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Approximation of isometric function spaces by entire functions of exponential type

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In this article, sufficient conditions are established for mappings by convolution operators of spaces of function defined on the n-dimensional Euclidean space into spaces of entire functions of exponential type. Integral representations for these operators are obtained. Equalities of approximation characteristics in isometric functional spaces of many variables are obtained using equalities of approximation characteristics in isometric function spaces of one variable.

Key words and phrases: isometric space, linear operator, approximation characteristic, convolution space, delta-like kernel, space of entire functions of exponential type.

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1 Introduction

In the paper [10], some subspaces of real functions of $n+k$ variables were constructed that are isometric to spaces of real functions defined on the n-dimensional Euclidean space. Since isometry of functional spaces with different numbers of variables is a rare phenomenon and was previously known only for spaces of complex-valued functions (see [10, p. 1027]), we consider its applications.

In papers [1, 12], subspaces of solutions to the Laplace and heat equations were found, being isometric to spaces of real functions of one variable.

To construct subspaces of solutions to differential equations and their systems that are isometric to spaces of real functions, it was necessary to establish conditions for the convergence of the convolution of the function with a delta-like kernel to this function that was done in [11].

In paper [13], the definitions of the main approximation characteristics were formulated, examples of their equality for isometric mappings of spaces of real functions of $n+m$ variables into spaces of real 2π-periodic functions in each of the n variables were presented, and some examples of their applications in the theory of function approximation were given.

In the paper [9], the results of work [13] were extended to isometric mappings in spaces of non-periodic functions. Integral representations were found for function spaces isometric to spaces of entire functions of exponential type, which are necessary for establishing the equality of approximation characteristics in isometric function spaces. The topic related to the study of approximation characteristics in spaces of entire functions and vectors of exponential type is

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relevant and is developing intensively (see, e.g., [15,16,25]). Problems of approximation theory also play an important role in applied research (see, e.g., [3,5,18–20,26]).

2 Main notations, definitions and statements

Let us denote by C^n, L^n_∞, $L^n_{\bar p}$, $\hat L^n_{\bar p}$, $\bar p = (p_1, \dots, p_n)$, the spaces of real functions defined on E^n that are continuous, bounded, essentially bounded and measurable, respectively, with the following norms:

$$\|f\|_{C^n} = \sup_{x\in E^n} |f(x)|,$$

$$\|f\|_{L^n_\infty} = \sup_{x\in E^n} \operatorname{vrai} |f(x)|,$$

$$\|f\|_{L^n_{\bar p}} = \Big\| \dots \big\| \|f(x_1,\dots,x_n)\|_{p_1,x_1} \big\|_{p_2,x_2} \dots \Big\|_{p_n,x_n}$$
$$= \left(\int_{-\infty}^{\infty} \left(\dots \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x_1,x_2,\dots,x_n)|^{p_1}\, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \dots \right)^{\frac{p_n}{p_{n-1}}} dx_n \right)^{\frac{1}{p_n}},$$

$$\| f\|_{\hat L^n_{\bar p}} = \Big\| \dots \big\| \|f(x_1,\dots,x_n)\|_{\hat p_1,x_1} \big\|_{\hat p_2,x_2} \dots \Big\|_{\hat p_n,x_n}$$
$$= \sup_{a_n\in E} \left(\int\limits_{a_n}^{a_n+2\pi} \left(\dots \sup_{a_2\in E} \left(\int\limits_{a_2}^{a_2+2\pi} \left(\sup_{a_1\in E} \int\limits_{a_1}^{a_1+2\pi} |f(x_1,\dots,x_n)|^{p_1}\, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \dots \right)^{\frac{p_n}{p_{n-1}}} dx_n \right)^{\frac{1}{p_n}}.$$

Let X^n be one of the spaces C^n, L^n_∞, $L^n_{\bar p}$, $\hat L^n_{\bar p}$ with $\bar p = (p_1,\dots,p_n)$, $1 \le p_i < \infty$, $i = 1,\dots,n$. Let

$$F^{X^n}_{\bar\sigma} = \left\{ T_{\bar\sigma^n}(z) = T_{\bar\sigma^n}(x_1 + it_1, \dots, x_n + it_n) : T_{\bar\sigma^n}(x) \in X^n \right\}$$

be the space of entire functions of exponential type not larger than $\bar\sigma^n = (\sigma_1,\dots,\sigma_n)$ (see [23, pp. 118, 119]), which belong to the space X^n on the real n-dimensional Euclidean space E^n. Let $X^n \supset F^{X^n}_{\bar\sigma}(E^n) = \left\{ T_{\bar\sigma^n}(x) : T_{\bar\sigma^n}(z) \in F^{X^n}_{\bar\sigma} \right\}$ be their restriction to the space E^n.

Let the inequality $\bar y = (y_1,\dots,y_m) \ge \bar 0 = (0,\dots,0)$ mean that the coordinates of the vector $\bar y$ are non-negative, and $\bar y > \bar 0$ mean that at least one of them is positive. Denote by

$$\Pi^+_{n,m} = \{(x,y) = (\bar x, \bar y) = (x_1,\dots,x_n,y_1,\dots,y_m) \in E^{n+m} : \bar y > \bar 0\},$$

$$\bar\Pi^+_{n,m} = \{(x,y) \in E^{n+m} : \bar y \ge \bar 0\}$$

the subsets of the real $(n+m)$-dimensional Euclidean space E^{n+m}.

Let $I^n_{\bar y^m}(x)$ be delta-like kernels (see [10, equation (9), (10)]), defined on $\Pi^+_{n,m}$. Let

$$\{F^{X^n}_{\bar\sigma} * I^n_{\bar y^m}\} = \left\{ (T_{\bar\sigma^n} * I^n_{\bar y^m})(z) = \int_{E^n} I^n_{\bar y^m}(t) T_{\bar\sigma^n}(z-t)\, dt : (T_{\bar\sigma^n} \in F^{X^n}_{\bar\sigma}) \wedge (\bar y > \bar 0) \right\},$$

$$\{\overline{F^{X^n}_{\bar\sigma} * I^n_{\bar y^m}}\} = \left\{ IT_{\bar\sigma^n}(z,\bar y) = \begin{Bmatrix} (T_{\bar\sigma^n} * I^n_{\bar y^m})(z), & \bar y > \bar 0, \\ T_{\bar\sigma^n}(z), & \bar y = \bar 0 \end{Bmatrix} : T_{\bar\sigma^n} \in F^{X^n}_{\bar\sigma} \right\}$$

be convolution spaces of entire functions of exponential type not larger than $\bar\sigma^n$ with delta-like kernels, and let

$$\{F^{X^n}_{\bar\sigma}(E^n) * I^n_{\bar y^m}\} = \left\{ (T_{\bar\sigma^n} * I^n_{\bar y^m})(x) : (T_{\bar\sigma^n}(x) \in F^{X^n}_{\bar\sigma}(E^n)) \wedge (\bar y > \bar 0) \right\},$$

$$\{\overline{F_{\bar{\sigma}}^{X^n}(E^n) * I_{\bar{y}^m}^n}\} = \left\{ IT_{\bar{\sigma}^n}(x,y) = \begin{Bmatrix} (T_{\bar{\sigma}^n} * I_{\bar{y}^m}^n)(x), & \bar{y} > \bar{0}, \\ T_{\bar{\sigma}^n}(x), & \bar{y} = \bar{0} \end{Bmatrix} : T_{\bar{\sigma}^n}(x) \in F_{\bar{\sigma}}^{X^n}(E^n) \right\} \tag{1}$$

be their restrictions to E^n.

If $n = 1$, then the index n and the overlines above the vectors $\bar{\sigma}$, $\bar{p}$ etc. will be omitted.

Let us denote by

$$U_{\bar{\sigma}^n}(\Lambda, f, x) = (\hat{\lambda}_{\bar{\sigma}^n} * f)(x) = \int_{E^n} \hat{\lambda}_{\bar{\sigma}^n}(x - t) f(t)\, dt \tag{2}$$

a linear operator defined by convolution with the kernel $\hat{\lambda}_{\bar{\sigma}^n}(t)$.

We will determine sufficient conditions under which the operator $U_{\bar{\sigma}^n}(\Lambda, f, x)$ maps the space X^n into the subspace $F_{\bar{\sigma}}^{X^n}(E^n)$.

Let $\mathbb{Z}^n = \{\bar{m} = (m_1, \ldots, m_i, \ldots, m_n) : m_i \in \mathbb{Z}, i = 1, \ldots, n\}$ be the integer lattice in the space E^n, and let $Q_{\bar{m}}^n = [m_1, m_1 + 1] \times \cdots \times [m_n, m_n + 1] \subset E^n$ be the n-dimensional cube in E^n corresponding to the vector $\bar{m} \in \mathbb{Z}^n$. Let $x_{\bar{m}} \in Q_{\bar{m}}^n$ be an arbitrary element of this cube. The following statement holds.

Lemma 1. *If the function f, together with all its derivatives and mixed partial derivatives $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$, $k = 1, 2, \ldots, n$, of at most first order with respect to each variable, is absolutely integrable on E^n, then*

$$\sum_{\bar{m} \in Z^n} |f(x_{\bar{m}})| \leq \int_{E^n} |f(x)| dx + \sum_{k=1}^{n} \sum_{1 \leq m_1 < \cdots < m_k \leq n} \int_{E^n} \left| \frac{\partial^k f(x)}{\partial x_{m_1} \ldots \partial x_{m_k}} \right| dx. \tag{3}$$

Proof. Let us show that inequality (3) holds for $n = 1$. In this case, $Q_{\bar{m}}^1 = [m_1, m_1 + 1]$ and $x_{\bar{m}} = x^{m_1} \in [m_1, m_1 + 1]$, where $m_1 \in \mathbb{Z}$, and

$$\sum_{m_1 \in \mathbb{Z}} |f(x^{m_1})| \leq \int_{-\infty}^{\infty} |f(x)|\, dx + \int_{-\infty}^{\infty} |f'(x)|\, dx. \tag{4}$$

Since the function $f(x)$ is differentiable on the entire real axis, the function $|f(x)|$ is continuous on E, and

$$\min_{x \in [m_1, m_1 + 1]} |f(x)| = |f(y_{m_1})|, \tag{5}$$

where $y_{m_1} \in [m_1, m_1 + 1]$. Then, due to the differentiability of the function $f(x)$, and considering that both x^{m_1} and y_{m_1} belong to the segment $[m_1, m_1 + 1]$, using relation (5), we get

$$\begin{aligned} |f(x^{m_1})| &\leq |f(x^{m_1}) - f(y_{m_1})| + |f(y_{m_1})| \\ &= \int_{m_1}^{m_1+1} |f(y_{m_1})|\, dx + \left| \int_{x^{m_1}}^{y_{m_1}} f'(x)\, dx \right| \leq \int_{m_1}^{m_1+1} |f(x)|\, dx + \int_{m_1}^{m_1+1} |f'(x)|\, dx. \end{aligned} \tag{6}$$

From inequality (6), using the additive property of the integral and the equality

$$\bigcup_{m_1 \in \mathbb{Z}} [m_1, m_1 + 1] = (-\infty, +\infty),$$

inequality (4) follows.

Let us prove that inequality (3) holds for $n=2$. That is, $Q^2_{\bar{m}} = [m_1, m_1+1] \times [m_2, m_2+1]$, $x_{\bar{m}} = (x_1^{\bar{m}}, x_2^{\bar{m}}) \in Q^2_{\bar{m}}$, and

$$\sum_{\bar{m}\in\mathbb{Z}^2} |f(x_1^{\bar{m}}, x_2^{\bar{m}})| \le \int_{E^2} \left(|f(x_1,x_2)| + |f'_{x_1}(x_1,x_2)| + |f'_{x_2}(x_1,x_2)| + |f''_{x_1x_2}(x_1,x_2)|\right) dx_1 dx_2.$$

Since the functions $f(x)$ and $f'_{x_1}(x_1,x_2)$ are absolutely integrable on E^2, by Fubini theorem (see, e.g., [22, pp. 85, 86]), for each fixed x_2 these functions are absolutely integrable in the variable x_1 on the entire real axis. Therefore, using inequality (6), we obtain

$$|f(x_1^{\bar{m}}, x_2^{\bar{m}})| \le \int_{m_1}^{m_1+1} |f(x_1, x_2^{\bar{m}})|\, dx_1 + \int_{m_1}^{m_1+1} |f'_{x_1}(x_1, x_2^{\bar{m}})|\, dx_1. \tag{7}$$

Reasoning in the same way as in proving inequality (7), we establish

$$|f'_{x_1}(x_1, x_2^{\bar{m}})| \le \int_{m_2}^{m_2+1} |f'_{x_1}(x_1,x_2)|\, dx_2 + \int_{m_2}^{m_2+1} |f''_{x_1x_2}(x_1,x_2)|\, dx_2, \tag{8}$$

$$|f(x_1, x_2^{\bar{m}})| \le \int_{m_2}^{m_2+1} |f'_{x_2}(x_1,x_2)|\, dx_2 + \int_{m_2}^{m_2+1} |f(x_1,x_2)|\, dx_2. \tag{9}$$

From inequalities (7)–(9), it follows that

$$|f(x_1^{\bar{m}}, x_2^{\bar{m}})| \le \int_{m_1}^{m_1+1}\int_{m_2}^{m_2+1} \left(|f(x_1,x_2)| + |f'_{x_1}(x_1,x_2)| + |f'_{x_2}(x_1,x_2)| + |f''_{x_1x_2}(x_1,x_2)|\right) dx_1 dx_2.$$

Using the method of mathematical induction, we can prove that for any natural number n the following inequality

$$|f(x_{\bar{m}})| = |f(x_1^{\bar{m}}, \dots, x_n^{\bar{m}})| \le \int_{Q^n_{\bar{m}}} \left(|f(x)| + \sum_{k=1}^{n} \sum_{1\le m_1 < \dots < m_k \le n} \left| \frac{\partial^k f(x)}{\partial x_{m_1} \cdots \partial x_{m_k}} \right| \right) dx$$

holds, from which, using the additive property of the integral and the equality $\bigcup\limits_{\bar{m}} Q^n_{\bar{m}} = E^n$, inequality (3) follows. Lemma 1 is proved. ☐

From Lemma 1 and the Zygmund inequality (see, e.g., [23, pp. 138–140]), the following statement follows.

Corollary 1. *If $T_{\bar{\sigma}^n}(x)$ is an entire function of exponential type not larger than $\bar{\sigma}^n$ and is absolutely integrable on E^n, then*

$$\sum_{\bar{m}\in\mathbb{Z}^n} |T_{\bar{\sigma}^n}(x_{\bar{m}})| \le \left(1 + \sum_{k=1}^{n} \sum_{1\le m_1<\dots<m_k\le n} \sigma_{m_1}\cdots\sigma_{m_k}\right) \|T_{\bar{\sigma}^n}\|_{L_1^n}. \tag{10}$$

Proof. Using Zygmund inequality for entire functions of exponential type not larger than $\bar{\sigma}^n$, we obtain the inequalities:

$$\left| \frac{\partial^k T_{\bar{\sigma}^n}(x)}{\partial x_{m_1}\cdots\partial x_{m_k}} \right| \le \sigma_{m_1}\cdots\sigma_{m_k} \left|T_{\bar{\sigma}^n}(x)\right|,$$

$$\int_{E^n} \left| \frac{\partial^k T_{\bar{\sigma}^n}(x)}{\partial x_{m_1}\cdots\partial x_{m_k}} \right| dx \le \sigma_{m_1}\cdots\sigma_{m_k} \int_{E^n} \left|T_{\bar{\sigma}^n}(x)\right|\, dx = \sigma_{m_1}\cdots\sigma_{m_k} \|T_{\bar{\sigma}^n}\|_{L_1^n},$$

from which, together with Lemma 1, inequality (10) follows. ☐

We should note that inequality (10) for the case $x_{\bar{m}} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ follows from [23, Theorem 3.3.2].

Let us establish sufficient conditions under which the operator $U_{\bar{\sigma}^n}(\Lambda, f, z)$ maps the space X^n into the subspace $F_{\bar{\sigma}}^{X^n}$.

Theorem 1. *Let X^n be one of the spaces C^n, L_∞^n, $L_{\bar{p}}^n$, or $\hat{L}_{\bar{p}}^n$, where $\bar{p} = (p_1, \dots, p_n)$, $1 \le p_i < \infty$, $i = 1, \dots, n$. Let $\hat{\lambda}_{\bar{\sigma}^n}(z)$ be an entire function of exponential type not larger than $\bar{\sigma}^n$, belonging to the space $F_{\bar{\sigma}}^{1^n} := F_{\bar{\sigma}}^{L_1^n}$, and let $f \in X^n$. Then the function*

$$U_{\bar{\sigma}^n}(\Lambda, f, z) = (\hat{\lambda}_{\bar{\sigma}^n} * f)(z) = \int_{E^n} \hat{\lambda}_{\bar{\sigma}^n}(z - t) f(t)\, dt$$

is an entire function of exponential type not larger than $\bar{\sigma}^n$, which belongs to the space X^n, i.e. $U_{\bar{\sigma}^n}(\Lambda, f, z) \in F_{\bar{\sigma}}^{X^n}$.

Proof. For the spaces C^n and L_p^n, $1 \le p \le \infty$, Theorem 1 was established in [23, pp. 162, 163].

Let us prove that

$$X^n \subseteq \hat{L}^n := \hat{L}_1^n. \tag{11}$$

If $f \in \hat{L}_{\bar{p}}^n$ with $\bar{p} = (p_1, \dots, p_n)$, then using the definition of the norm in the space $\hat{L}_{\bar{p}}^n$ and Hölder's inequality, we obtain

$$\|f\|_{\hat{L}^n} \le \prod_{i=1}^{n} (2\pi)^{1/q_i} \|f\|_{\hat{L}_{\bar{p}}^n}, \tag{12}$$

where $\frac{1}{p_i} + \frac{1}{q_i} = 1$, i.e.

$$\hat{L}_{\bar{p}}^n \subseteq \hat{L}^n. \tag{13}$$

From the definitions of norms in the spaces C^n, L_∞^n, $\hat{L}_{\bar{p}}^n$ and L_p^n it follows that $C^n \subset L_\infty^n \subset L_{\bar{p}}^n$ and $L_{\bar{p}}^n \subset \hat{L}_{\bar{p}}^n$. Therefore, from relations (12) and (13) we obtain (11).

Since $\hat{\lambda}_{\bar{\sigma}^n}(z)$ is an entire function of exponential type not larger than $\bar{\sigma}^n$, we get the following Taylor series expansion (see [23, p. 163])

$$\begin{aligned} \hat{\lambda}_{\bar{\sigma}^n}(z + u) &= \hat{\lambda}_{\bar{\sigma}^n}(z_1 + u_1, \dots, z_n + u_n) \\ &= \sum_{\bar{k} \in \mathbb{Z}_+^n} \frac{\hat{\lambda}_{\bar{\sigma}^n}^{(\bar{k})}(u)}{\bar{k}!} \bar{z}^{\bar{k}} = \sum_{\bar{k} \in \mathbb{Z}_+^n} \frac{\partial^{k_1 + \dots + k_n} \hat{\lambda}_{\bar{\sigma}^n}(u)}{\partial u_1^{k_1} \cdots \partial u_n^{k_n} k_1! \cdots k_n!} z_1^{k_1} \cdots z_n^{k_n}, \end{aligned} \tag{14}$$

which converges absolutely for any $u \in E^n$ and $\bar{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, where $\mathbb{C}_n$ is the complex n-dimensional Euclidean space, $\mathbb{Z}_+^n = \{\bar{k} = (k_1, \dots, k_n) : k_i \in \mathbb{Z}, k_i \ge 0, i = 1, \dots, n\}$ is the integer lattice in E^n with non-negative coordinates and $\bar{z}^{\bar{k}} := z_1^{k_1} \cdots z_n^{k_n}$.

Using equality (14) and a change of variables, we obtain

$$\begin{aligned} |(\hat{\lambda}_{\bar{\sigma}^n} * f)(z)| &\le \int_{E^n} |\hat{\lambda}_{\bar{\sigma}^n}(z - u) f(u)| du \\ &= \int_{E^n} |\hat{\lambda}_{\bar{\sigma}^n}(z + u) f(-u)| du \le \sum_{\bar{k} \in \mathbb{Z}_+^n} \int_{E^n} \left| \hat{\lambda}_{\bar{\sigma}^n}^{(\bar{k})}(u) f(-u) \right| du \frac{|\bar{z}^{\bar{k}}|}{\bar{k}!}. \end{aligned} \tag{15}$$

Let us denote by $Q_1^n = [0, 1]^n \subset E^n$ and $\pi^n = [-\pi, \pi)^n \subset E^n$ the cubes in the space E^n with side lengths 1 and 2π, respectively. Let $[0, 1]^n + \bar{m} = Q_{\bar{m}}^n$ and $\pi^n - 2\pi\bar{m} = \pi^n(\bar{m})$ denote the shifts of these cubes by the vectors $\bar{m} \in \mathbb{Z}^n$ and $2\pi\bar{m}$, respectively. Since the coordinates of the vectors $\bar{m}$ and $2\pi\bar{m}$ are multiples of the side lengths of the corresponding cubes, different cubes from the families $Q^n_{\bar{m}}$ and $\pi^n(\bar{m})$ do not have common interior points. Therefore, the following relations hold:

$$\bigcup_{\bar{m}\in\mathbb{Z}^n} Q^n_{\bar{m}} = E^n = \bigcup_{\bar{m}\in\mathbb{Z}^n} \pi^n(\bar{m}) \quad \text{and} \quad \mathring{\pi}^n(\bar{m}_1) \cap \mathring{\pi}^n(\bar{m}_2) = \varnothing = \mathring{Q}^n_{\bar{m}_1} \cap \mathring{Q}^n_{\bar{m}_2} \quad \forall \bar{m}_1 \neq \bar{m}_2, \tag{16}$$

where $\mathring{A}$ is the set of all interior points of the set A.

Since $6 < 2\pi < 7$, it follows from relations (16) that each cube $\pi^n(\bar{m})$ contains exactly 6^n cubes $Q^n_{\bar{m}_1}$, and each point of the cube $\pi^n(\bar{m})$ lies in some cube $Q^n_{\bar{m}_2}$. Therefore, the maximum value of the function defined on the cube $\pi^n(\bar{m})$ is less than the sum of all maximum values of this function on the cubes that intersect with this cube. Taking this into account, and using the definition of the norm in the space $\hat{L}^n$ and relations (16), for any function $f \in \hat{L}^n$ we have

$$\begin{aligned}\int_{E^n} \left|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u) f(-u)\right| du &\leq \sum_{\bar{m}\in\mathbb{Z}^n} \max_{u\in\pi^n(\bar{m})} \left|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u)\right| \int_{\pi^n(\bar{m})} |f(-u)|\, du \\ &< \|f(-u)\|_{\hat{L}^n} \sum_{\bar{m}\in\mathbb{Z}^n} \max_{u\in Q^n_{\bar{m}}} \left|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u)\right|.\end{aligned} \tag{17}$$

Since the function $\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u)$ is continuous on the set E^n, there exists a point $x_{\bar{m}} \in Q^n_{\bar{m}}$ such that

$$\max_{u\in Q^n_{\bar{m}}} |\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u)| = |\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(x_{\bar{m}})|. \tag{18}$$

Using Corollary 1 and Bernstein inequality (see [23, p. 163]), we obtain

$$\begin{aligned}\sum_{\bar{m}\in\mathbb{Z}^n} \left|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(x_{\bar{m}})\right| &\leq \left(1 + \sum_{k=1}^{n} \sum_{1\leq m_1<\ldots<m_k\leq n} \sigma_{m_1}\cdots\sigma_{m_k}\right) \left\|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}\right\|_{L^n_1} \\ &\leq \left(1 + \sum_{k=1}^{n} \sum_{1\leq m_1<\ldots<m_k\leq n} \sigma_{m_1}\cdots\sigma_{m_k}\right) \prod_{i=1}^{n} \sigma_i^{k_i} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1} \\ &= \left(1 + \sum_{k=1}^{n} \sum_{1\leq m_1<\ldots<m_k\leq n} \sigma_{m_1}\cdots\sigma_{m_k}\right) \bar{\sigma}^{\bar{k}} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1}.\end{aligned} \tag{19}$$

From (17)–(19), it follows that

$$\int_{E^n} \left|\hat{\lambda}^{(\bar{k})}_{\bar{\sigma}^n}(u) f(-u)\right| du \leq K_1 \|f\|_{\hat{L}^n_1} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1} \bar{\sigma}^{\bar{k}}, \qquad \bar{\sigma}^{\bar{k}} = \prod_{i=1}^{n} \sigma_i^{k_i}. \tag{20}$$

And from (15) and (20), we have

$$\left|\left(\hat{\lambda}_{\bar{\sigma}^n} * f\right)(z)\right| \leq K_1 \|f\|_{\hat{L}^n_1} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1} \sum_{\bar{k}\in\mathbb{Z}^n_+} \frac{|\bar{z}^{\bar{k}}|\bar{\sigma}^{\bar{k}}}{\bar{k}!} = K_1 \|f\|_{\hat{L}^n_1} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1} e^{\sum\limits_{i=1}^{n} \sigma_i |z_i|}. \tag{21}$$

Therefore, if $f \in \hat{L}^n_1$, then from inequality (21) (see also [23, p. 163]) it follows that $(\hat{\lambda}_{\bar{\sigma}^n} * f)(z)$ is an entire function of exponential type not larger than $\bar{\sigma}^n$. If $f \in X^n$, then, due to relation (11), $(\hat{\lambda}_{\bar{\sigma}^n} * f)(z)$ is also an entire function of exponential type not larger than $\bar{\sigma}^n$. Using the generalized Minkowski inequality, we obtain

$$\left\|f * \hat{\lambda}_{\bar{\sigma}^n}\right\|_{X^n} \leq \left\|f\right\|_{X^n} \left\|\hat{\lambda}_{\bar{\sigma}^n}\right\|_{L^n_1}.$$

Hence, if $f \in X^n$, then $(\hat{\lambda}_{\bar{\sigma}^n} * f)(z) \in F^{X^n}_{\bar{\sigma}}$. Theorem 1 is proved. □

From the integral representations (see [9, equations (21), (25)]) for functions from the space $F^{2^n}_{\bar\sigma} := F^{L^n_2}_{\bar\sigma}$, and taking into account that $F^{1^n}_{\bar\sigma} \subset F^{2^n}_{\bar\sigma}$, it follows that

$$\begin{aligned}\hat\lambda_{\bar\sigma^n}(t) &= \frac{1}{(2\pi)^n}\int_{\Delta^n_{\bar\sigma}} \lambda(u)e^{-iut}\,du = \frac{1}{(2\pi)^n}\int_{\Delta^n_{\bar\sigma}} (a(u)+ib(u))e^{-iut}\,du = \mathcal{F}^{-1}(\lambda_{\bar\sigma^n})(u)\\ &= \frac{1}{(2\pi)^n}\int_{E^n} \lambda_{\bar\sigma^n}(u)e^{-iut}\,du = \frac{1}{(2\pi)^n}\int_{E^n} (a_{\bar\sigma^n}(u)+ib_{\bar\sigma^n}(u))e^{-iut}\,du,\end{aligned} \tag{22}$$

where on the set $\Delta^n_{\bar\sigma}$, the equalities $\lambda_{\bar\sigma^n}(u) = \lambda(u)$, $a_{\bar\sigma^n}(u) = a(u)$ and $b_{\bar\sigma^n}(u) = b(u)$ hold, while on the set $\{E^n \setminus \Delta^n_{\bar\sigma}\}$ we have $\lambda_{\bar\sigma^n}(u) = a_{\bar\sigma^n}(u) = b_{\bar\sigma^n}(u) = 0$. The function $a(u)$ is a real even function, and $b(u)$ is a real odd function with respect to each variable $u_1,\dots,u_n$, both belonging to the space $L_{2(\Delta^n_{\bar\sigma})}$.

From the necessary condition for the absolute integrability of the Fourier transform, it follows that the function $\lambda_{\bar\sigma^n}(u)$, and thus the functions $a_{\bar\sigma^n}(u)$ and $b_{\bar\sigma^n}(u)$, are continuous on E^n. Therefore, according to the definition of these functions, the function $\lambda(u)$, and hence $a(u)$ and $b(u)$, are continuous on $\Delta^n_{\bar\sigma}$ and equal to zero on the set

$$\Gamma(\Delta^n_{\bar\sigma}) = \{x = (x_1,\dots,x_n) \in E^n : |x_i| = \sigma_i,\, i = 1,\dots,n\}.$$

Let us establish sufficient conditions for the absolute integrability of the kernel $\hat\lambda_{\bar\sigma^n}(x)$ on E^n.

We denote by $L_{r(a,b)}$ the normed space of functions defined on the interval (a,b) with the norm

$$\|f\|_{L_{r(a,b)}} = \left(\int_a^b |f(x)|^r\,dx\right)^{1/r}.$$

Let us prove the auxiliary statement.

Lemma 2. *Let $\lambda(u) = a(u) + ib(u)$, where $a(u)$ is an even real-valued function and $b(u)$ is an odd real-valued function, both absolutely continuous on the segment $[-\sigma,\sigma]$, and*

$$a(\sigma) = a(-\sigma) = b(\sigma) = b(-\sigma) = 0. \tag{23}$$

Then, for any $p > 1$, the function $\hat\lambda_\sigma(z)$, which for $n = 1$ is defined by equation (22), belongs to the space F^p_σ. Moreover, if the derivatives $a'(x)$ and $b'(x)$ of the functions $a(x)$ and $b(x)$ belong to the space $L_{q(-\sigma,\sigma)}$, where $q > 1$, then the function $\hat\lambda_\sigma(z)$ belongs to the space F^1_σ.

Proof. Since the function $\lambda(u)$ is continuous on $[-\sigma,\sigma]$, it follows that $\lambda(u) \in L_{2(-\sigma,\sigma)}$, and by the Wiener-Paley theorem (see [23, pp. 130, 131]), the function $\hat\lambda_\sigma(z)$ belongs to the space F^2_σ, and $F^2_\sigma \supseteq F^p_\sigma \supseteq F^1_\sigma$, where $1 \le p \le 2$. Since the functions $a(x)$ and $b(x)$ are absolutely continuous on the segment $[-\sigma,\sigma]$, integrating by parts and using equalities (22), (23), we obtain

$$\hat\lambda_\sigma(t) = \frac{1}{2\pi i t}\int_{-\sigma}^{\sigma} \lambda(u)e^{-iut}\,du = \frac{1}{it}\mathcal{F}^{-1}(\lambda_\sigma)(t) = \frac{1}{2\pi i t}\int_{-\infty}^{\infty} \lambda_\sigma(u)e^{-iut}\,du, \tag{24}$$

where on the interval $(-\sigma,\sigma)$ we have $\lambda_\sigma(u) = \lambda'(u)$, and for $|u| > \sigma$ the equality $\lambda'_\sigma(u) = 0$ holds. Since the functions $a(u)$ and $b(u)$ are absolutely continuous on $[-\sigma,\sigma]$, the function $\lambda'(u) = a'(u) + ib'(u)$ belongs (see, e.g., [21, pp. 252–255]) to the space $L_{1(-\sigma,\sigma)} = L_{(-\sigma,\sigma)}$, and the function $\lambda'_\sigma(u)$ belongs to the space $L = L_{(-\infty,+\infty)}$. Therefore, the function $\mathcal{F}^{-1}(\lambda'_\sigma)(t)$ is continuous and bounded on the entire real axis, as the inverse Fourier transform of a function that is absolutely integrable on $(-\infty, +\infty)$, that is,

$$\left\|\mathcal{F}^{-1}(\lambda'_\sigma)\right\|_C < K_2. \tag{25}$$

For an arbitrary $p_1 \geq 1$ and fixed $a > 0$, using the boundedness of the function $|\lambda(u)|$ on the segment $[-\sigma, \sigma]$ and the generalized Minkowski inequality, we obtain

$$\begin{aligned}\left(\int_{-a}^{a} |\hat{\lambda}_\sigma(t)|^{p_1}\, dt\right)^{\frac{1}{p_1}} &\leq \frac{1}{2\pi}\int_{-\sigma}^{\sigma} |\lambda(u)| \left(\int_{-a}^{a} |e^{-iut}|^{p_1}\, dt\right)^{\frac{1}{p_1}} du \\ &\leq (2a)^{\frac{1}{p_1}} \cdot \frac{1}{2\pi}\int_{-\sigma}^{\sigma} |\lambda(u)|\, du < K_3.\end{aligned} \tag{26}$$

For any $p > 1$, by virtue of inequality (25), we get

$$\left(\int_{|t|>a} \left|\frac{1}{t}\mathcal{F}^{-1}(\lambda'_\sigma)(t)\right|^p dt\right)^{\frac{1}{p}} \leq K_2 \left(\int_{|t|>a} \frac{dt}{|t|^p}\right)^{\frac{1}{p}} < K_4. \tag{27}$$

From relations (24), (26), and (27), it follows that for any $p > 1$, the function $\hat{\lambda}_\sigma(t)$ belongs to the space L_p. Therefore, for any $p > 1$, we have $\hat{\lambda}_\sigma(z) \in F^p_\sigma$.

If, in addition, the functions $a'(u)$ and $b'(u)$ belong to the space $L_{q(-\sigma,\sigma)}$ for some $q > 1$, then we get $\lambda'(u) = a'(u) + ib'(u) \in L_{q(-\sigma,\sigma)} \subset L_{r(-\sigma,\sigma)}$, where $1 < r \leq \min\{2, q\}$, and hence $\lambda'_\sigma(u) \in L_r$. According to the Hausdorff-Young theorem (see, e.g., [23, p. 201]), it follows that $\mathcal{F}^{-1}(\lambda'_\sigma) \in L_{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$. Therefore,

$$\left(\int_{|t|>a} \left|\mathcal{F}^{-1}(\lambda'_\sigma)(t)\right|^{r'} dt\right)^{\frac{1}{r'}} < K_5. \tag{28}$$

Using Hölder inequality, relation (28), and taking into account that $r > 1$, we obtain

$$\int_{|t|>a} \left|\frac{1}{it}\mathcal{F}^{-1}(\lambda'_\sigma)(t)\right| dt \leq \left(\int_{|t|>a} \left|\mathcal{F}^{-1}(\lambda'_\sigma)(t)\right|^{r'} dt\right)^{\frac{1}{r'}} \left(\int_{|t|>a} \frac{dt}{|t|^r}\right)^{\frac{1}{r}} < K_6. \tag{29}$$

From relations (24), (26), and (29), it follows that $\hat{\lambda}_\sigma(t) \in L$ and $\hat{\lambda}_\sigma(z) \in F^1_\sigma$, which completes the proof of Lemma 2. □

Let us note that a similar statement holds for Fourier coefficients (see, e.g., [7, p. 173] or [8, pp. 162–163]). If the 2π-periodic absolutely continuous function $f(x)$ has the derivative $f'(x)$ that belongs to the space $\tilde{L}_p$ with the norm

$$\|f\|_{\tilde{L}_p} = \left(\int_{-\pi}^{\pi} |f(x)|^p\, dx\right)^{1/p}, \quad p > 1,$$

then

$$\frac{|a_0(f)|}{2} + \sum_{k=1}^{\infty} (|a_k(f)| + |b_k(f)|) < K_7, \tag{30}$$

but there exist absolutely continuous functions for which the series (30) diverges. The condition that the derivative $\lambda'(u)$ belongs to the space $L_{q(-\sigma,\sigma)}$ is not necessary for the absolute integrability of the function $\hat{\lambda}_\sigma(t)$. If $\lambda(u) = a(|u|)$, where $a(u) = -\ln(\sigma^{-1}u)$, then on the interval $(0, \sigma)$ we have $a'(u) = -\frac{1}{u} < 0$ and $a''(u) = \frac{1}{u^2} > 0$. Therefore (see, e.g., [2, p. 125]), the function $\hat{\lambda}_\sigma(t) = \frac{1}{2\pi}\int\limits_{-\sigma}^{\sigma} a(|u|)e^{-iut}\, du = \frac{1}{\pi}\int\limits_{0}^{\sigma} a(u)\cos(ut)\, du$ belongs to the space L, but the derivative $a'(u)$ does not belong to any of the spaces $L_{p(0,\sigma)}$ for $p > 1$.

Theorem 2. *Let*
$$\lambda(u) = \prod_{j=1}^{n} \lambda^j(u_j) = \prod_{j=1}^{n} \left(a^j(u_j) + ib^j(u_j) \right), \tag{31}$$
where the functions $a^j(u_j)$ are real and even, and $b^j(u_j)$ are real and odd, and all are absolutely continuous on the segments $[-\sigma_j, \sigma_j]$ for $j = 1, \dots, n$ with the boundary conditions $a^j(|\sigma_j|) = b^j(|\sigma_j|) = 0$. Then for any $p > 1$ the function
$$\hat{\lambda}_{\bar{\sigma}^n}(z) = \frac{1}{(2\pi)^n} \int_{\Delta^n_{\bar{\sigma}}} \lambda(u) e^{-iuz} \, du \tag{32}$$
belongs to the space $F^{p^n}_{\bar{\sigma}}$. Moreover, if the derivatives $(a^j(u_j))'$ and $(b^j(u_j))'$ belong to the space $L_{q(-\sigma_j,\sigma_j)}$ for some $q > 1$, then $\hat{\lambda}_{\bar{\sigma}^n}(z) \in F^{1^n}_{\bar{\sigma}}$.

Proof. Since the functions $a^j(u_j)$ and $b^j(u_j)$ are absolutely continuous on the segments $[-\sigma_j, \sigma_j]$, these functions belong to the space $L_{2(-\sigma_j,\sigma_j)}$, and hence the functions $\lambda^j(u_j) = a^j(u_j) + ib^j(u_j)$ also belong to $L_{2(-\sigma_j,\sigma_j)}$. From equation (31) it follows that the function $\lambda(u)$ belongs to the space $L_{2(\Delta^n_{\bar{\sigma}})}$. Therefore, by the Wiener-Paley theorem, the function $\hat{\lambda}_{\bar{\sigma}^n}(z)$, defined by equation (32), belongs to the space $F^{2^n}_{\bar{\sigma}}$, that is, $\hat{\lambda}_{\bar{\sigma}^n}(z)$ is an entire function of exponential type not large then $\bar{\sigma}^n$.

From equations (31) and (32), it follows that
$$\hat{\lambda}_{\bar{\sigma}^n}(z) = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int_{-\sigma_j}^{\sigma_j} \lambda^j(u_j) e^{-iu_j z_j} \, du_j = \prod_{j=1}^{n} \mathcal{F}^{-1}(\lambda^j_{\sigma_j})(z_j), \tag{33}$$
where
$$\lambda^j_{\sigma_j}(u_j) = \begin{cases} \lambda^j(u_j), & |u_j| \le \sigma_j, \\ 0, & |u_j| > \sigma_j. \end{cases}$$

Since, according to Lemma 2, for any $p > 1$, we have $\mathcal{F}^{-1}(\lambda^j_{\sigma_j})(x_j) \in L_p$, then from equality (33), by virtue of the definition of the norm in the space L^n_p, it follows that $\hat{\lambda}_{\bar{\sigma}^n}(x) \in L^n_p$. Hence, for any $p > 1$, we get $\hat{\lambda}_{\bar{\sigma}^n}(z) \in F^{p^n}_{\bar{\sigma}}$. Moreover, if the derivatives $a^{j\prime}(u_j)$ and $b^{j\prime}(u_j)$ belong to the space $L_{q(-\sigma_j,\sigma_j)}$ with $q > 1$, then, by Lemma 2, the functions $\mathcal{F}^{-1}(\lambda^j_{\sigma_j})(x_j)$ belong to the space L. Therefore, from equation (33) it follows that $\hat{\lambda}_{\bar{\sigma}^n}(x) \in L^n$, i.e. $\hat{\lambda}_{\bar{\sigma}^n}(z) \in F^{1^n}_{\bar{\sigma}}$. Theorem 2 is proved. □

Let us find the integral representations for the operators
$$U_{\bar{\sigma}^n}(\Lambda, f, x) = (\hat{\lambda}_{\bar{\sigma}^n} * f)(x) = \int_{E^n} \hat{\lambda}_{\bar{\sigma}^n}(x - t) f(t) \, dt.$$

Theorem 3. *Let the kernel $\hat{\lambda}_{\bar{\sigma}^n}(t)$ be defined by equation (22) and belong to the space L^n. Let $f \in L^n_p$, where $1 \le p \le 2$. Then, for the operators $U_{\bar{\sigma}^n}(\Lambda, f, x)$ given by (2), the following integral representation*
$$U_{\bar{\sigma}^n}(\Lambda, f, x) = \frac{1}{(2\pi)^n} \int_{E^n} \lambda_{\bar{\sigma}^n}(u) \mathcal{F}(f)(u) e^{-iux} \, du = \frac{1}{(2\pi)^n} \int_{\Delta^n_{\bar{\sigma}}} \lambda(u) \mathcal{F}(f)(u) e^{-iux} \, du \tag{34}$$
holds for all $x \in E^n$.

Proof. If $f \in L^n_p$, $1 \le p \le 2$, then by the Hausdorff-Young theorem, we get $F(f)(u) \in L^n_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, according to equation (22) and the definition of the function $\lambda_{\bar{\sigma}^n}(u)$, we obtain $\mathcal{F}(f)(u)\lambda_{\bar{\sigma}^n}(u) \in L^n$. Then, by [10, Lemma 7], equality (34) holds for all $x \in E^n$. Theorem 3 is proved. □

3 Examples of equality of approximation characteristics in function spaces that are isometric to spaces of real non-periodic functions

To simplify the notation, we first present examples of the equality of approximation characteristics in function spaces that are isometric to spaces of real non-periodic functions of one variable.

Let $\Psi_{\bar{y}^m}(x)$ be delta-like kernels (see [10, equations (9), (10)]), defined on $\Pi^+_{1,m}$ with respect to a single variable x.

Let us introduce the linear operator $U_\sigma(\Lambda * \Psi_{\bar{y}^m}, f, x) = \big((\hat{\lambda}_\sigma * \Psi_{\bar{y}^m}) * f\big)(x)$, which, for $n = 1$, coincides with the operator defined by (2), where the kernel $\hat{\lambda}_\sigma(z) = \frac{1}{2\pi}\mathcal{F}^{-1}(\lambda_\sigma)(z)$ is described by formulas (22) and (24), and belongs to the function space F^1_σ.

We define the following approximation characteristics:

- $\rho_X(f, U_\sigma(\Lambda, f)) = \|f - U_\sigma(\Lambda, f)\|_X$ is the approximation of the function $f \in X$ by the linear operator $U_\sigma(\Lambda, f)$,
- $\rho_{XM_m}(f * \Psi_{\bar{y}^m}, U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})) = \big\|f * \Psi_{\bar{y}^m} - U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})\big\|_{XM_m}$ is the approximation of the function $f * \Psi_{\bar{y}^m} \in XM_m$ by the linear operator $U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})$,
- $\rho_X(M, U_\sigma(\Lambda, M)) = \sup\limits_{f\in M} \rho_X(f, U_\sigma(\Lambda, f))$ is the approximation of the set $M \subset X$ by the operator $U_\sigma(\Lambda, f)$,
- $\rho_{XM_m}\big(\{M * \Psi_{\bar{y}^m}\}, U_\sigma(\Lambda, \{M * \Psi_{\bar{y}^m}\})\big) = \sup\limits_{f\in M} \rho_{XM_m}\big(f * \Psi_{\bar{y}^m}, U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})\big)$ is the approximation of the set $\{M * \Psi_{\bar{y}^m}\} \subset XM_m$ by the operator $U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})$,
- $E_\sigma(f)_X = \rho_X(f, F^X_\sigma(E)) = \inf\limits_{T_\sigma \in F^X_\sigma(E)} \|f(x) - T_\sigma(x)\|_X$ is the best approximation of the function $f \in X$ (entire function of exponential type not large than σ) from the space F^X_σ,
- $E_\sigma(f * \Psi_{\bar{y}^m})_{XM_m} = \rho_{XM_m}(f * \Psi_{\bar{y}^m}, F^X_\sigma(E) * \Psi_{\bar{y}^m})$ is the best approximation of the function $f \in XM_m$ (entire function of exponential type not large than σ) from the space $\{F^X_\sigma(E) * \Psi_{\bar{y}^m}\}$,
- $E_\sigma(M)_X = \sup\limits_{f\in M} E_\sigma(f)_X$ is the best approximation of the set $M \subset X$ by functions from F^X_σ,
- $E_\sigma\big(\{M * \Psi_{\bar{y}^m}\}\big)_{XM_m} = \sup\limits_{f\in M} E_\sigma\big(f * \Psi_{\bar{y}^m}\big)_{XM_m}$ is the best approximation of the set $\{M * \Psi_{\bar{y}^m}\} \subset XM_m$ by functions from the space $\{F^X_\sigma(E) * \Psi_{\bar{y}^m}\}$,
- $\lambda(M, F^X_\sigma(E))_X$ is the best linear approximation (see [13]) of the set $M \subset X$ by linear operators from the set $L(X, F^X_\sigma(E))$, which consists of all linear operators mapping the space X into the subspace $F^X_\sigma(E)$,
- $\lambda\big(\{M * \Psi_{\bar{y}^m}\}, \{F^X_\sigma(E) * \Psi_{\bar{y}^m}\}\big)_{XM_m}$ is the best linear approximation of the image $\{M * \Psi_{\bar{y}^m}\}$ by linear operators from the set $L(\{X * \Psi_{\bar{y}^m}\}, \{F^X_\sigma(E) * \Psi_{\bar{y}^m}\})_{XM_m}$, i.e. the set of all linear operators that map the space $\{X * \Psi_{\bar{y}^m}\}$ into the subspace $\{F^X_\sigma(E) * \Psi_{\bar{y}^m}\}$.

Taking into account the isometric property of the convolution operator with a non-negative delta-like kernel (see [10, Corollary 1]), we can state the following propositions.

Proposition 1. *Let $\Psi_{\bar{y}^m}(x)$ be a non-negative delta-like kernel. If X is either one of the spaces L_p, $1 \le p < \infty$, or the space C_r of uniformly continuous functions defined on E, then the space $X * \Psi_{\bar{y}^m}$ is isometric to the space X, and the subspace $F_\sigma^X(E)$ is isometric to the subspace $F_\sigma^X(E) * \Psi_{\bar{y}^m}$. Then, due to this isometry, the following equalities hold:*

$$\rho_X(f, U_\sigma(\Lambda, f)) = \rho_{XM_m}(f * \Psi_{\bar{y}^m}, U_\sigma(\Lambda * \Psi_{\bar{y}^m}, f)),$$

$$\rho_X(M, U_\sigma(\Lambda, M)) = \rho_{XM_m}(\{M * \Psi_{\bar{y}^m}\}, U_\sigma(\Lambda, \{\Psi_{\bar{y}^m} * M\})), \tag{35}$$

$$E_\sigma(f)_X = E_\sigma(f * \Psi_{\bar{y}^m})_{XM_m}, \qquad E_\sigma(M)_X = E_\sigma(\{M * \Psi_{\bar{y}^m}\})_{XM_m}, \tag{36}$$

$$\lambda(M, F_\sigma^X(E))_X = \lambda(\{M * \Psi_{\bar{y}^m}\}, \{F_\sigma^X(E) * \Psi_{\bar{y}^m}\})_{XM_m}. \tag{37}$$

Proposition 2. *If X is one of the spaces L_p, $\hat{L}_p$, $1 \le p \le \infty$, or C, then the same equalities hold for the isometric spaces X and $\overline{X * \Psi\bar{y}^m}$, as well as for their isometric subspaces $F_\sigma^X(E)$ and $\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}$ (see (1)), with the replacement of the set $M * \Psi_{\bar{y}^m}$ by $\overline{M * \Psi_{\bar{y}^m}}$, and the norm in the space XM_m by the norm in $X\bar{M}_m$, where XM_m and $X\bar{M}_m$ are the function spaces (see [10, equations (7), (8)]).*

Proposition 3. *Let the delta-like kernel $\Psi_{\bar{y}^m}(x)$ be not non-negative. If X is one of the subspaces of L_p, $\hat{L}_p$, $1 \le p \le \infty$, or C, then, according to [10, Corollary 1], the spaces $\{\overline{X * \Psi_{\bar{y}^m}}\}$ are isomorphic to the space X, and the subspaces $\{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\}$ are isomorphic to the subspace $F_\sigma^X(E)$.*

Proof. Using [10, inequalities (22)], we obtain:

$$\|f - U_\sigma(\Lambda, f)\|_X \le \left\|f * \Psi_{\bar{y}^m} - U_\sigma(\Lambda, f * \Psi_{\bar{y}^m})\right\|_{X\bar{M}_m} \le \left\|\Psi_{\bar{y}^m}\right\|_{1\bar{M}_m} \cdot \|f - U_\sigma(\Lambda, f)\|_X, \tag{38}$$

$$\begin{aligned} \rho_X(M, U_\sigma(\Lambda, M)) &\le \rho_{X\bar{M}_m}(\{M * \Psi_{\bar{y}^m}\}, U_\sigma(\Lambda, \{\Psi_{\bar{y}^m} * M\})) \\ &\le \left\|\Psi_{\bar{y}^m}\right\|_{1\bar{M}_m} \cdot \rho_X(M, U_\sigma(\Lambda, M)), \end{aligned} \tag{39}$$

$$E_\sigma(f)_X \le E_\sigma(f * \Psi_{\bar{y}^m})_{X\bar{M}_m} \le \left\|\Psi_{\bar{y}^m}\right\|_{1\bar{M}_m} \cdot E_\sigma(f)_X, \tag{40}$$

$$E_\sigma(M)_X \le E_\sigma(\{\overline{M * \Psi_{\bar{y}^m}}\})_{X\bar{M}_m} \le \left\|\Psi_{\bar{y}^m}\right\|_{1\bar{M}_m} \cdot E_\sigma(M)_X, \tag{41}$$

$$\lambda(M, F_\sigma^X(E))_X \le \lambda(\{\overline{M * \Psi_{\bar{y}^m}}\}, \{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\})_{X\bar{M}_m} \le \left\|\Psi_{\bar{y}^m}\right\|_{1\bar{M}_m} \cdot \lambda(M, F_\sigma^X(E))_X. \tag{42}$$

□

If X is one of the spaces L_p, $1 \le p < \infty$, or C_r, then these inequalities are valid for the isomorphic spaces $\{X * \Psi_{\bar{y}^m}\}$ and X, as well as for their isomorphic subspaces $\{F_\sigma^X(E) * \Psi_{\bar{y}^m}\}$ and $F_\sigma^X(E)$. In this case, the set $\{\overline{M * \Psi_{\bar{y}^m}}\}$ from the space $X\bar{M}_m$ is replaced by the set $\{M * \Psi_{\bar{y}^m}\}$ from the space XM_m, and the norm of the space $X\bar{M}_m$ is replaced by the norm of the space XM_m.

Proceeding analogously to the proof of the Zygmund inequality, we establish that the Zygmund inequality holds in the spaces $\{F_\sigma^X(E) * \Psi_{\bar{y}^m}\}$ and $\{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\}$, which are isometric to the space $F_\sigma^X(E)$.

Proposition 4. *For all functions* $(\Psi_{\bar{y}^m} * T_\sigma)(x) \in \{F_\sigma^X(E) * \Psi_{\bar{y}^m}\}$ *and* $\psi T_\sigma(x,y) \in \{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\}$, *the following inequalities hold:*

$$\left\| \frac{\delta^{(k)}(\Psi_{\bar{y}^m} * T_\sigma)(x)}{\delta x^k} \right\|_{XM_m} \le \sigma^k \left\| (\Psi_{\bar{y}^m} * T_\sigma)(x) \right\|_{XM_m}, \tag{43}$$

$$\left\| \frac{\delta^{(k)}(\psi T_\sigma(x,y))}{\delta x^k} \right\|_{X\bar{M}_m} \le \sigma^k \left\| \psi T_\sigma(x,y) \right\|_{X\bar{M}_m}, \tag{44}$$

where X *is one of the spaces* C, L_p, $\hat{L}_p$, $1 \le p \le \infty$.

From the criterion of the best approximation of the function $f \in L_p$, $1 < p < \infty$, (entire function of exponential type not large than σ) by elements of the space F_σ^p (see, e.g., [24, p. 84]), it follows the criterion for the best approximation of the isometric image $f * \Psi_{\bar{y}^m}$ by elements of the space $\{\overline{F_\sigma^p(E) * \Psi_{\bar{y}^m}}\}$, which is isometric to the space $F_\sigma^p(E)$.

Using the criterion for the element of best approximation (see [10, equation (15)]), we obtain the following assertion.

Proposition 5. *The element* $\psi T_\sigma^*(x,y) \in \{\overline{F_\sigma^p(E) * \Psi_{\bar{y}^m}}\}$ *is the best approximation element for* $v(x,y) \in \{\overline{L_p * \Psi_{\bar{y}^m}}\}$, *which is defined by the identity*

$$E_\sigma(v)_{L_p\bar{M}_m} = \| v(x,y) - \psi T_\sigma^*(x,y) \|_{p\bar{M}_m}$$

if and only if, for every element $\psi T_\sigma(x,y) \in \{\overline{F_\sigma^p(E) * \Psi_{\bar{y}^m}}\}$, *the following condition*

$$\int_{-\infty}^{\infty} \psi T_\sigma(x,0) \cdot |v(x,0) - \psi T_\sigma^*(x,0)|^{p-1} \operatorname{sgn}(v(x,0) - \psi T_\sigma^*(x,0))\, dx = 0$$

holds, where $\psi T_\sigma(x,0) = T_\sigma(x)$ *and* $\psi T_\sigma^*(x,0) = T_\sigma^*(x)$.

If we replace the spaces X, XM_m, $X\bar{M}_m$ and their norms respectively by the spaces X^n, $X^n M_m$, $X^n \bar{M}_m$ and their norms, the delta-like kernels $\Psi_{\bar{y}^m}(x)$ by the delta-like kernels $I_{\bar{y}^m}^n(x)$, the subspaces $F_\sigma^X(E)$, $\{F_\sigma^X(E) * \Psi_{\bar{y}^m}\}$ and $\{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\}$ by the corresponding subspaces $F_{\bar{\sigma}}^{X^n}(E^n)$, $\{F_{\bar{\sigma}}^{X^n}(E^n) * I_{\bar{y}^m}^n\}$ and $\{\overline{F_{\bar{\sigma}}^{X^n}(E^n) * I_{\bar{y}^m}^n}\}$, and the operators $U_\sigma(\Lambda, f, x)$, $U_\sigma(\Lambda, f * \Psi_{\bar{y}^m}, x)$ by the corresponding operators $U_{\bar{\sigma}^n}(\Lambda, f, x)$, $U_{\bar{\sigma}^n}(\Lambda, f * I_{\bar{y}^m}^n, x)$, the best approximations E_σ of subspaces $F_\sigma^X(E)$ by the best approximations $E_{\bar{\sigma}^n}$ of subspaces $F_{\bar{\sigma}}^{X^n}(E^n)$, and the best approximations $E_\sigma(f * \Psi_{\bar{y}^m})_{XM_m}$ and $E_\sigma(f * \Psi_{\bar{y}^m})_{X\bar{M}_m}$ of the respective subspaces $\{F_\sigma^X(E) * \Psi_{\bar{y}^m}\}$ and $\{\overline{F_\sigma^X(E) * \Psi_{\bar{y}^m}}\}$ by $E_{\bar{\sigma}^n}(f * I_{\bar{y}^m}^n)_{X^n M_m}$ and $E_{\bar{\sigma}^n}(f * I_{\bar{y}^m}^n)_{X^n \bar{M}_m}$ of the respective subspaces $\{F_{\bar{\sigma}}^{X^n}(E^n) * I_{\bar{y}^m}^n\}$ and $\{\overline{F_{\bar{\sigma}}^{X^n}(E^n) * I_{\bar{y}^m}^n}\}$, then for the function spaces X^n, $X^n M_m$, and $X^n \bar{M}_m$, under the appropriate conditions of [10, Corollary 1], the equalities (35)–(37), inequalities (38)–(42), and Zygmund inequalities (43), (44) hold. Namely,

$$\left\| \frac{\partial^{k_1+\dots+k_n} \left(T_{\bar{\sigma}^n} * I_{\bar{y}^m}^n \right)(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{X^n M_m} \le \prod_{i=1}^n \sigma_i^{k_i} \left\| T_{\bar{\sigma}^n} * I_{\bar{y}^m}^n \right\|_{X^n M_m}, \tag{45}$$

$$\left\| \frac{\partial^{k_1+\dots+k_n} I T_{\bar{\sigma}^n}(x,y)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{X^n \bar{M}_m} \le \prod_{i=1}^n \sigma_i^{k_i} \left\| I T_{\bar{\sigma}^n}(x,y) \right\|_{X^n \bar{M}_m}, \tag{46}$$

where X^n is one of the spaces C^n, L^n_p, $1 \le p \le \infty$, $L^n_{\bar{p}}$, $\hat{L}^n_{\bar{p}}$, $\bar{p} = (p_1, \dots, p_n)$, $1 \le p_i < \infty$, $i = 1, \dots, n$, and $I^n_{\bar{y}^m}(x)$ is a non-negative delta-like kernel.

We should note that inequalities (45), (46) were established in [23, pp. 138–140] for arbitrary function spaces with a norm invariant under shifts in the variables $x_1, \dots, x_n$, under the condition that norm convergence in the space and almost everywhere convergence imply equality of the limit elements.

Given the growing interest of researchers in applied mathematics in the methods of approximation theory (see, e.g., [4,6,14,17]), the results obtained in this work may prove useful in studies related to mathematical modeling, optimization methods, and mathematical physics.

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Бушев Д.М., Кальчук І.В., Харкевич Ю.І. *Наближення цілими функціями експоненціального типу ізометричних функціональних просторів* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 679–692.

У цій статті встановлені достатні умови для відображень операторами згорток просторів функцій, заданих на n-вимірному евклідовому просторі, на простори цілих функцій експоненціального типу. Знайдені інтегральні представлення для цих операторів. Використовуючи рівності апроксимаційних характеристик в ізометричних просторах функцій від однієї змінної, представлено рівності апроксимаційних характеристик в ізометричних функціональних просторах багатьох змінних.

Ключові слова і фрази: ізометричний простір, лінійний оператор, апроксимативна характеристика, простір згорток, дельтаподібне ядро, простір цілих функцій експоненціального типу.