_1} = \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix}, \quad Q_{S_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ -1 & c^2 \end{bmatrix}, \quad P_{T_1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad Q_{T_1} = I.$$

Each left divisor of S_1 has a nontrivial left common divisor with A by Theorem 2 and $(A, T_1)_l = I$ by Fact 1 (ii), so the decomposition $B = S_1T_1$ also satisfies Gatalevych's definition. However, S is the left divisor of S_1 , because $S_1 = S \begin{bmatrix} 1 & 0 \\ 1 & d^2 \end{bmatrix}$.

Note that the decompositions $B = ST$ and $B = S_1T_1$ also have an additional undesirable property. Let us consider the cosets $SGL_2(R)$ and $S_1GL_2(R)$, i.e. the sets of all right strongly associated matrices to the matrices S and S_1 , respectively. According to Fact 1 (ii), each left divisor of the matrices from $SGL_2(R)$ and $S_1GL_2(R)$ has a nontrivial left common divisor with the matrix A . However, if $U, V \in GL_2(R)$ and

$$B = (SU)(U^{-1}T) = (S_1V)(V^{-1}T_1),$$

then it does not necessarily follow that $(A, U^{-1}T)_l = I$ and $(A, V^{-1}T_1)_l = I$. Indeed, if

$$U := \begin{bmatrix} 1 & 0 \\ 1-d & 1 \end{bmatrix} \quad \text{and} \quad V := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

then

$$T' := U^{-1}T = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \cdot \text{diag}(1, d^2c^2) \quad \text{and} \quad T'_1 := V^{-1}T_1 = \text{diag}(1, c^2).$$

It is easy to see that $(A, T')_l \neq I$ and $(A, T'_1)_l \neq I$.

Remark. The impetus for the research presented in this article was the communication with Professor Bohdan Zabasky, whose scholarly have had a significant impact on the development of the theory of elementary divisor rings and, in particular, adequate rings.

Acknowledgments

The authors are grateful to the reviewer for his helpful comments and advice, which led to the improvement of the quality of our article.

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Received 31.05.2025

Revised 18.09.2025

Бовді В., Шедрик В. Адекватність неособливих матриць над комутативними областями головних ідеалів // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 236–249.

Поняття адекватності для комутативних областей було введено О. Гелмером у [Bull. Amer. Math. Soc. 1943, **49** (4), 225–236]. У цій статті ми поширюємо поняття адекватності на некомутивні кільця Безу. Показано, що множина неособливих матриць розміру 2×2 над комутативною областю головних ідеалів є адекватною.

Ключові слова і фрази: адекватне кільце, область головних ідеалів, дільник матриці.