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Symmetric polynomials on Cartesian products of Banach spaces of Lebesgue integrable functions

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The work is devoted to the study of complex-valued continuous symmetric polynomials on Cartesian products of complex Banach spaces of Lebesgue integrable functions. Let L_p, where $p \in [1;+\infty)$, be the complex Banach space of all complex-valued functions on $[0;1]$, the pth powers of absolute values of which are Lebesgue integrable. Let $\Xi_{[0;1]}$ be the set of all bijections $\sigma : [0;1] \to [0;1]$ such that both σ and σ^{-1} are measurable and preserve Lebesgue measure, i.e. $\mu(\sigma(E)) = \mu(\sigma^{-1}(E)) = \mu(E)$ for every Lebesgue measurable set $E \subset [0;1]$, where μ is Lebesgue measure. A function f on the Cartesian product $L_{p_1} \times \cdots \times L_{p_n}$, where $p_1, \ldots, p_n \in [1;+\infty)$, is called symmetric if $f((x_1 \circ \sigma, \ldots, x_n \circ \sigma)) = f((x_1, \ldots, x_n))$ for every $\sigma \in \Xi_{[0;1]}$ and $(x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$. We construct an algebraic basis of the algebra of all complex-valued continuous symmetric polynomials on $L_{p_1} \times \cdots \times L_{p_n}$. Also we construct some isomorphisms of Fréchet algebras of complex-valued entire symmetric functions of bounded type on $L_{p_1} \times \cdots \times L_{p_n}$.

Key words and phrases: polynomial, entire function, symmetric function, algebraic basis, Banach space of Lebesgue integrable functions.

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Introduction

Symmetric functions on infinite dimensional spaces were introduced in [17] as an important tool in the theory of uniform approximation of continuous functions on separable Hilbert spaces by smooth functions and, in particular, by polynomials. In general, if a vector space has some symmetric structure, it is natural to consider the set of operators that preserve this structure and to define a symmetric function on this space as a function that is invariant under the action on its argument of the above-mentioned operators [2]. Important examples of such spaces are rearrangement-invariant function spaces (see [15, Definition 2.a.1, p. 117]), e.g., spaces L_p (see definition below). As mentioned in [9], sequence spaces with symmetric Schauder basis (see [14, Definition 3.a.1, p. 113]), such as spaces ℓ_p of absolutely summable in a power p sequences, can be considered as rearrangement-invariant function spaces on $\mathbb{N}$. Symmetric continuous polynomials on real separable rearrangement-invariant function spaces were applied to the investigation of the problem of the existence of separating polynomials on these spaces in [9]. Also in [9] there were constructed not more than countable algebraic bases (see definition below) of algebras of symmetric continuous polynomials on these spaces. Analogical results for the case of complex spaces implies the fact that corresponding topolog-

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ical algebras of analytic symmetric functions on these spaces are finitely or countably generated, which is, in particular, useful for description of spectra (sets of all nontrivial complex-valued linear multiplicative functionals, also called characters) of these algebras. The terms of Taylor series expansions of such functions are algebraic combinations (see definition below) of elements of an algebraic basis of the corresponding algebra of symmetric polynomials. Consequently, every character on such an algebra of analytic symmetric functions is completely defined by its values on elements of the algebraic basis. This approach was firstly introduced in [1] for the study of spectra of algebras of analytic symmetric functions on ℓ_1. The complete description of the spectrum of the Fréchet algebra of entire symmetric functions of bounded type on ℓ_1 was obtained in [6] (see also [4, 5]). In the continuous case the above-mentioned approach was used in [7] to describe the spectrum of the Fréchet algebra of entire symmetric functions of bounded type on L_∞, which made it possible to represent this algebra as the algebra of entire functions on its spectrum [8]. Some general results on spectra of countably generated algebras of entire functions on Banach spaces were established in [10, 18].

Similar to algebras of continuous symmetric polynomials on the above-mentioned spaces, algebras of continuous symmetric polynomials on Cartesian products of these spaces have not more than countable algebraic bases (see, e.g., [3, 11, 13] for the case of sequence spaces and [19–21, 26] for the case of function spaces). Note that natural groups of symmetry on these Cartesian products of spaces induce some groups of symmetry on the spaces themselves that are proper subgroups of the respective natural groups of symmetry on these spaces (see, e.g., [25]). Functions that are symmetric with respect to such groups are called block-symmetric. Since the condition of block-symmetry is, in general, weaker than the condition of symmetry, it follows that algebras of block-symmetric polynomials and analytic functions contain the respective algebras of symmetric polynomials and analytic functions as proper subalgebras. Thus, the technique developed for algebras of symmetric functions in the classical sense can be extended to broader classes of algebras of functions. The limiting cases of such algebras are the algebras of weakly symmetric functions (see, e.g., [27]).

In the work, we construct an algebraic basis of the algebra of all complex-valued continuous symmetric polynomials on the Cartesian product $L_{p_1} \times \cdots \times L_{p_n}$, where $p_1, \ldots, p_n \in [1; +\infty)$. Also we construct some isomorphism between Fréchet algebras of complex-valued entire symmetric functions of bounded type on this Cartesian product.

1 Preliminaries

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers. For every $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$ we define $|k|$ as $|k| = k_1 + \cdots + k_n$.

Polynomials. A mapping $P : X \to Y$, where X and Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, is called an N-homogeneous polynomial, where $N \in \mathbb{N}$, if there exists an N-linear mapping $A_P : X^N \to Y$ such that $P(x) = A_P\big(\underbrace{x, \ldots, x}_{N}\big)$ for every $x \in X$.

A mapping $P : X \to Y$ is called a polynomial of degree at most N if it can be represented in the form

$$P = P_0 + P_1 + \cdots + P_N, \tag{1}$$

where $P_0 \in Y$ and $P_j : X \to Y$ is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$.

It is known that a polynomial $P : X \to Y$ is continuous if and only if $\|P\| < +\infty$, where

$$\|P\| = \sup_{\|x\|_X \le 1} \|P(x)\|_Y.$$

Consequently, for a continuous N-homogeneous polynomial $P : X \to Y$, we have

$$\|P(x)\|_Y \le \|P\| \|x\|_X^N$$

for every $x \in X$.

Algebraic basis. Let A be a unital commutative algebra over the field $\mathbb{C}$. For every polynomial $Q : \mathbb{C}^n \to \mathbb{C}$ of the form

$$Q(z_1, \ldots, z_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1} \cdot \ldots \cdot z_n^{k_n},$$

where $\alpha_{(k_1,\ldots,k_n)} \in \mathbb{C}$ and Ω is some nonempty finite subset of $\mathbb{Z}_+^n$, let us define the mapping $Q_A : A^n \to A$ by

$$Q_A(a_1, \ldots, a_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} a_1^{k_1} \cdot \ldots \cdot a_n^{k_n}, \tag{2}$$

where $a_1, \ldots, a_n \in A$ (we consider the zeroth power a_j^0 of an element a_j to be the unit element of A).

Definition 1. *Let $a, a_1, \ldots, a_n \in A$. The element a is called an algebraic combination of $a_1, \ldots, a_n$, if there exists a polynomial $Q : \mathbb{C}^n \to \mathbb{C}$ such that $a = Q_A(a_1, \ldots, a_n)$.*

Definition 2. *A nonempty set $B \subset A$ is called a generating system of A if every element of A can be represented as an algebraic combination of some elements of B.*

Definition 3. *A nonempty set $B \subset A$ is called an algebraic basis of A if every element of A can be uniquely represented as an algebraic combination of some elements of B.*

Definition 4. *A finite nonempty set $\{a_1, \ldots, a_n\} \subset A$ is called algebraically independent if the equality $Q_A(a_1, \ldots, a_n) = 0$ is possible only if the polynomial Q is identically equal to 0. An infinite set $A_0 \subset A$ is called algebraically independent if every its finite nonempty subset is algebraically independent.*

Evidently, every algebraic basis is algebraically independent. Furthermore, every algebraically independent generating system is an algebraic basis.

The algebra $H_b(X)$. Let X be a complex Banach space. Let $H_b(X)$ be the Fréchet algebra of all entire functions $f : X \to \mathbb{C}$, which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. Let $\|f\|_r = \sup_{\|x\|\le r} |f(x)|$ for $f \in H_b(X)$ and $r > 0$. The topology of $H_b(X)$ can be generated by an arbitrary set of norms $\{\|\cdot\|_r : r \in \Gamma\}$, where Γ is any unbounded subset of $(0, +\infty)$.

Symmetric mappings. Let A, B be arbitrary nonempty sets. Let S be an arbitrary fixed set of mappings that act from A to itself. A mapping $f : A \to B$ is called *S-symmetric* if $f(s(a)) = f(a)$ for every $a \in A$ and $s \in S$.

The algebras $H_{b,S}(X)$ and $\mathcal{P}_S(X)$. Let X be a complex Banach space. Let S be a set of operators on X. Let $H_{b,S}(X)$ be the subalgebra of all S-symmetric elements of $H_b(X)$. By [22, Lemma 3], $H_{b,S}(X)$ is closed in $H_b(X)$. So, $H_{b,S}(X)$ is a Fréchet algebra. Let $\mathcal{P}_S(X)$ be the subalgebra of $H_{b,S}(X)$ that consists of all S-symmetric continuous polynomials on X.

Isomorphisms of Fréchet algebras of entire symmetric functions. We will use the following result [28, Theorem 2] for the item a) and [24, Theorem 4] for the items b) and c).

Theorem 1. *Let X and Y be complex Banach spaces. Let S_1 and S_2 be semigroups of operators on X and Y, respectively. Let $\iota_{X,Y} : X \to Y$ be an isomorphism such that*

1) *for every $x \in X$ and $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $\iota_{X,Y}(s_1(x)) = s_2(\iota_{X,Y}(x))$;*
2) *for every $y \in Y$ and $s_2 \in S_2$ there exists $s_1 \in S_1$ such that $\iota^{-1}_{X,Y}(s_2(y)) = s_1(\iota^{-1}_{X,Y}(y))$.*

Then

a) *the mapping $I : f \in H_{b,S_2}(Y) \mapsto f \circ \iota_{X,Y} \in H_{b,S_1}(X)$ is an isomorphism, i.e. I is a continuous linear multiplicative bijection;*
b) *the restriction of I to $\mathcal{P}_{S_2}(Y)$ is an isomorphism between algebras $\mathcal{P}_{S_2}(Y)$ and $\mathcal{P}_{S_1}(X)$;*
c) *if $\mathcal{P}_{S_2}(Y)$ has some algebraic basis B, then $I(B)$ is an algebraic basis in $\mathcal{P}_{S_1}(X)$.*

Spaces of Lebesgue measurable functions. Let L_p, where $p \in [1;+\infty)$, be the Banach space of measurable functions $x : [0;1] \to \mathbb{C}$ for which the pth power of the absolute value is Lebesgue integrable, i.e. there exists finite integral $\int_0^1 |x(t)|^p\,dt$, with norm

$$\|x\|_{L_p} = \left(\int_0^1 |x(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded functions $x : [0,1] \to \mathbb{C}$ with norm

$$\|x\|_{L_\infty} = \operatorname*{ess\,sup}_{t\in[0,1]} |x(t)|.$$

We define a norm on the Cartesian product $L_{p_1} \times \cdots \times L_{p_n}$, where $p_1, \ldots, p_n \in [1;+\infty]$, as

$$\|x\|_{L_{p_1}\times\cdots\times L_{p_n}} = \max_{i\in\{1,\ldots,n\}} \|x_i\|_{L_i} \tag{3}$$

for every $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$.

Symmetric functions on $L_{p_1} \times \cdots \times L_{p_n}$. Let $p_1, \ldots, p_n \in [1;+\infty]$. Let $\Xi_{[0;1]}$ be the set of all bijections $\sigma : [0;1] \to [0;1]$ such that both σ and σ^{-1} are measurable and preserve Lebesgue measure, i.e.

$$\mu(\sigma(E)) = \mu(\sigma^{-1}(E)) = \mu(E)$$

for every Lebesgue measurable set $E \subset [0;1]$, where μ is Lebesgue measure. For $\sigma \in \Xi_{[0;1]}$ and $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$, let

$$x \circ \sigma = (x_1 \circ \sigma, \ldots, x_n \circ \sigma).$$

For $\sigma \in \Xi_{[0;1]}$, let the operator g_σ be defined by

$$g_\sigma : x \in L_{p_1} \times \cdots \times L_{p_n} \mapsto x \circ \sigma \in L_{p_1} \times \cdots \times L_{p_n}. \tag{4}$$

Let

$$G_{p_1,\ldots,p_n} = \{g_\sigma : \sigma \in \Xi_{[0;1]}\}. \tag{5}$$

It can be verified that $G_{p_1,\ldots,p_n}$ is a group of operators on $L_{p_1} \times \cdots \times L_{p_n}$.

In what follows, let us refer to a $G_{p_1,\ldots,p_n}$-symmetric function $f : L_{p_1} \times \cdots \times L_{p_n} \to \mathbb{C}$ as symmetric, i.e. a function $f : L_{p_1} \times \cdots \times L_{p_n} \to \mathbb{C}$ is called symmetric if $f(x) = f(x \circ \sigma)$ for every $x \in L_{p_1} \times \cdots \times L_{p_n}$ and $\sigma \in \Xi_{[0;1]}$.

Algebras of symmetric polynomials on $L_{p_1} \times \cdots \times L_{p_n}$. Let $p_1, \ldots, p_n \in [1; +\infty)$. For every $\alpha = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$ we define the corresponding sum $S_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ by

$$S_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})} = \sum_{i=1}^{n} \frac{k_i}{p_i}. \tag{6}$$

In cases where the space $L_{p_1} \times \cdots \times L_{p_n}$ is clear from the context, we will denote $S_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ simply as S_α. Let

$$\aleph_{L_{p_1} \times \cdots \times L_{p_n}} = \{\alpha \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\} : S_\alpha \le 1\}. \tag{7}$$

For every $\alpha = (k_1, \ldots, k_n) \in \aleph_{L_{p_1} \times \cdots \times L_{p_n}}$, let $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})} : L_{p_1} \times \cdots \times L_{p_n} \to \mathbb{C}$ be defined by

$$R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}((x_1, \ldots, x_n)) = \int_0^1 x_1^{k_1}(t) \cdot \ldots \cdot x_n^{k_n}(t)\, dt, \tag{8}$$

where $x_i \in L_{p_i}$ for $i \in \{1, \ldots, n\}$. In the case $k_i = 0$ and $x_i \equiv 0$ we will consider $x_i^{k_i}$ to be equal to 1. Note that $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ is a $(k_1 + \ldots + k_n)$-homogeneous polynomial. It was already shown in [19, Theorem 2.10] that such polynomials in the case $p_1 = \ldots = p_n$ are well defined if and only if $S_\alpha \le 1$. In Theorem 3, we will generalize this result on arbitrary $p_1, \ldots, p_n \in [1; +\infty)$. In Theorem 5, it will be shown that for every $\beta \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$ such that $\beta \notin \aleph_{L_{p_1} \times \cdots \times L_{p_n}}$ the corresponding polynomial $R_\beta^{(L_{p_1} \times \cdots \times L_{p_n})}$ is not well defined.

In [19, Theorem 2.10], it was established the structure of symmetric continuous polynomials on the space $L_p^n := L_p \times \cdots \times L_p$, endowed with the norm

$$\|x\|_{p,n} = \left(\sum_{i=1}^{n} \|x_i\|_{L_p}^p \right)^{\frac{1}{p}}, \tag{9}$$

where $x = (x_1, \ldots, x_n) \in L_p^n$. By definitions (3) and (9), we get

$$\|x\|_{L_p^n} \le \|x\|_{p,n} \le n^{\frac{1}{p}} \|x\|_{L_p^n}.$$

Thus, norms (3) and (9) are equivalent on the space L_p^n. Therefore from [19, Theorem 2.10] the following theorem follows.

Theorem 2. *For $p \in [1;+\infty)$, every m-homogeneous symmetric continuous polynomial $P : L_p^n \to \mathbb{C}$, where L_p^n is endowed with the norm* (3)*, can be uniquely represented as an algebraic combination of elements of the set*

$$\Big\{R_\alpha^{(L_p^n)} : \alpha \in \mathbb{Z}_+^n \text{ such that } 1 \le |\alpha| \le \min(m, \lfloor p \rfloor)\Big\}.$$

We will use the following results.

Lemma 1 ([12, Lemma 11])**.** *Let $m \in \mathbb{N}$, $G \subset \mathbb{C}^m$ and $\chi : G \to \mathbb{C}^{m-1}$ be an orthogonal projection $\chi((x_1, x_2, \ldots, x_m)) = (x_2, x_3, \ldots, x_m)$. Let $G_1 = \chi(G)$, $\operatorname{int} G_1 \neq \varnothing$ and for every open set $U \subset G_1$, the set $\chi^{-1}(U)$ is unbounded. If a complex-valued polynomial $Q(x_1, \ldots, x_m)$ is bounded on G, then Q does not depend on x_1.*

Lemma 2 ([23, Theorem 3])**.** *Let $n \in \mathbb{N}$. There exists $K_n > 0$ such that for every mapping $c : \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\} \to \mathbb{C}$ such that*

$$\sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}} |c(k)|^{\frac{1}{|k|}} < \infty,$$

there exists $x_c = (x_1, \ldots, x_n) \in L_\infty^n$ such that

$$\int_0^1 x_1^{k_1}(t) x_2^{k_2}(t) \cdot \ldots \cdot x_n^{k_n}(t)\, dt = c(k)$$

for every $k \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$ and

$$\|x_c\|_{L_\infty^n} \le K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}} |c(k)|^{\frac{1}{|k|}}.$$

2 Symmetric polynomials on $L_{p_1} \times \cdots \times L_{p_n}$

In the current section, we consider $p_1, \ldots, p_n \in [1;+\infty)$ to be fixed numbers and $L_{p_1} \times \cdots \times L_{p_n}$ to be the Cartesian product of spaces $L_{p_1}, \ldots, L_{p_n}$ endowed with the norm (3). We begin with proving couple of relatively simple general assertions. Firstly, let us prove the following lemmas.

Lemma 3. *Let $a > 1$ and $a > b > 0$. Then there exists $\lambda \in (0;1)$ such that $\lambda a > 1 > \lambda b$.*

Proof. Let $0 < \varepsilon < \min\big(\frac{a-1}{a}; \frac{a-b}{ab}\big)$. Let $\lambda = \frac{1}{a} + \varepsilon$. Let us check that λ satisfies all conditions of the lemma. Firstly, $\lambda < \frac{1}{a} + \frac{a-1}{a} = 1$ and $\lambda > \frac{1}{a} > 0$. Therefore $\lambda \in (0;1)$. Secondly, $\lambda a = 1 + a\varepsilon > 1$. Therefore $\lambda a > 1$. Thirdly, $\lambda b = \frac{b}{a} + b\varepsilon < \frac{b}{a} + \frac{a-b}{a} = 1$. Therefore $1 > \lambda b$. Consequently, $\lambda a > 1 > \lambda b$. This completes the proof. □

Lemma 4. *The set L_∞^n is dense in $L_{p_1} \times \cdots \times L_{p_n}$ with respect to the norm* (3)*.*

Proof. Let $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$. Since $x_i \in L_{p_i}$ and $L_{p_i} \subset L_1$ for every $p_i \ge 1$, we can conclude that $x_i \in L_1$. Consequently, x_i is Lebesgue integrable on $[0;1]$ for every $i \in \{1, \ldots, n\}$. Therefore, for every $i \in \{1, \ldots, n\}$ and $\varepsilon > 0$, there exists a measurable function $y_i : [0;1] \to \mathbb{C}$ such that $y_i([0;1])$ is not more than countable and $|x_i(t) - y_i(t)| < \varepsilon$ for every $t \in [0;1]$. Let us show that $y_i \in L_{p_i}$. Since $p_i \ge 1$, we obtain

$$\begin{aligned} |y_i(t)|^{p_i} &\le \big(|x_i(t)| + |x_i(t) - y_i(t)|\big)^{p_i} \le \max\big((2|x_i(t)|)^{p_i}, (2|x_i(t) - y_i(t)|)^{p_i}\big) \\ &\le 2^{p_i}\big(|x_i(t)|^{p_i} + |x_i(t) - y_i(t)|^{p_i}\big) \le 2^{p_i}|x_i(t)|^{p_i} + 2^{p_i}\varepsilon. \end{aligned}$$

Since $x_i \in L_{p_i}$, the right-hand side of the latter inequality is Lebesgue integrable. Consequently, $|y_i(t)|^{p_i}$ is Lebesgue integrable. So, $y_i \in L_{p_i}$.

Let $y = (y_1, \dots, y_n)$. Then we have

$$\begin{aligned} \|x - y\|_{L_{p_1} \times \cdots \times L_{p_n}} &= \max_{i \in \{1,\dots,n\}} \|x_i - y_i\|_{L_{p_i}} \\ &= \max_{i \in \{1,\dots,n\}} \left(\int_0^1 |x_i(t) - y_i(t)|^{p_i} \, dt \right)^{\frac{1}{p_i}} < \max_{i \in \{1,\dots,n\}} \left(\int_0^1 \varepsilon^{p_i} \, dt \right)^{\frac{1}{p_i}} = \varepsilon. \end{aligned} \tag{10}$$

Since $y_i \in L_{p_i}$ and $y_i([0;1])$ is not more than countable, there exists a measurable function with finite number of values $y_i^{(0)} : [0;1] \to \mathbb{C}$ such that $\|y_i^{(0)} - y_i\|_{L_{p_i}} < \varepsilon$. Let $y^{(0)} = \big(y_1^{(0)}, \dots, y_n^{(0)}\big)$. Then we get

$$\big\|y^{(0)} - y\big\|_{L_{p_1} \times \cdots \times L_{p_n}} = \max_{i \in \{1,\dots,n\}} \big\|y_i^{(0)} - y_i\big\|_{L_{p_i}} < \varepsilon. \tag{11}$$

Note that $y_i^{(0)} \in L_\infty$ for every $i \in \{1, \dots, n\}$. Thus, $y^{(0)} \in L_\infty^n$. Consequently, by (10) and (11), we obtain

$$\big\|x - y^{(0)}\big\|_{L_{p_1} \times \cdots \times L_{p_n}} < 2\varepsilon. \tag{12}$$

So, for every $x \in L_{p_1} \times \cdots \times L_{p_n}$ and $\varepsilon > 0$ there exists $y^{(0)} \in L_\infty^n$ such that (12) holds. Therefore L_∞^n is dense in $L_{p_1} \times \cdots \times L_{p_n}$. ☐

Lemma 5. *Let $n \in \mathbb{N}$. There exists $K_n > 0$ such that for every map $c : \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\} \to \mathbb{C}$, such that*

$$\sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\dots,0)\}} |c(k)|^{\frac{1}{|k|}} < \infty,$$

there exists $x_c = (x_{1;c}, \dots, x_{n;c}) \in L_\infty^n$, such that

(i) $\int_0^1 x_{1;c}^{k_1}(t) x_{2;c}^{k_2}(t) \cdot \ldots \cdot x_{n;c}^{k_n}(t) \, dt = c(k)$ *for every* $k \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$;

(ii) $x_{i;c}(t) = 0$ *for every* $t \in [0; \frac{1}{2})$ *and* $i \in \{1, \dots, n\}$;

(iii) $\|x_c\|_{L_\infty^n} \le 2K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\dots,0)\}} |c(k)|^{\frac{1}{|k|}}$.

Proof. Let K_n be given by Lemma 2. Let $c' : \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\} \to \mathbb{C}$ be the mapping defined by $c'(k) = 2c(k)$ for $k \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$. Firstly, according to Lemma 2, there exists $u_{c'} = (u_{1;c'}, \dots, u_{n;c'}) \in L_\infty^n$ such that

$$\int_0^1 u_{1;c'}^{k_1}(t) u_{2;c'}^{k_2}(t) \cdot \ldots \cdot u_{n;c'}^{k_n}(t) \, dt = c'(k)$$

for every $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$ and

$$\|u_{c'}\|_{L_\infty^n} \le K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\dots,0)\}} |c'(k)|^{\frac{1}{|k|}}. \tag{13}$$

Let

$$x_{i;c}(t) = \begin{cases} u_{i,c'}(2t - 1), & \text{if } t \in [\frac{1}{2}; 1], \\ 0, & \text{if } t \in [0; \frac{1}{2}) \end{cases} \tag{14}$$

for $i \in \{1, \ldots, n\}$. Let $x_c = (x_{1;c}, \ldots, x_{n;c})$. Note that $x_c \in L_\infty^n$ and

$$\|x_c\|_{L_\infty^n} \leq \|u_{c'}\|_{L_\infty^n}. \tag{15}$$

Let us prove Condition *(i)*. By (14), for every $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$, we have

$$\int_0^1 x_{1;c}^{k_1}(t) x_{2;c}^{k_2}(t) \cdot \ldots \cdot x_{n;c}^{k_n}(t)\, dt = \int_{\frac{1}{2}}^1 u_{1;c}^{k_1}(2t-1) u_{2;c}^{k_2}(2t-1) \cdot \ldots \cdot u_{n;c}^{k_n}(2t-1)\, dt. \tag{16}$$

Let $s = 2t - 1$. Then, by (13), we obtain

$$\begin{aligned} &\int_{\frac{1}{2}}^1 u_{1;c}^{k_1}(2t-1) u_{2;c}^{k_2}(2t-1) \cdot \ldots \cdot u_{n;c}^{k_n}(2t-1)\, dt \\ &\qquad = \frac{1}{2}\int_0^1 u_{1;c}^{k_1}(s) u_{2;c}^{k_2}(s) \cdot \ldots \cdot u_{n;c}^{k_n}(s)\, ds = \frac{1}{2} c'(k) = c(k). \end{aligned} \tag{17}$$

By (16) and (17), we get

$$\int_0^1 x_{1;c}^{k_1}(t) x_{2;c}^{k_2}(t) \cdot \ldots \cdot x_{n;c}^{k_n(t)}\, dt = c(k)$$

for every $k \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$. Thus, Condition *(i)* is proven.

Condition *(ii)* is an immediate consequence of (14).

Let us prove Condition *(iii)*. By (15) and (13), taking into account the equality $c'(k) = 2c(k)$ and the inequality $|k| \geq 1$, we can conclude

$$\|x_c\|_{L_\infty^n} \leq \|u_{c'}\|_{L_\infty^n} \leq K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}} |c'(k)|^{\frac{1}{|k|}} \leq 2K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}} |c(k)|^{\frac{1}{|k|}}.$$

Thus, Condition *(iii)* is proven. Therefore all conditions of the lemma are proven. ☐

Theorem 3. *Let $\alpha \in \aleph_{L_{p_1} \times \cdots \times L_{p_n}}$, where the set $\aleph_{L_{p_1} \times \cdots \times L_{p_n}}$ is defined by* (7)*. The polynomial $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ is well defined by* (8)*, continuous and $\|R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}\|_{L_{p_1} \times \cdots \times L_{p_n}} = 1$.*

Proof. Let $\alpha = (k_1, \ldots, k_n)$, $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$. Let us show that $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}(x)$ exists. By (7), we have $\sum_{i=1}^n \frac{k_i}{p_i} \leq 1$. So,

$$k_1 p_2 p_3 \ldots p_n + k_2 p_1 p_3 \ldots p_n + \ldots + k_n p_1 p_2 \ldots p_{n-1} \leq \prod_{i=1}^n p_i.$$

Then $s \geq 0$, where

$$s = \prod_{i=1}^n p_i - (k_1 p_2 p_3 \ldots p_n + k_2 p_1 p_3 \ldots p_n + \ldots + k_n p_1 p_2 \ldots p_{n-1}). \tag{18}$$

The weighted inequality of arithmetic and geometric means states that for every natural number r, nonnegative numbers $a_1, \ldots, a_r$ and nonnegative weights $w_1, \ldots, w_r$ such that $w = w_1 + \ldots + w_r > 0$, the following inequality holds

$$(a_1^{w_1} \cdot \ldots \cdot a_r^{w_r})^{\frac{1}{w}} \leq \frac{a_1 w_1 + \ldots + a_r w_r}{w}.$$

Here the convention $0^0 = 1$ is used. Let us use this inequality for $r = n+1$ and numbers

$$a_1 = |x_1(t)|^{p_1},\ a_2 = |x_2(t)|^{p_2}, \ldots,\ a_n = |x_n(t)|^{p_n},\ a_{n+1} = 1,$$

where $t \in [0;1]$, with weights

$$w_1 = k_1 p_2 p_3 \ldots p_n,\ w_2 = k_2 p_1 p_3 \ldots p_n, \ldots,\ w_n = k_n p_1 p_2 \ldots p_{n-1},\ w_{n+1} = s,$$

where s is defined by (18). This gives the following inequality

$$\left(\prod_{i=1}^{n} |x_i(t)|^{k_i \prod_{i=1}^{n} p_i}\right)^{\frac{1}{\prod_{i=1}^{n} p_i}} \leq \frac{k_1 p_2 p_3 \ldots p_n |x_1(t)|^{p_1} + \ldots + k_n p_1 p_2 \ldots p_{n-1} |x_n(t)|^{p_n} + s}{\prod_{i=1}^{n} p_i}$$

for every $t \in [0;1]$. Therefore

$$\prod_{i=1}^{n} |x_i(t)|^{k_i} \leq \sum_{i=1}^{n} \frac{k_i}{p_i} |x_i(t)|^{p_i} + \frac{s}{\prod_{i=1}^{n} p_i} \tag{19}$$

for every $t \in [0;1]$. Since $x_i \in L_{p_i}$, the right-hand side of the above inequality is integrable. Thus, the left-hand side of (19) is Lebesgue integrable. Consequently, taking into account (8), we obtain that $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ is well defined. Moreover, if $\|x_i\|_{L_{p_i}} \leq 1$ for every $i \in \{1, \ldots, n\}$, then $\int_0^1 |x_i(t)|^{p_i}\, dt \leq 1$ and, by (19) and (18), we obtain

$$R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}(x) \leq \sum_{i=1}^{n} \frac{k_i}{p_i} + \frac{s}{\prod_{i=1}^{n} p_i} = 1.$$

Thus,

$$\|R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}\|_{L_{p_1} \times \cdots \times L_{p_n}} \leq 1. \tag{20}$$

So, $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}$ is bounded, and therefore it is continuous.

Let us show that

$$\|R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}\|_{L_{p_1} \times \cdots \times L_{p_n}} = 1.$$

Let $y_i(t) = 1$ for every $i \in \{1, \ldots, n\}$ and every $t \in [0;1]$. Let $y = (y_1, \ldots, y_n)$. Then $y \in L_{p_1} \times \cdots \times L_{p_n}$ and $\|y\|_{L_{p_1} \times \cdots \times L_{p_n}} = 1$. Note that $R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}(y) = 1$. Consequently,

$$\|R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}\|_{L_{p_1} \times \cdots \times L_{p_n}} \geq 1. \tag{21}$$

By (20) and (21), we obtain

$$\|R_\alpha^{(L_{p_1} \times \cdots \times L_{p_n})}\|_{L_{p_1} \times \cdots \times L_{p_n}} = 1.$$

This completes the proof. □

Theorem 4. *Let $\alpha \in \aleph_{L_{p_1}\times\cdots\times L_{p_n}}$, where the set $\aleph_{L_{p_1}\times\cdots\times L_{p_n}}$ is defined by* (7)*. The polynomial $R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}$, defined by* (8)*, is symmetric.*

Proof. Let $\alpha = (k_1,\ldots,k_n)$. Let us show that for every $\sigma \in \Xi_{[0;1]}$ and $x \in L_{p_1} \times \cdots \times L_{p_n}$, we have

$$R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) = R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma).$$

Let us assume that there is $\sigma_0 \in \Xi_{[0;1]}$ and $x = (x_1,\ldots,x_n) \in L_{p_1}\times\cdots\times L_{p_n}$, such that

$$R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) \neq R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma_0).$$

Let us choose an arbitrary ε such that

$$0 < \varepsilon < \frac{1}{2}\Big|R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) - R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma_0)\Big|.$$

By Theorem 3, $R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}$ is well defined. Therefore, by (8), we obtain that the function $t \in [0;1] \mapsto x_1^{k_1}(t)\cdot\ldots\cdot x_n^{k_n}(t)$ is Lebesgue integrable. Consequently, there exists a measurable function $u : [0;1] \to \mathbb{C}$ such that $u([0;1])$ is not more than countable and

$$|x_1^{k_1}(t)\cdot\ldots\cdot x_n^{k_n}(t) - u(t)| < \varepsilon \tag{22}$$

for every $t \in [0;1]$. Consequently,

$$\left|R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) - \int_0^1 u(t)\,dt\right| < \varepsilon. \tag{23}$$

By (22), we have

$$\big|x_1^{k_1}(\sigma_0(t))\cdot\ldots\cdot x_n^{k_n}(\sigma_0(t)) - u(\sigma_0(t))\big| < \varepsilon$$

for every $t \in [0;1]$. Consequently,

$$\left|R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma_0) - \int_0^1 u(\sigma_0(t))\,dt\right| < \varepsilon. \tag{24}$$

Let us show that

$$\int_0^1 u(\sigma_0(t))\,dt = \int_0^1 u(t)\,dt. \tag{25}$$

Since u is a measurable function, such that $u([0;1])$ is not more than countable, there exist a sequence of complex numbers $\{a_j\}_{j=1}^\infty$ and a sequence of disjoint sets $\{E_j\}_{j=1}^\infty$ that are measurable subsets of $[0;1]$ such that

$$u(t) = \sum_{j=1}^{\infty} a_j 1_{E_j}(t)$$

for $t \in [0;1]$, where $1_{E_1}, 1_{E_2}, \ldots$ are indicator functions. Let $v_r(t) = \sum_{j=1}^r a_j 1_{E_j}(t)$. Since $\{v_r\}_{r=1}^\infty$ and $\{v_r\circ\sigma_0\}_{r=1}^\infty$ converge pointwise to u and $u\circ\sigma_0$, respectively, by Dominated convergence theorem, we get

$$\lim_{r\to\infty}\int_0^1 v_r(t)\,dt = \int_0^1 u(t)\,dt \quad\text{and}\quad \lim_{r\to\infty}\int_0^1 v_r(\sigma_0(t))\,dt = \int_0^1 u(\sigma_0(t))\,dt. \tag{26}$$

Let us show that

$$\int_0^1 v_j(\sigma_0(t))\,dt = \int_0^1 v_j(t)\,dt \tag{27}$$

for every $j \in \mathbb{N}$. It is enough to prove that

$$\int_0^1 1_{E_j}(t)\,dt = \int_0^1 1_{E_j}(\sigma(t))\,dt$$

for every $j \in \mathbb{N}$. Since 1_{E_j} is an indicator function and $\sigma_0 \in \Xi_{[0;1]}$, we obtain

$$\int_0^1 1_{E_j}(t)\,dt = \mu(E_j) = \mu(\sigma_0(E_j)) = \int_0^1 1_{E_j}(\sigma_0(t))\,dt.$$

Thus, (27) is proven. By (26) and (27), (25) is correct. By (23), (24) and (25), we get

$$\left| R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) - R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma_0) \right| < 2\varepsilon.$$

This directly contradicts the choice of ε. Consequently, for every $\sigma \in \Xi_{[0;1]}$ and every $x \in L_{p_1} \times \cdots \times L_{p_n}$, we obtain

$$R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x) = R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}(x\circ\sigma).$$

So, $R_\alpha^{(L_{p_1}\times\cdots\times L_{p_n})}$ is symmetric. This completes the proof. □

Theorem 5. *For every $\beta = (q_1, \ldots, q_n) \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$ such that $\beta \notin \aleph_{L_{p_1}\times\cdots\times L_{p_n}}$, where $\aleph_{L_{p_1}\times\cdots\times L_{p_n}}$ is defined by* (7)*, there exists $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$ such that*

$$\int_0^1 x_1^{q_1}(t) \cdot \ldots \cdot x_n^{q_n}(t)\,dt = \infty.$$

Proof. By (6), $S_\beta = \sum_{i=1}^n \frac{q_i}{p_i}$. Since $\beta \notin \aleph_{L_{p_1}\times\cdots\times L_{p_n}}$, it follows that $S_\beta > 1$. Let

$$x_i(t) = t^{-\frac{1}{p_i} + \frac{S_\beta - 1}{n(q_i+1)}} \tag{28}$$

for every $i \in \{1, \ldots, n\}$, where $t \in [0;1]$. Then

$$|x_i(t)|^{p_i} = t^{-1 + \frac{p_i(S_\beta-1)}{n(q_i+1)}}$$

for every $t \in [0;1]$. Note that

$$-1 + \frac{p_i(S_\beta - 1)}{n(q_i+1)} > -1.$$

Therefore $x_i \in L_{p_i}$ for every $i \in \{1, \ldots, n\}$. So, $x = (x_1, \ldots, x_n) \in L_{p_1} \times \cdots \times L_{p_n}$.

By (28), we have

$$\int_0^1 x_1^{q_1}(t) \cdot \ldots \cdot x_n^{q_n}(t)\,dt = \int_0^1 t^{-\sum_{i=1}^n \frac{q_i}{p_i} + \sum_{i=1}^n \frac{q_i(S_\beta-1)}{n(q_i+1)}}\,dt > \int_0^1 t^{-\sum_{i=1}^n \frac{q_i}{p_i} + \sum_{i=1}^n \frac{(q_i+1)(S_\beta-1)}{n(q_i+1)}}\,dt$$
$$= \int_0^1 t^{-\sum_{i=1}^n \frac{q_i}{p_i} + \sum_{i=1}^n \frac{(S_\beta-1)}{n}}\,dt = \int_0^1 t^{-S_\beta + S_\beta - 1}\,dt = \int_0^1 t^{-1}\,dt = \infty.$$

This completes the proof. □

3 Algebraic basis of the algebra of all continuous symmetric complex-valued polynomials on $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_n}$, where $p_1 \geq p_2 \geq \cdots \geq p_n$

In the current section we will assume that

$$p_1 \geq p_2 \geq \cdots \geq p_n. \tag{29}$$

Let

$$L = L_{p_1} \times L_{p_2} \times \cdots \times L_{p_n}. \tag{30}$$

For any q_1, q_2 such that $1 \leq q_1 \leq q_2 \leq \infty$, it is well known that

$$L_{q_2} \subset L_{q_1} \quad \text{and} \quad \|x\|_{L_{q_1}} \leq \|x\|_{L_{q_2}} \text{ for every } x \in L_{q_2}. \tag{31}$$

Consequently,

$$L_\infty^n \subset L_{p_1}^n \subset L \tag{32}$$

and

$$\|x\|_{L_{p_1}^n} \geq \|x\|_L, \quad x \in L_{p_1}^n. \tag{33}$$

Lemma 6. *The restriction to $L_{p_1}^n$ of any m-homogeneous symmetric continuous polynomial $P : L \to \mathbb{C}$ is also an m-homogeneous symmetric continuous polynomial.*

Proof. It can be checked that this restriction is a symmetric m-homogeneous polynomial. Let us prove its continuity. Consider any sequence $\{x_j\}_{j=1}^\infty \subset L_{p_1}^n$ that converges to some $x \in L_{p_1}^n$. By (32), $x_j \in L$ and $x \in L$. Let us prove that $\{x_j\}_{j=1}^\infty$ converges to x in L. Suppose $\{x_j\}_{j=1}^\infty$ does not converge to x in L. Then there exists $\delta > 0$ such that $\|x - x_j\|_L > \delta$ for infinitely many j. By (33), we have $\|x - x_j\|_{L_{p_1}^n} \geq \|x - x_j\|_L > \delta$ and therefore the sequence $\{x_j\}_{j=1}^\infty$ does not converge to x in the space $L_{p_1}^n$. This contradicts our assumption. So, $\{x_j\}_{j=1}^\infty$ converges to x in the space L. Consequently, by the continuity of P, $\{P(x_j)\}_{j=1}^\infty$ converges to $P(x)$. Therefore the restriction of P to $L_{p_1}^n$ is continuous. This completes the proof. □

For $m \in \mathbb{N}$, let

$$\aleph_L^{(m)} = \big\{\alpha \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\} : 1 \leq |\alpha| \leq \min(m, \lfloor p_1 \rfloor)\big\}. \tag{34}$$

Lemma 7. *The restriction of a symmetric continuous m-homogeneous polynomial $P : L \to \mathbb{C}$ to the space $L_{p_1}^n$ can be uniquely represented as an algebraic combination of elements of the set*

$$\big\{R_\alpha^{(L_{p_1}^n)} : \alpha \in \aleph_L^{(m)}\big\}, \tag{35}$$

where $R_\alpha^{(L_{p_1}^n)}$ is defined by (8) *and $\aleph_L^{(m)}$ is defined by* (34).

Proof. Let P_1 be the restriction of P to $L_{p_1}^n$. By Lemma 6, P_1 is an m-homogeneous symmetric continuous polynomial. By Theorem 2, P_1 can be uniquely represented as an algebraic combination of elements of the set

$$\Big\{R_\alpha^{(L_{p_1}^n)} : \alpha \in \mathbb{Z}_+^n \text{ such that } 1 \leq |\alpha| \leq \min(m, \lfloor p_1 \rfloor)\Big\}.$$

By (34), this set is exactly the set (35). This completes the proof. □

Note that $\aleph_L^{(m)}$ is a finite set. For every $\alpha = (k_1, \ldots, k_n) \in \aleph_L^{(m)}$ and $i \in \{1, \ldots, n\}$, we get

$$m \geq k_i. \tag{36}$$

For multi-indices $\beta = (k_{\beta;1}, \ldots, k_{\beta;n})$, $\alpha = (k_{\alpha;1}, \ldots, k_{\alpha;n}) \in \aleph_L^{(m)}$, let

$$\beta > \alpha \tag{37}$$

if one of the following conditions holds true:

1) $S_\beta > S_\alpha$, where S_β and S_α are defined by (6);

2) $S_\beta = S_\alpha$ and there exists $t \in \{1, \ldots, n\}$ such that for any $l \in \{1, \ldots, t-1\}$, $k_{\beta;l} = k_{\alpha;l}$ and $k_{\beta;t} < k_{\alpha,t}$, i.e. β precedes α in lexicographic order.

Let us provide an example of inequality between multi-indices.

Example 1. *Let $m = 7$, $n = 2$, $p_1 = 6$, $p_2 = 4$, $\alpha = (3,2), \beta = (6,0); \gamma = (5,0)$. By (34), we have $\alpha, \beta, \gamma \in \aleph^{(7)}_{L_6 \times L_4}$. Note that $S_\alpha = 1$, $S_\beta = 1$, $S_\gamma = \frac{5}{6}$. By Condition 2), we get $\alpha > \beta$. By Condition 1), we obtain $\beta > \gamma$.*

Lemma 8. *The relation* (37) *is a strict linear order on $\aleph_L^{(m)}$.*

Proof. Let $\alpha = (k_{\alpha;1}, \ldots, k_{\alpha;n})$, $\beta = (k_{\beta;1}, \ldots, k_{\beta;n})$, $\gamma = (k_{\gamma;1}, \ldots, k_{\gamma;n}) \in \aleph_L^{(m)}$.

Let us prove irreflexivity. Assume $\alpha > \alpha$. Then one of the two conditions, Condition 1) or Condition 2), must hold. If Condition 1) is satisfied, then $S_\alpha > S_\alpha$, which is clearly contradictory. If Condition 2) is satisfied, then there exists $t \in \{1, \ldots, n\}$ such that $k_{\alpha;t} < k_{\alpha;t}$, which is clearly contradictory. Thus, our initial assumption is false. Therefore, the relation is irreflexive.

Let us prove asymmetry. Assume $\beta > \alpha$ and $\alpha > \beta$, then one of the two conditions, Condition 1) or Condition 2), must hold for both inequalities. Thus, $S_\beta \geq S_\alpha$ and $S_\alpha \geq S_\beta$. Therefore, $S_\alpha = S_\beta$. Consequently, the condition that is satisfied for both inequalities is Condition 2). So, since $\beta > \alpha$, there exists $t \in \{1, \ldots, n\}$ such that for any $l \in \{1, \ldots, t-1\}$, $k_{\beta;l} = k_{\alpha;l}$ and $k_{\beta;t} < k_{\alpha;t}$. So, there does not exist any $t_1 \in \{1, \ldots, n\}$ such that for any $l_1 \in \{1, \ldots, t_1 - 1\}$, $k_{\beta;l_1} = k_{\alpha;l_1}$ and $k_{\beta;t_1} > k_{\alpha,t_1}$. Thus, for $\alpha > \beta$ Condition 2) is not satisfied. So, our initial assumption is false. Therefore, the relation is asymmetric.

Let us prove that the relation is transitive. Let $\alpha > \beta$ and $\beta > \gamma$. Then, $S_\alpha \geq S_\beta \geq S_\gamma$. There are two cases: $S_\alpha > S_\gamma$ and $S_\alpha = S_\beta = S_\gamma$. Consider the case $S_\alpha > S_\gamma$. In this case, $\alpha > \gamma$ by Condition 1). Consider the case $S_\alpha = S_\beta = S_\gamma$. In this case, there exists $t_1 \in \{1, \ldots, n\}$, such that for any $l_1 \in \{1, \ldots, t_1 - 1\}$, $k_{\alpha;l_1} = k_{\beta;l_1}$ and $k_{\alpha;t_1} < k_{\beta,t_1}$. Moreover, there exists $t_2 \in \{1, \ldots, n\}$ such that for any $l_2 \in \{1, \ldots, t_2 - 1\}$, $k_{\beta;l_2} = k_{\gamma;l_2}$ and $k_{\beta;t_2} < k_{\gamma,t_2}$. Let $t = \min(t_1; t_2)$. Then for any $l \in \{1, \ldots, t-1\}$, $k_{\gamma;l} = k_{\beta;l} = k_{\alpha;l}$ and $k_{\gamma;t} > k_{\alpha,t}$. Thus, Condition 2) is satisfied, so, $\alpha > \gamma$. Hense, the relation is transitive.

Let us prove that the relation is connected. Let α, β be such that $\alpha \neq \beta$. Let us show that either $\alpha > \beta$ or $\beta > \alpha$. There are three cases: $S_\alpha > S_\beta$, $S_\alpha < S_\beta$ and $S_\alpha = S_\beta$. In the first case, $\alpha > \beta$ by Condition 1). In the second case, $\alpha < \beta$ by Condition 1). Consider the case $S_\alpha = S_\beta$. Since $\alpha \neq \beta$, there exists $j \in \{1, \ldots, n\}$ such that $k_{\alpha;j} \neq k_{\beta;j}$. Let

$$t = \min \{ j \in \{1, \ldots, n\} : k_{\alpha;j} \neq k_{\beta;j} \}.$$

Therefore, for every $l \in \{1, \ldots, t-1\}$, $k_{\beta;l} = k_{\alpha;l}$ and $k_{\beta;t} \neq k_{\alpha;t}$. If $k_{\beta;t} > k_{\alpha;t}$, then $\alpha > \beta$ by Condition 2). If $k_{\beta;t} < k_{\alpha;t}$, then $\alpha < \beta$ by Condition 2). Thus, the relation is connected.

Hence, the relation is a strict linear order. □

Let q be the cardinality of the set $\aleph_L^{(m)}$, i.e.

$$q = |\aleph_L^{(m)}|. \tag{38}$$

Let us enumerate elements of $\aleph_L^{(m)}$ in the descending order with respect to the relation (37) that is a strict linear order by Lemma 8, namely

$$\alpha_1 > \alpha_2 > \cdots > \alpha_q. \tag{39}$$

By (37), we have

$$S_{\alpha_k} \le S_{\alpha_l} \quad \text{for every} \quad k, l \in \{1, \ldots, q\} \quad \text{such that} \quad k \ge l. \tag{40}$$

Lemma 9. *The lowest element α_q in $\aleph_L^{(m)}$ is equal to $(1, 0, \ldots, 0)$.*

Proof. Let us show that every element $\beta = (k_{\beta;1}, \ldots, k_{\beta;n}) \in \aleph_L^{(m)}$ such that $\beta \neq (1, 0, \ldots, 0)$ is greater than $(1, 0, \ldots, 0)$. There are two cases:

(i) there is $j \in \{2, \ldots, n\}$ such that $k_{\beta;j} \neq 0$;

(ii) for every $j \in \{2, \ldots, n\}$, $k_{\beta;j} = 0$.

Consider the Case (ii). By (6), we have

$$S_\beta = \frac{k_{\beta;1}}{p_1}, \tag{41}$$

since $k_{\beta;j} = 0$ for every $j \in \{2, \ldots, n\}$, i.e. $\beta = (k_{\beta;1}, 0, \ldots, 0)$. Let us show that $k_{\beta;1} > 1$. By (34), we can conclude that $\beta \neq (0, \ldots, 0)$. Consequently, $k_{\beta;1} \neq 0$. By the assumption, $\beta \neq (1, 0, \ldots, 0)$. Consequently, $k_{\beta;1} \neq 1$. Thus, $k_{\beta;1} > 1$. Therefore, by (41), $S_\beta > \frac{1}{p_1}$. By (6), we have $S_{(1,\ldots,0)} = \frac{1}{p_1}$. Consequently, $S_\beta > S_{(1,\ldots,0)}$. Therefore $\beta > (1, 0, \ldots, 0)$ by Condition 1).

Consider the Case (i). Since there exists $j \in \{2, \ldots, n\}$ such that $k_{\beta;j} \neq 0$, it follows that $\frac{k_{\beta;j}}{p_j} \ge \frac{1}{p_j}$. By (6), we have $S_\beta \ge \frac{k_{\beta;j}}{p_j}$ and $\frac{1}{p_1} = S_{(1,0,\ldots,0)}$. By (29), we get $\frac{1}{p_j} \ge \frac{1}{p_1}$. Therefore we can conclude that

$$S_\beta \ge \frac{k_{\beta;j}}{p_j} \ge \frac{1}{p_j} \ge \frac{1}{p_1} = S_{(1,0,\ldots,0)}.$$

Then this case can be divided into two subcases: $S_\beta > S_{(1,0,\ldots,0)}$ and $S_\beta = S_{(1,0,\ldots,0)}$.

Consider the subcase $S_\beta > S_{(1,0,\ldots,0)}$. We have $\beta > (1, 0, \ldots, 0)$ by Condition 1). Consider the subcase $S_\beta = S_{(1,0,\ldots,0)}$. We obtain

$$S_\beta = \frac{k_{\beta;j}}{p_j} = \frac{1}{p_j} = \frac{1}{p_1} = S_{(1,0,\ldots,0)}$$

by the equality $S_\beta = \frac{k_{\beta;j}}{p_j}$ and by (6), taking into account that $j \ge 2$, $k_{\beta;1} = 0$. Consequently, $k_{\beta,1}$ is less than the first component of the multi-index $(1, 0, \ldots, 0)$. Therefore $\beta > (1, 0, \ldots, 0)$ by Condition 2). □

Let $\eta : L^n_{p_1} \to \mathbb{C}^q$ be defined by

$$\eta(x) = \Big(R^{(L^n_{p_1})}_{\alpha_1}(x), \ldots, R^{(L^n_{p_1})}_{\alpha_q}(x)\Big) \tag{42}$$

for $x \in L^n_{p_1}$, where q, α_j and $R^{(L^n_{p_1})}_{\alpha_j}$ are defined by (38), (39) and (8), respectively.

Lemma 10. *Let $P : L \to \mathbb{C}$ be an m-homogeneous symmetric continuous polynomial. Let P_1 be the restriction of P to the space $L^n_{p_1}$. There exists a polynomial $Q : \mathbb{C}^q \to \mathbb{C}$ such that $Q(\eta(x)) = P_1(x)$ for every $x \in L^n_{p_1}$, where η is defined by* (42)*.*

Proof. According to Lemma 7, taking into account (39), P_1 is an algebraic combination of $R^{(L^n_{p_1})}_{\alpha_1}, \ldots, R^{(L^n_{p_1})}_{\alpha_q}$. Therefore, by the definition of algebraic combination, there exists a polynomial $Q : \mathbb{C}^q \to \mathbb{C}$ such that

$$Q\Big(R^{(L^n_{p_1})}_{\alpha_1}(x), \ldots, R^{(L^n_{p_1})}_{\alpha_q}(x)\Big) = P_1(x)$$

for every $x \in L^n_{p_1}$. Consequently, taking into account (42), we obtain $Q(\eta(x)) = P_1(x)$ for every $x \in L^n_{p_1}$. ☐

Our goal is to prove that Q, given by Lemma 10, does not depend on the several first variables, for which the respective sums S_{α_j} are greater than 1. Now let us prove this in three steps.

Lemma 11. *Let $m \in \mathbb{N}$. Let $r \in \mathbb{Z}_+$ be such that $r < q$, where q is defined by* (38)*. Let*

$$\alpha_j = (k_{\alpha_j;1}, \ldots, k_{\alpha_j;n}) \in \aleph^{(m)}_L, \quad j \in \{1, \ldots, q\}, \tag{43}$$

be defined by (39)*. Suppose $S_{\alpha_{r+1}} > 1$, where S_α is defined by* (6)*. Then there exists $C_1 > 0$ such that for every $C_2 > 0$ there exists $(u_1, u_2, \ldots, u_n) \in L^n_{p_1}$ such that the following conditions are satisfied:*

a) $u_i(t) = 0$ for every $t \in [\frac{1}{2}; 1]$ and $i \in \{1, 2, \ldots, n\}$;

b) $\|u_i\|_{L_{p_i}} < C_1$ for every $i \in \{1, 2, \ldots, n\}$;

c) $\Big|R^{(L^n_{p_1})}_{\alpha_j}((u_1, \ldots, u_n))\Big| < C_1$ for every $j \in \{r+2, r+3, \ldots, q\}$;

d) $\Big|R^{(L^n_{p_1})}_{\alpha_{r+1}}((u_1, \ldots, u_n))\Big| > C_2$.

Proof. Let

$$l = \min\big\{j \in \{1, \ldots, q - r - 1\} : S_{\alpha_{r+1}} > S_{\alpha_{r+j+1}}\big\}.$$

Such l exists since $S_{\alpha_q} = \frac{1}{p_1} < 1$ by Lemma 9 and (6) while $S_{\alpha_{r+1}} > 1$ by assumptions of the Lemma. Consequently,

$$S_{\alpha_{r+1}} = S_{\alpha_{r+j}}, \quad j \in \{1, \ldots, l\}, \tag{44}$$

and $S_{\alpha_{r+1}} > S_{\alpha_{r+l+1}}$. By (6), we have $S_{\alpha_{r+l+1}} > 0$. Then, by Lemma 3, there exists λ, $0 < \lambda < 1$, such that

$$\lambda S_{\alpha_{r+1}} > 1 > \lambda S_{\alpha_{r+l+1}}. \tag{45}$$

Let

$$x_i(t) = \begin{cases} t^{-\frac{\lambda}{p_i}+\delta_i}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0, \end{cases} \tag{46}$$

where $i \in \{1, 2, \ldots, n\}$, $\delta_i > 0$ and $t \in [0; 1]$.

Let us show that $x_i \in L_{p_i}$. Note that

$$\|x_i\|_{L_{p_i}} = \left(\int_0^1 \left(t^{-\frac{\lambda}{p_i}+\delta_i} \right)^{p_i} dt \right)^{\frac{1}{p_i}} = \left(\int_0^1 t^{-\lambda+\delta_i p_i} \, dt \right)^{\frac{1}{p_i}} < \infty$$

since $-\lambda + \delta_i p_i > -1$. Therefore

$$x_i \in L_{p_i}. \tag{47}$$

Note that

$$x_1^{k_{\alpha_j;1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_j;n}}(t) = t^{-\frac{\lambda k_{\alpha_j;1}}{p_1}+\delta_1 k_{\alpha_j;1}} \cdot \ldots \cdot t^{-\frac{\lambda k_{\alpha_j;n}}{p_n}+\delta_n k_{\alpha_j;n}} = t^{-\lambda S_{\alpha_j}+\sum_{i=1}^n \delta_i k_{\alpha_j;i}} \tag{48}$$

for every $j \in \{1, \ldots, q\}$, where $(k_{\alpha_j;1}, \ldots, k_{\alpha_j;n})$ is defined by (43) and $t \in (0; 1]$.

Let $f \in \mathbb{N}$ be such that

$$r + l + 1 \le r + f \le q.$$

Let us show that for arbitrary $\delta_1 > 0, \ldots, \delta_n > 0$ we get

$$\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_{r+f};n}}(t) \, dt < \infty, \tag{49}$$

where $(k_{\alpha_{r+f};1}, \ldots, k_{\alpha_{r+f};n}) = \alpha_{r+f}$. By (48), we have

$$\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_{r+f};n}}(t) \, dt = \int_0^1 t^{-\lambda S_{\alpha_{r+f}}+\sum_{i=1}^n \delta_i k_{\alpha_{r+f};i}} \, dt. \tag{50}$$

By (40), $S_{\alpha_{r+f}} \le S_{\alpha_{r+l+1}}$ since $r + l + 1 \le r + f$. Therefore, taking into account that $\lambda > 0$, we obtain $-\lambda S_{\alpha_{r+f}} \ge -\lambda S_{\alpha_{r+l+1}}$. Consequently,

$$-\lambda S_{\alpha_{r+f}} + \sum_{i=1}^n \delta_i k_{\alpha_{r+f};i} \ge -\lambda S_{\alpha_{r+l+1}} + \sum_{i=1}^n \delta_i k_{\alpha_{r+f};i}. \tag{51}$$

By (45), we get $\lambda S_{\alpha_{r+l+1}} < 1$. Therefore $-\lambda S_{\alpha_{r+l+1}} > -1$. For every $i \in \{1, \ldots, n\}$ we have $\delta_i > 0$ and $k_{\alpha_{r+f};i} \ge 0$, therefore

$$\sum_{i=1}^n \delta_i k_{\alpha_{r+f};i} \ge 0.$$

By the latter two inequalities, we obtain

$$-\lambda S_{\alpha_{r+l+1}} + \sum_{i=1}^n \delta_i k_{\alpha_{r+f};i} > -1. \tag{52}$$

By (51) and (52), we can conclude

$$-\lambda S_{\alpha_{r+f}} + \sum_{i=1}^n \delta_i k_{\alpha_{r+f};i} > -1.$$

Consequently, the integral (50) is finite.

Let us show that there exist $\delta_1 > 0, \ldots, \delta_n > 0$ such that following n conditions are satisfied:

$$\begin{aligned} &1)\ \sum_{i=1}^{n} \delta_i k_{\alpha_{r+1};i} = \lambda S_{\alpha_{r+1}} - 1;\\ &2)\ \delta_1 > m \sum_{i=2}^{n} \delta_i;\\ &3)\ \delta_2 > m \sum_{i=3}^{n} \delta_i;\\ &\ldots\\ &n)\ \delta_{n-1} > m\delta_n. \end{aligned} \tag{53}$$

Let us arbitrarily define $\phi_n > 0$. Assume that we have already defined every ϕ_j, $j \in \{s+1, \ldots, n\}$ for some $s \in \{1, \ldots, n-1\}$. Let us choose an arbitrary ϕ_s such that

$$\phi_s > m \sum_{i=s+1}^{n} \phi_i. \tag{54}$$

So, we have defined ϕ_j for $j \in \{s, \ldots, n\}$. Therefore we can define $\phi_1, \ldots, \phi_n$ such that for every $s \in \{1, \ldots, n-1\}$ condition (54) is satisfied.

Set $\delta_i = \gamma\phi_i$ for $i \in \{1, \ldots, n\}$, where

$$\gamma = \frac{\lambda S_{\alpha_{r+1}} - 1}{\sum_{i=1}^{n} \phi_i k_{\alpha_{r+1};i}}.$$

Then

$$\sum_{i=1}^{n} \delta_i k_{\alpha_{r+1};i} = \gamma \sum_{i=1}^{n} \phi_i k_{\alpha_{r+1};i} = \lambda S_{\alpha_{r+1}} - 1,$$

i.e. Condition 1) of (53) is satisfied. By (54), Condition $j)$ of (53) for every $j \in \{2, \ldots, n\}$ is satisfied, indeed

$$\delta_{j-1} = \gamma\phi_{j-1} > \gamma m \sum_{i=j}^{n} \phi_i = m \sum_{i=j}^{n} \gamma\phi_i = m \sum_{i=j}^{n} \delta_i.$$

Fix $\delta_1 > 0, \ldots, \delta_n > 0$ that satisfy (53). By (48), taking into account Condition 1) of (53), we obtain

$$x_1^{k_{\alpha_{r+1};1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_{r+1};n}}(t) = t^{-1} \tag{55}$$

for $t \in (0;1]$.

Let $f \in \mathbb{N}$ be such that

$$r+1 < r+f < r+l+1.$$

Let us show that

$$\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_{r+f};n}}(t)\, dt < \infty, \tag{56}$$

where $(k_{\alpha_{r+f};1}, \ldots, k_{\alpha_{r+f};n}) = \alpha_{r+f}$. By (48), taking into account (44), we obtain

$$\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t) \cdot \ldots \cdot x_n^{k_{\alpha_{r+f};n}}(t)\, dt = \int_0^1 t^{-\lambda S_{\alpha_{r+f}} + \sum_{i=1}^{n} \delta_i k_{\alpha_{r+f};i}}\, dt = \int_0^1 t^{-\lambda S_{\alpha_{r+1}} + \sum_{i=1}^{n} \delta_i k_{\alpha_{r+f};i}}\, dt.$$

So, it is enough to prove that

$$-\lambda S_{\alpha_{r+1}} + \sum_{i=1}^{n} \delta_i k_{\alpha_{r+f};i} > -1.$$

By Condition 1) of (53), the latter inequality is equivalent to

$$\sum_{i=1}^{n} \delta_i k_{\alpha_{r+1};i} < \sum_{i=1}^{n} \delta_i k_{\alpha_{r+f};i}. \tag{57}$$

Let us prove (57). First let us prove some auxiliary result.

Let us show that there exists $j \in \{1, \ldots, n-1\}$ such that

$$k_{\alpha_{r+1};j} < k_{\alpha_{r+f};j} \tag{58}$$

and

$$k_{\alpha_{r+1};j_1} = k_{\alpha_{r+f};j_1} \tag{59}$$

for every $j_1 \in \mathbb{N}$ such that $j_1 < j$. By (39), we have $\alpha_{r+1} > \alpha_{r+f}$ since $r+1 < r+f$. By (44), we get $S_{\alpha_{r+1}} = S_{\alpha_{r+f}}$ since $r+f < r+l+1$.

Therefore, by Condition 2) of (53), there exists $j \in \{1, \ldots, n\}$ such that (58) and (59) hold for every $j_1 \in \mathbb{N}$ such that $j_1 < j$. Let us show that $j < n$. Suppose $j = n$. Then $k_{\alpha_{r+1};i} = k_{\alpha_{r+f};i}$ for every $i \in \{1, \ldots, n-1\}$ and $k_{\alpha_{r+1};n} \neq k_{\alpha_{r+f};n}$. Therefore, by (6), $S_{\alpha_{r+1}} \neq S_{\alpha_{r+f}}$, which contradicts the fact that $S_{\alpha_{r+1}} = S_{\alpha_{r+f}}$. Consequently, $j \in \{1, \ldots, n-1\}$.

By (59), the inequality (57) is equivalent to

$$\sum_{i=j}^{n} \delta_i k_{\alpha_{r+1};i} < \sum_{i=j}^{n} \delta_i k_{\alpha_{r+f};i},$$

which is equivalent to the inequality

$$\sum_{i=j+1}^{n} \delta_i k_{\alpha_{r+1};i} < \sum_{i=j}^{n} \delta_i k_{\alpha_{r+f};i} - \delta_j k_{\alpha_{r+1};j}. \tag{60}$$

Since, in particular, $\delta_i > 0$ and $k_{\alpha_{r+f;i}} \geq 0$ for $i \in \{j+1, \ldots, n\}$, it follows that $\sum_{i=j+1}^{n} \delta_i k_{\alpha_{r+f};i} \geq 0$. Consequently,

$$\sum_{i=j}^{n} \delta_i k_{\alpha_{r+f};i} = \delta_j k_{\alpha_{r+f};j} + \sum_{i=j+1}^{n} \delta_i k_{\alpha_{r+f};i} \geq \delta_j k_{\alpha_{r+f};j}.$$

Therefore

$$\sum_{i=j}^{n} \delta_i k_{\alpha_{r+f};i} - \delta_j k_{\alpha_{r+1};j} \geq \delta_j k_{\alpha_{r+f};j} - \delta_j k_{\alpha_{r+1};j} = (k_{\alpha_{r+f};j} - k_{\alpha_{r+1};j})\delta_j. \tag{61}$$

By (58), $k_{\alpha_{r+f};j} - k_{\alpha_{r+1};j} > 0$. Therefore, since $k_{\alpha_{r+f};j} - k_{\alpha_{r+1};j}$ is integer, it follows that $k_{\alpha_{r+f};j} - k_{\alpha_{r+1};j} \geq 1$. Consequently,

$$(k_{\alpha_{r+f};j} - k_{\alpha_{r+1};j})\delta_j \geq \delta_j. \tag{62}$$

By Condition $j+1$) of (53), taking into account (36), for every $i \in \{1, \ldots, n\}$ we have

$$\delta_j > m \sum_{i=j+1}^{n} \delta_i = \sum_{i=j+1}^{n} \delta_i m \geq \sum_{i=j+1}^{n} \delta_i k_{\alpha_{r+1};i}. \tag{63}$$

By (61), (62) and (63), the inequality (60) is proven and, since it is equivalent to (57), it follows that (57) is proven. Consequently, the integral (56) is finite.

Combining (49) and (56) we can conclude that

$$\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t)\cdot\ldots\cdot x_n^{k_{\alpha_{r+f};n}}(t)\,dt<\infty \tag{64}$$

for every $f\in\mathbb{N}$ such that $r+1<r+f\le q$.

Let

$$C_1=1+\max\left(\max_{j\in\{1,\ldots,n\}}\|x_j\|_{L_{p_j}},\max_{f\in\{2,\ldots,q-r\}}\left|\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t)\cdot\ldots\cdot x_n^{k_{\alpha_{r+f};n}}(t)\,dt\right|\right). \tag{65}$$

By (47) and (64), such C_1 exists.

Let

$$u_i(t)=\begin{cases} x_i(t), & \text{if } t\in[\xi;\frac{1}{2}),\\ 0, & \text{if } t\in[0;\xi)\cup[\frac{1}{2};1],\end{cases} \tag{66}$$

where $i\in\{1,\ldots,n\}$, $0<\xi<\frac{1}{2}$ and $t\in[0;1]$. Note that

$$\|u_j\|_{L_j}\le\|x_j\|_{L_j} \tag{67}$$

for every $j\in\{1,\ldots,n\}$. Let us show that $u_j\in L_{p_1}$ for every $\xi\in(0;\frac{1}{2})$ and $j\in\{1,\ldots,n\}$. By (46) and (66), we get

$$\int_0^1|u_j(t)|^{p_1}\,dt=\int_\xi^{\frac{1}{2}}(x_j(t))^{p_1}\,dt=\int_\xi^{\frac{1}{2}}t^{-\frac{\lambda p_1}{p_j}+p_1\delta_j}\,dt<\infty$$

for every $\xi\in(0;\frac{1}{2})$ and $j\in\{1,\ldots,n\}$. So, $u_j\in L_{p_1}$.

Elements $u_1,\ldots,u_n$ satisfy Condition *a)* of the Lemma for every $\xi\in(0;\frac{1}{2})$.

Let us check Condition *b)*. By (65) and (67), we have

$$C_1>\|x_j\|_{L_{p_j}}\ge\|u_j\|_{L_{p_j}}.$$

So, C_1 and $u_1,\ldots,u_n$ satisfy Condition *b)* of the Lemma for every $\xi\in(0;\frac{1}{2})$.

Let us check Condition *c)*. By (8) and (66), we obtain

$$R_{\alpha_j}^{(L_{p_1}^n)}((u_1,\ldots,u_n))=\int_0^1 u_1^{k_{\alpha_j;1}}(t)\cdot\ldots\cdot u_n^{k_{\alpha_j;n}}(t)\,dt=\int_\xi^{\frac{1}{2}}x_1^{k_{\alpha_j;1}}(t)\cdot\ldots\cdot x_n^{k_{\alpha_j;n}}(t)\,dt \tag{68}$$

for every $j\in\{1,\ldots,q\}$. By (65) and (68), we get

$$\begin{aligned}C_1>\left|\int_0^1 x_1^{k_{\alpha_{r+f};1}}(t)\cdot\ldots\cdot x_n^{k_{\alpha_{r+f};n}}(t)\,dt\right|&\ge\left|\int_\xi^{\frac{1}{2}}x_1^{k_{\alpha_{r+f};1}}(t)\cdot\ldots\cdot x_n^{k_{\alpha_{r+f};n}}(t)\,dt\right|\\&=\left|R_{\alpha_{r+f}}^{(L_{p_1}^n)}((u_1,\ldots,u_n))\right|\end{aligned}$$

for $f\in\mathbb{N}$ such that $r+1<r+f\le q$. So, C_1 and $u_1,\ldots,u_n$ satisfy Condition *c)* of the Lemma for every $\xi\in(0;\frac{1}{2})$.

Let us check Condition *d)* of the Lemma. Let $C_2>0$. By (68) and (55), we have

$$R_{\alpha_{r+1}}^{(L_{p_1}^n)}((u_1,\ldots,u_n))=\int_\xi^{\frac{1}{2}}t^{-1}\,dt. \tag{69}$$

Now let us fix $\xi \in (0; \frac{1}{2})$ such that

$$\int_{\xi}^{\frac{1}{2}} t^{-1}\,dt > C_2. \tag{70}$$

Then, by (69) and (70), $u_1, \ldots, u_n$ satisfy Condition *d)* of the Lemma by guaranteeing $R_{\alpha_{r+1}}^{(L_{p_1}^n)}((u_1, \ldots, u_n)) > C_2$. Therefore C_1 and $u_1, \ldots, u_n$ satisfy all conditions of the Lemma. This completes the proof. □

Theorem 6. *Let L be defined by* (30)*. Let $P : L \to \mathbb{C}$ be a symmetric continuous m-homogeneous polynomial. Let P_1 be the restriction of P to the space $L_{p_1}^n$. Let $Q : \mathbb{C}^q \to \mathbb{C}$ be the polynomial given by Lemma 10, where q is defined by* (38)*. Suppose there exists $r \in \{0, \ldots, q-1\}$ such that following conditions are satisfied:*

1) there exists $h \in \{r+1, r+2, \ldots, q\}$ such that $S_{\alpha_h} > 1$, where α_h is defined by (39) *and S_{α_h} is defined by* (6)*;*

2) if $r > 0$, then Q does not depend on its first r variables.

Then Q does not depend on its $(r+1)$th variable.

Proof. Let N be the difference between the cardinality of $\aleph_L^{(m)}$ and r, i.e. $N = q - r$. Note that $S_{\alpha_q} = \frac{1}{p_1} < 1$ by Lemma 9 and (6) while $S_{\alpha_h} > 1$ by Condition *1)* of the Theorem. Thus, $h \neq q$. Therefore, since both h and q belong to the set $\{r+1, \ldots, r+N\}$ we can conclude that the cardinality of this set in not less than 2 and, consequently, $N \geq 2$.

Let $\tau : \mathbb{C}^{r+N} \to \mathbb{C}^N$ be the projection defined by

$$\tau((a_1, \ldots, a_{r+N})) = (a_{r+1}, \ldots, a_{r+N})$$

for $(a_1, \ldots, a_{r+N}) \in \mathbb{C}^{r+N}$. Let us define $Q_\tau : \mathbb{C}^N \to \mathbb{C}$ as

$$Q_\tau((a_{r+1}, \ldots, a_{r+N})) = Q((0, \ldots, 0, a_{r+1}, \ldots, a_{r+N})).$$

By Condition *2)*, Q does not depend on its first r variables, therefore

$$Q(a) = Q_\tau(\tau(a)) \tag{71}$$

for every $a \in \mathbb{C}^{r+N}$.

Let us prove that Q_τ does not depend on its first variable. We will use Lemma 1, where we set N instead of m. Let $\chi : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the projection defined by

$$\chi((a_1, a_2, \ldots, a_N)) = (a_2, \ldots, a_N).$$

Let $C > 1$ and

$$M = 2K_n C, \tag{72}$$

where K_n is given by Lemma 5. Define some auxiliary sets. Let

$$A_1 = \Big\{ u = (u_1, \ldots, u_n) \in L_{p_1}^n : \ \|u_i\|_{L_{p_i}} < M \text{ for every } i \in \{1, \ldots, n\} \Big\} \tag{73}$$

and

$$A_2 = \Big\{ u \in L_{p_1}^n : \ \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(u)\big| < C \text{ for every } j \in \{2, \ldots, N\} \Big\}. \tag{74}$$

Now define the set

$$G = \Big\{ \big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}(u), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(u) \big) : u \in A_1 \cap A_2 \Big\}. \tag{75}$$

Note that $G \subset \mathbb{C}^N$. Let us check conditions of Lemma 1 for the set G, the polynomial Q_τ and the projection χ.

Firstly, $\chi(G)$ is not an empty set since G is not an empty set (for example, $(0, \ldots, 0) \in G$).

Let us show that $Q_\tau(G)$ is bounded. By (73), the set A_1 is bounded in L. Thus, $A_1 \cap A_2$ is bounded in L. Therefore P is bounded on $A_1 \cap A_2$ since P is continuous on L, i.e.

$$\sup_{u \in A_1 \cap A_2} |P(u)| < \infty.$$

Consequently, taking into account that $A_1, A_2 \subset L_{p_1}^n$, we obtain

$$\sup_{u \in A_1 \cap A_2} |P_1(u)| = \sup_{u \in A_1 \cap A_2} |P(u)| < \infty, \tag{76}$$

where P_1 is the restriction of P to the space $L_{p_1}^n$. By Condition *2)* of the Theorem, Q does not depend on its first r variables. Thus, taking into account (71), we get

$$\begin{aligned} Q\Big(\Big(R_{\alpha_1}^{(L_{p_1}^n)}(u), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(u)\Big)\Big) &= Q\Big(\Big(0, \ldots, 0, R_{\alpha_{r+1}}^{(L_{p_1}^n)}(u), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(u)\Big)\Big) \\ &= Q_\tau\Big(\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}(u), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(u)\Big)\Big) \end{aligned} \tag{77}$$

for every $u \in L_{p_1}^n$. By the definition of Q from Lemma 10 and by (77), we obtain

$$P_1(u) = Q_\tau\Big(\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}(u), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(u)\Big)\Big)$$

for every $u \in L_{p_1}^n$. Therefore, taking into account (75), we get

$$\sup_{u \in A_1 \cap A_2} |P_1(u)| = \sup_{g \in G} |Q_\tau(g)|. \tag{78}$$

By (78) and (76), we can conclude that Q_τ is bounded on G.

Let us prove that $\chi^{-1}(U)$ is unbounded for any open set $U \subset \chi(G)$. It is enough to prove this fact for every open ball U with respect to the metric

$$\rho(x, y) = \|x - y\|_\infty \tag{79}$$

on $\mathbb{C}^{N-1}$, defined by the norm $\|(a_2, \ldots, a_N)\|_\infty = \max_{j \in \{2, \ldots, N\}} |a_j|$. Let $U \subset \chi(G)$ be the open ball with respect to (79). Let $\varepsilon > 0$ be the radius of U. Since $U \subset \chi(G)$, taking into account (75), there exists

$$v \in A_1 \cap A_2 \tag{80}$$

such that $\Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}(v), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(v)\Big)$ is the centre of the ball U. Since $v \in A_1 \cap A_2$, by (74), we have

$$\Big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\Big| < C \tag{81}$$

for every $j \in \{2, \dots, N\}$. Let $c_U : \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\} \to \mathbb{C}$ be defined by

$$c_U(k) = \begin{cases} R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v), & \text{if } k = \alpha_{r+j} \text{ for some } j \in \{2, \dots, N\}, \\ 0, & \text{otherwise} \end{cases} \tag{82}$$

for every $k \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$. By (81) and (82), we have

$$|c_U(k)| < C$$

for every $k \in \{\alpha_{r+2}, \dots, \alpha_{r+N}\}$. Therefore, by (82), taking into account that $C > 1$, we obtain

$$\sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\dots,0)\}} |c_U(k)|^{\frac{1}{|k|}} = \max_{k \in \{\alpha_{r+2},\dots,\alpha_{r+N}\}} |c_U(k)|^{\frac{1}{|k|}} < C. \tag{83}$$

Consequently, by Lemma 5, there exists

$$y = (y_1, \dots, y_n) \in L_\infty^n \tag{84}$$

such that

$$\int_0^1 y_1^{k_1}(t) y_2^{k_2}(t) \cdot \ldots \cdot y_n^{k_n}(t)\, dt = c_U(k) \tag{85}$$

for every $k \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$ and

$$\|y\|_{L_\infty^n} \le 2K_n \sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\dots,0)\}} |c_U(k)|^{\frac{1}{|k|}} \tag{86}$$

while

$$y_i(t) = 0 \quad \text{for every} \quad i \in \{1, \dots, n\} \quad \text{and} \quad t \in \left[0; \tfrac{1}{2}\right). \tag{87}$$

By (86), (83) and (72), we obtain

$$\|y\|_{L_\infty^n} < M. \tag{88}$$

By Condition *1)* of the Theorem, $S_{\alpha_{r+j}} > 1$ for some $j \in \{1, \dots, N\}$. Consequently, we can conclude that $S_{\alpha_{r+1}} > 1$ since, by (40), $S_{\alpha_{r+1}} \ge S_{\alpha_{r+j}}$. Therefore, by Lemma 11, there exists $C_1 > 0$ such that for every $C_2 > 0$ there exists $x^{(C_2)} = \left(x_1^{(C_2)}, \dots, x_n^{(C_2)}\right) \in L_{p_1}^n$ such that the following conditions are satisfied:

(a) $x_i^{(C_2)}(t) = 0$ for every $t \in [\frac{1}{2}; 1]$ and $i \in \{1, \dots, n\}$;

(b) $\|x_i^{(C_2)}\|_{L_{p_i}} < C_1$ for every $i \in \{1, \dots, n\}$;

(c) $\left| R_{\alpha_j}^{(L_{p_1}^n)}\left(x^{(C_2)}\right) \right| < C_1$ for every $j \in \{r+2, \dots, q\}$;

(d) $\left| R_{\alpha_{r+1}}^{(L_{p_1}^n)}\left(x^{(C_2)}\right) \right| > C_2$.

For arbitrary $C_2 > 0$, let

$$z^{(C_2;\beta)} = y + \beta x^{(C_2)}, \tag{89}$$

where y is defined by (84) and $\beta > 0$ is arbitrary. We will specify parameters C_2 and β later. Since $x^{(C_2)} \in L_{p_1}^n$ and $y \in L_\infty^n$, we can conclude that $z^{(C_2;\beta)} \in L_{p_1}^n$. By Property (a) of $x^{(C_2)}$ and by (87), we get

$$z_i^{(C_2;\beta)}(t) = \begin{cases} \beta x_i^{(C_2)}(t), & \text{if } t \in [0; \frac{1}{2}), \\ y_i(t), & \text{if } t \in [\frac{1}{2}; 1] \end{cases}$$

for every $i \in \{1, \ldots, n\}$. Therefore

$$R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big) = R_{\alpha_{r+j}}^{(L_{p_1}^n)}(y) + R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(\beta x^{(C_2)}\big) \tag{90}$$

for every $j \in \{1, \ldots, N\}$. Note that

$$R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(\beta x^{(C_2)}\big) = \beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big) \tag{91}$$

for every $j \in \{1, \ldots, N\}$ since $R_{\alpha_{r+j}}^{(L_{p_1}^n)}$ is a $|\alpha_{r+j}|$-homogeneous polynomial. By (90) and (91), we obtain

$$R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big) = R_{\alpha_{r+j}}^{(L_{p_1}^n)}(y) + \beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big) \tag{92}$$

for every $j \in \{1, \ldots, N\}$.

Let us show that $z^{(C_2;\beta)} \in A_1$ for every $\beta > 0$ such that

$$\beta < \min_{i \in \{1, \ldots, n\}} \frac{M - \|y_i\|_{L_{p_i}}}{C_1} \tag{93}$$

regardless of C_2. Note that this minimum is positive by (88) and (31). By (89) and Property (b) of $x^{(C_2)}$, we obtain

$$\|z_i^{(C_2;\beta)}\|_{L_{p_i}} = \|\beta x_i^{(C_2)} + y_i\|_{L_{p_i}} \le \beta \|x_i^{(C_2)}\|_{Lp_i} + \|y_i\|_{L_{p_i}} < \beta C_1 + \|y_i\|_{L_{p_i}} \tag{94}$$

for every $i \in \{1, \ldots, n\}$. Note that for every $\beta > 0$ that satisfies (93) and for every $i \in \{1, \ldots, n\}$, we have

$$\beta C_1 < M - \|y_i\|_{L_{p_i}}$$

and, consequently, taking into account (94), we get

$$\|z_i^{(C_2;\beta)}\|_{Lp_i} < M$$

for every $i \in \{1, \ldots, n\}$, i.e. by (73), $z^{(C_2;\beta)} \in A_1$. So, $z^{(C_2;\beta)} \in A_1$ for every $\beta > 0$ that satisfies (93).

Let us show that $z^{(C_2;\beta)} \in A_2$ for every $\beta > 0$ such that

$$\beta < \min_{j \in \{2, \ldots, N\}} \left(\frac{C - \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big|}{C_1} \right)^{\frac{1}{|\alpha_{r+j}|}} \tag{95}$$

regardless of C_2, where v is defined by (80). Note that this minimum is positive by (74) since $v \in A_1 \cap A_2$. By (92), (85), (82) and Property (c) of $x^{(C_2)}$, we have

$$\begin{aligned} \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\big| &= \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(y) + \beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| \\ &= \big|c_U(\alpha_{r+j}) + \beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| \\ &= \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v) + \beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| \\ &\le \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big| + \beta^{|\alpha_{r+j}|} \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| \\ &\le \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big| + \beta^{|\alpha_{r+j}|} C_1 \end{aligned} \tag{96}$$

for every $j \in \{2, \ldots, N\}$. Note that for every $\beta > 0$ that satisfies (95) and for every $j \in \{2, \ldots, N\}$ we have

$$\beta^{|\alpha_{r+j}|} C_1 < C - \big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big|,$$

and, consequently, taking into account (96), we get

$$\big|R_{\alpha_{r+j}}^{(L_{p_1}^n)}(z^{(C_2;\beta)})\big| < C$$

for every $j \in \{2, \ldots, N\}$, i.e. by (74), $z^{(C_2;\beta)} \in A_2$. So, $z^{(C_2;\beta)} \in A_2$ for every $\beta > 0$ that satisfies (95).

So, it is enough to choose $\beta > 0$ that satisfies conditions (93) and (95) for $z^{(C_2;\beta)}$ to be an element of $A_1 \cap A_2$ regardless of C_2.

Let us fix $\beta > 0$ such that

$$\beta < \min_{j \in \{2,\ldots,N\}} \left(\frac{\varepsilon}{C_1}\right)^{\frac{1}{|\alpha_{r+j}|}} \tag{97}$$

that also satisfies (93) and (95). Note that none of these conditions depend on C_2. Let us show that

$$\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big) \in \chi^{-1}(U), \tag{98}$$

i.e. the projection of $\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big)$ belongs to the ball U. Remind that $\Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}(v), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(v)\Big)$ is the centre of U and ε is its radius. Since β satisfies (93) and (95), $z^{(C_2;\beta)} \in A_1 \cap A_2$. Therefore, by (75), we obtain

$$\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big) \in G.$$

Consequently, we conclude

$$\Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big) \in \chi(G).$$

By (79), (92), (85), (82), Property (c) of $x^{(C_2)}$ and (97), we obtain

$$\begin{aligned}
\rho\Big(\Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big), \Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}(v), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}(v)\Big)\Big) \\
= \max_{j \in \{2,\ldots,N\}} \big|\beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big) + R_{\alpha_{r+j}}^{(L_{p_1}^n)}(y) - R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big| \\
= \max_{j \in \{2,\ldots,N\}} \big|\beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big) + c_U(\alpha_{r+j}) - R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big| \\
= \max_{j \in \{2,\ldots,N\}} \big|\beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big) + R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v) - R_{\alpha_{r+j}}^{(L_{p_1}^n)}(v)\big| \\
= \max_{j \in \{2,\ldots,N\}} \big|\beta^{|\alpha_{r+j}|} R_{\alpha_{r+j}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| < \big|\beta^{|\alpha_{r+j}|} C_1\big| < \varepsilon.
\end{aligned}$$

Thus, distance between the point $\Big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big)$ and the centre of the ball U is lesser than ε. So,

$$\big(R_{\alpha_{r+2}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big), \ldots, R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\big) \in U.$$

Consequently, (98) holds.

Let us show that the set $\chi^{-1}(U)$ is unbounded. It is enough to show that the set

$$\Big\{\Big(R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big),\ldots,R_{\alpha_{r+N}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\Big) : C_2\in(0;+\infty)\Big\} \tag{99}$$

is unbounded since, by (98), this set is a subset of $\chi^{-1}(U)$. By (92), (85), (82) and Property (d) of $x^{(C_2)}$, we have

$$\begin{aligned}\big|R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\big| &= \big|R_{\alpha_{r+1}}^{(L_{p_1}^n)}(y)+\beta^{|\alpha_{r+1}|}R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big|\\ &= \big|c_U(\alpha_{r+1})+\beta^{|\alpha_{r+1}|}R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| = \big|\beta^{|\alpha_{r+1}|}R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(x^{(C_2)}\big)\big| > \beta^{|\alpha_{r+1}|}C_2\end{aligned}$$

for every $C_2\in(0;+\infty)$. Therefore $\big|R_{\alpha_{r+1}}^{(L_{p_1}^n)}\big(z^{(C_2;\beta)}\big)\big|$ can be made greater than any given positive number by choosing C_2 sufficiently large, which means that the set (99) is unbounded. Thus, $\chi^{-1}(U)$ is unbounded.

So, we have checked all the conditions of Lemma 1 for the set G, the polynomial Q_τ and the projection χ. Consequently, according to Lemma 1, the polynomial Q_τ does not depend on its first variable. Therefore, by (71), the polynomial Q does not depend on its $(r+1)$th variable. This completes the proof. □

Theorem 7. *Let L be defined by* (30)*. Let $P : L\to\mathbb{C}$ be a symmetric continuous m-homogeneous polynomial. Let P_1 be the restriction of P to the space $L_{p_1}^n$. Let*

$$J=\{j:\ j\in\{1,\ldots,q\}\ \textit{such that}\ S_{\alpha_j}\le 1\}, \tag{100}$$

where α_j is defined by (39) *and S_{α_j} is defined by* (6)*. Then P_1 is an algebraic combination of elements of the set*

$$\Big\{R_{\alpha_j}^{(L_{p_1}^n)} :\ j\in J\Big\}. \tag{101}$$

Furthermore, the set J is equal to the set

$$\{j^*,\ldots,q\}, \tag{102}$$

where

$$j^*=\min\{j\in\{1,\ldots,q\}:\ S_{\alpha_j}\le 1\}. \tag{103}$$

Proof. Let $Q:\mathbb{C}^q\to\mathbb{C}$ be the polynomial given by Lemma 10. So,

$$P_1(x)=Q(\eta(x)), \tag{104}$$

where η is defined by (42) and $x\in L_{p_1}^n$. Thus, P_1 is an algebraic combination of elements of the set

$$\Big\{R_{\alpha_j}^{(L_{p_1}^n)} :\ j\in\{1,\ldots,q\}\Big\}. \tag{105}$$

Let us consider two cases.

Case 1: there is no $j\in\{1,\ldots,q\}$ such that $S_{\alpha_j}>1$. In this case, the set (101) is equal to the set (105). So, P_1 is an algebraic combination of elements of the set (101). Note that in this case $j^*=1$. Therefore the set (100) is equal to the set (102).

Case 2: there is some $j \in \{1, \dots, q\}$ such that $S_{\alpha_j} > 1$. Let

$$j_* = \max\{j \in \{1, \dots, q\} : S_{\alpha_j} > 1\}. \tag{106}$$

Let us show that Q does not depend on its mth variable for every $m \in \{1, \dots, j_*\}$. Suppose this is not the case. Let $m_1 \in \{1, \dots, j_*\}$ be the minimal index such that Q depends on its m_1th variable. Then Q does not depend on its first $m_1 - 1$ variables. This provides Condition *2)* of Theorem 6 for $r = m_1 - 1$ and $h = j_*$. Note that (106) provides Condition *1)* of Theorem 6 for $r = m_1 - 1$ and $h = j_*$. Thus, by Theorem 6, Q does not depend on its m_1th variable. This contradicts our assumption.

By (104), P_1 is an algebraic combination of elements of the set (105). And, as proven above, Q does not depend on its first j_* variables. Thus, P_1 is an algebraic combination of elements of the set

$$\left\{R_{\alpha_j}^{(L_{p_1}^n)} : j \in \{j_* + 1, \dots, q\}\right\}. \tag{107}$$

By (106), $S_{\alpha_j} \leq 1$ for every $j \in \{j_* + 1, \dots, q\}$. Thus, (107) is a subset of (101). By (106), we have $S_{\alpha_{j_*}} > 1$. Therefore, by (40), for every $j_0 \in \{1, \dots, q\}$ such that $S_{\alpha_{j_0}} \leq 1$ we can conclude $j_0 > j_*$. Consequently, (101) is a subset of (107). Therefore the set (101) is equal to the set (107). So, P_1 is an algebraic combination of elements of the set (101). By (106), (103) and (40), we get $j_* + 1 = j^*$. Consequently, the set (100) is equal to the set (102). This completes the proof. □

Thus, we may now present a series of final results.

Theorem 8. *Let L be defined by* (30)*. Every symmetric continuous m-homogeneous polynomial $P : L \to \mathbb{C}$ is an algebraic combination of elements of the set*

$$\left\{R_{\alpha_j}^{(L)} : \alpha_j \in \aleph_L^{(m)} \text{ such that } S_{\alpha_j} \leq 1\right\}, \tag{108}$$

where S_α is defined by (6)*, $\aleph_L^{(m)}$ is defined by* (34) *and the order on $\aleph_L^{(m)}$ is defined by* (39)*.*

Proof. Let P_1 be the restriction of the polynomial P to the space $L_{p_1}^n$. By Theorem 7, P_1 is an algebraic combination of elements of the set (101). By Theorem 7, the cardinality of the set (101) is equal to the cardinality of the set (102) which is equal to $q - j^* + 1$. Thus, there exists the polynomial $H : \mathbb{C}^{q-j^*+1} \to \mathbb{C}$ such that

$$P_1(x) = H\Big(\Big(R_{\alpha_{j^*}}^{(L_{p_1}^n)}(x), \dots, R_{\alpha_q}^{(L_{p_1}^n)}(x)\Big)\Big) \tag{109}$$

for every $x \in L_{p_1}^n$. Let $G : L \to \mathbb{C}$ be defined as

$$G = P - H_{\mathcal{P}_{G_{p_1,\dots,p_n}}(L)}\Big(R_{\alpha_{j^*}}^{(L)}, \dots, R_{\alpha_q}^{(L)}\Big), \tag{110}$$

where for a polynomial H the mapping $H_{\mathcal{P}_{G_{p_1,\dots,p_n}}(L)}$ is defined by (2).

By Theorem 3, all elements of (108) are continuous on L. Therefore, since H is a polynomial, $H_{\mathcal{P}_{G_{p_1,\dots,p_n}}(L)}\Big(R_{\alpha_{j^*}}^{(L)}, \dots, R_{\alpha_q}^{(L)}\Big)$ is continuous on L. Consequently, by (110), taking into account that P is continuous on L by assumption, G is continuous on L. By (110) and (109), for every $x \in L_{p_1}^n$, we have

$$G(x) = P(x) - H\Big(R_{\alpha_{j^*}}^{(L)}(x), \dots, R_{\alpha_q}^{(L)}(x)\Big) = P_1(x) - H\Big(R_{\alpha_{j^*}}^{(L_{p_1}^n)}(x), \dots, R_{\alpha_q}^{(L_{p_1}^n)}(x)\Big) = 0. \tag{111}$$

By (31), $L_\infty \subset L_{p_1}$ and $L_{p_1} \subset L_{p_i}$ for every $i \in \{1, \ldots, n\}$. Therefore $L_\infty^n \subset L_{p_1}^n \subset L$. Consequently, taking into account Lemma 4, $L_{p_1}^n$ is dense in L. Thus, taking into account (111), the continuous function G equals 0 on the dense subset $L_{p_1}^n$ of L. Consequently, G equals 0 everywhere on L. Therefore, by (110), we obtain

$$P(x) = H\Big(R^{(L)}_{\alpha_{j^*}}(x), \ldots, R^{(L)}_{\alpha_q}(x)\Big)$$

for every $x \in L$. So, taking into account equality of sets (102) and (100) granted by Theorem 7, P is an algebraic combination of elements of the set (108). This completes the proof. ☐

Lemma 12. *Let $P = P_0 + P_1 + \ldots + P_N$ be a symmetric continuous complex-valued polynomial on L, where $P_0 \in \mathbb{C}$, P_j is a j-homogeneous polynomial for $j \in \{1, \ldots, N\}$ and L is defined by* (30). *Then every P_j is symmetric and continuous, where $j \in \{0, \ldots, N\}$.*

Proof. This is immediate from Cauchy Integral Formula (see [16, Corollary 7.3, p. 47]) since $P_0 + \ldots + P_N$ is the Taylor series expansion of P at 0. ☐

Now, let us prove the final result of this section.

Theorem 9. *Let L be defined by* (30)*, i.e. $L = L_{p_1} \times \cdots \times L_{p_n}$, $p_i \in [1; \infty)$, $p_1 \geq \cdots \geq p_n$. The set of polynomials*

$$\{R^{(L)}_\alpha : \alpha \in \aleph_L\}, \tag{112}$$

where $R^{(L)}_\alpha$ are defined by (8) *and $\aleph_L$ is defined by* (7)*, is an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on L.*

Proof. Firstly, let us show that the set (112) is a generating system. Let $P : L \to \mathbb{C}$ be an arbitrary symmetric continuous polynomial. By (1), we have $P = P_0 + P_1 + \cdots + P_N$, where $N \in \mathbb{N}$, $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for $j \in \{1, \ldots, N\}$. By Lemma 12, taking into account that P is symmetric and continuous, P_j is symmetric and continuous for every $j \in \{1, \ldots, N\}$. By Theorem 8, where we set $m = j$, the polynomial P_j is an algebraic combination of elements of the set (108) for every $j \in \{1, \ldots, N\}$. By (7) and (34), the set (108), for $m = j$, is a subset of (112). Therefore P_j is an algebraic combination of elements of the set (112) for every $j \in \{1, \ldots, N\}$. Consequently, P is an algebraic combination of elements of the set (112).

Secondly, let us show the algebraic independence of the set (112). Let us assume that (112) is algebraically dependent. Then there exists a subset $\{R^{(L)}_{\beta_1}, \ldots, R^{(L)}_{\beta_r}\}$ of the set (112) and a nontrivial polynomial $Q : \mathbb{C}^r \to \mathbb{C}$ such that

$$Q\Big(R^{(L)}_{\beta_1}(x), \ldots, R^{(L)}_{\beta_r}(x)\Big) = 0 \tag{113}$$

for every $x \in L$. Let us show that Q cannot be nontrivial. Let $(z_1, \ldots, z_r)$ be an arbitrary element of $\mathbb{C}^r$. Let the mapping $c : \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\} \to \mathbb{C}$ be defined by

$$c(k) = \begin{cases} z_i, & \text{for } k = \beta_i \text{ such that } i \in \{1, \ldots, r\}, \\ 0, & \text{otherwise} \end{cases} \tag{114}$$

for every $k \in \mathbb{Z}_+^n \setminus \{(0, \ldots, 0)\}$. Note that

$$\sup_{k \in \mathbb{Z}_+^n \setminus \{(0,\ldots,0)\}} |c(k)|^{\frac{1}{|k|}} < \infty.$$

Consequently, by Lemma 2, taking into account (32), there exists $x_c = (x_1, \dots, x_n) \in L_\infty^n$ such that $R_k^{(L)}(x_c) = c(k)$ for every $k \in \mathbb{Z}_+^n \setminus \{(0, \dots, 0)\}$. Therefore, taking into account (114), we get

$$R_{\beta_i}^{(L)}(x_c) = z_i \tag{115}$$

for every $i \in \{1, \dots, r\}$. By substituting x_c into (113) and taking into account (115), we obtain

$$Q(z_1, \dots, z_r) = 0.$$

So, $Q(z_1, \dots, z_r) = 0$ for an arbitrary $(z_1, \dots, z_r) \in \mathbb{C}^r$. Thus, Q is trivial. This contradicts our assumption.

This completes the proof. □

4 Algebraic basis of the algebra of all continuous symmetric complex-valued polynomials on $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_n}$

Let $p_1, p_2, \dots, p_n \in [1; +\infty)$. Our goal is to prove the similar result to Theorem 9 without ordering p_i, i.e. without condition (29).

Let

$$L' = L_{p_1} \times L_{p_2} \times \cdots \times L_{p_n} \quad \text{and} \quad L^{(\tau)} = L_{p_{\tau(1)}} \times L_{p_{\tau(2)}} \times \cdots \times L_{p_{\tau(n)}}, \tag{116}$$

where $\tau : \{1, \dots, n\} \to \{1, \dots, n\}$ is some fixed permutation.

Let $\iota : L' \to L^{(\tau)}$ be defined by

$$\iota((x_1, \dots, x_n)) = (x_{\tau(1)}, \dots, x_{\tau(n)}), \tag{117}$$

where $(x_1, \dots, x_n) \in L'$.

Lemma 13. *The mapping ι, defined by* (117)*, is a well-defined isometric isomorphism.*

Proof. Let us show that ι is well-defined, i.e. $\iota(x) \in L^{(\tau)}$ for every $x = (x_1, x_2, \dots, x_n) \in L'$. By (117), $\iota(x) = (x_{\tau(1)}, \dots, x_{\tau(n)})$. For every $j \in \{1, \dots, n\}$, we have $x_{\tau(j)} \in L_{p_{\tau(j)}}$, since $x \in L'$. Therefore $\iota(x) \in L^{(\tau)}$.

Show that ι preserves norm. Let $x = (x_1, x_2, \dots, x_n) \in L'$. Note that

$$\|\iota(x)\|_{L^{(\tau)}} = \max_{j \in \{1,\dots,n\}} \|x_{\tau(j)}\|_{L_{p_{\tau(j)}}} = \max_{j \in \{1,\dots,n\}} \|x_j\|_{L_{p_j}} = \|x\|_{L'}.$$

It can be verified that ι is linear. Let us show that ι is a bijection. The mapping ι is linear and isometric, consequently, ι is an injection. Let $y = (y_1, y_2, \dots, y_n) \in L^{(\tau)}$. Let us construct $x \in L'$ such that $\iota(x) = y$. Set $x = (y_{\tau^{-1}(1)}, \dots, y_{\tau^{-1}(n)})$. Let us show that $x \in L'$, i.e. $y_{\tau^{-1}(i)} \in L_{p_i}$ for every $i \in \{1, \dots, n\}$. Note that, for every $i \in \{1, \dots, n\}$, we have $y_{\tau^{-1}(i)} \in L_{p_{\tau(\tau^{-1}(i))}} = L_{p_i}$ since $y \in L^{(\tau)}$. So, $x \in L'$. By (117), we obtain

$$\iota(x) = (y_{\tau(\tau^{-1}(1))}, \dots, y_{\tau(\tau^{-1}(n))}) = (y_1, \dots, y_n) = y.$$

Therefore ι is a surjection. This completes the proof. □

Let us define $\phi : \aleph_{L'} \to \aleph_{L^{(\tau)}}$ by

$$\phi((k_1, \dots, k_n)) = (k_{\tau(1)}, \dots, k_{\tau(n)}), \tag{118}$$

where $\aleph_{L'}$ and $\aleph_{L^{(\tau)}}$ are defined by (7).

Lemma 14. *The mapping ϕ, defined by* (118)*, is a well-defined bijection.*

Proof. Note that

$$S_{\phi(\alpha)}^{(L^{(\tau)})} = \sum_{i=1}^{n} \frac{k_{\tau(i)}}{p_{\tau(i)}} = \sum_{i=1}^{n} \frac{k_i}{p_i} = S_{\alpha}^{(L')} \tag{119}$$

for any $\alpha = (k_1, \ldots, k_n) \in \aleph_{L'}$, where $S_{\alpha}^{(L')}$ and $S_{\phi(\alpha)}^{(L^{(\tau)})}$ are defined by (6). Since $\alpha \in \aleph_{L'}$, according to the definition (7), we know that $0 < S_{\alpha}^{(L')} \le 1$. So, according to (119), we have $0 < S_{\phi(\alpha)}^{(L^{(\tau)})} \le 1$. Therefore, according to the definition (7), we get $\phi(\alpha) \in \aleph_{L^{(\tau)}}$.

Let us show that ϕ is a bijection. Let $\alpha = (k_1, \ldots, k_n), \beta = (q_1, \ldots, q_n) \in \aleph_{L'}$ be such that $\alpha \neq \beta$. Then there exists $j \in \{1, \ldots, n\}$ such that $k_j \neq q_j$. Therefore $k_{\tau(\tau^{-1}(j))} \neq q_{\tau(\tau^{-1}(j))}$ and, consequently, $\phi(\alpha)$ and $\phi(\beta)$ have different components with index $\tau^{-1}(j)$. So, $\phi(\alpha) \neq \phi(\beta)$. Thus, ϕ is an injection. Let $\beta = (q_1, \ldots, q_n) \in \aleph_{L^{(\tau)}}$. Let us construct $\alpha \in \aleph_{L'}$ such that $\phi(\alpha) = \beta$. Let $\alpha = (q_{\tau^{-1}(1)}, \ldots, q_{\tau^{-1}(n)})$. Let us show that $\alpha \in \aleph_{L'}$. Note that

$$S_{\alpha}^{(L')} = \sum_{i=1}^{n} \frac{q_{\tau^{-1}(i)}}{p_i} = \sum_{i=1}^{n} \frac{q_{\tau^{-1}(i)}}{p_{\tau(\tau^{-1}(i))}} = \sum_{i=1}^{n} \frac{q_i}{p_{\tau(i)}} = S_{\beta}^{(L^{(\tau)})}. \tag{120}$$

Since $\beta \in \aleph_{L^{(\tau)}}$, by the definition (7), we know that $0 < S_{\beta}^{(L^{(\tau)})} \le 1$. So, by (120), $0 < S_{\alpha}^{(L')} \le 1$. Therefore, by the definition (7), we get $\alpha \in \aleph_{L'}$. By (118), we obtain

$$\phi(\alpha) = (q_{\tau(\tau^{-1}(1))}, \ldots, q_{\tau(\tau^{-1}(n))}) = (q_1, \ldots, q_n) = \beta.$$

So, ϕ is a surjection. Thus, ϕ is a bijection. ☐

Theorem 10. *Let $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Let the spaces L' and $L^{(\tau)}$ be defined by* (116)*, ι be defined by* (117)*, $G_{p_1,\ldots,p_n}$ and $G_{p_{\tau(1)},\ldots,p_{\tau(n)}}$ be defined by* (5)*. Then*

a) the mapping $I : f \in H_{b,G_{p_{\tau(1)},\ldots,p_{\tau(n)}}}(L^{(\tau)}) \mapsto f \circ \iota \in H_{b,G_{p_1,\ldots,p_n}}(L')$ is an isomorphism, i.e. I is a continuous linear multiplicative bijection;

b) the restriction of I to the algebra $\mathcal{P}_{G_{p_{\tau(1)},\ldots,p_{\tau(n)}}}(L^{(\tau)})$ is an isomorphism between the algebras $\mathcal{P}_{G_{p_{\tau(1)},\ldots,p_{\tau(n)}}}(L^{(\tau)})$ and $\mathcal{P}_{G_{p_1,\ldots,p_n}}(L')$;

c) if $\mathcal{P}_{G_{p_{\tau(1)},\ldots,p_{\tau(n)}}}(L^{(\tau)})$ has some algebraic basis B, then $I(B)$ is an algebraic basis in $\mathcal{P}_{G_{p_1,\ldots,p_n}}(L')$.

Proof. Let us substitute into Theorem 1 $L', L^{(\tau)}, G_{p_1,\ldots,p_n}, G_{p_{\tau(1)},\ldots,p_{\tau(n)}}, \iota$ instead of $X, Y, G_1, G_2, \iota_{X,Y}$, respectively. Let us verify Condition *1)* of Theorem 1, which in our case will take the following form: for every $x \in L'$ and $g_1 \in G_{p_1,\ldots,p_n}$, there exists $g_2 \in G_{p_{\tau(1)},\ldots,p_{\tau(n)}}$ such that $\iota(g_1(x)) = g_2(\iota(x))$. Let $g_1 \in G_{p_1,\ldots,p_n}$. Let us define $g_2 : L^{(\tau)} \to L^{(\tau)}$ by

$$g_2(y) = \iota(g_1(\iota^{-1}(y))), \tag{121}$$

where $y \in L^{(\tau)}$. Then, for every $x \in L'$, we obtain

$$g_2(\iota(x)) = \iota(g_1(\iota^{-1}(\iota(x)))) = \iota(g_1(x)).$$

Let us show that $g_2 \in G_{p_{\tau(1)},\dots,p_{\tau(n)}}$. For every $y \in L^{(\tau)}$, by (117), we have $\iota^{-1}(y) \in L'$. Therefore, by the definition of g_1, we obtain $g_1(\iota^{-1}(y)) \in L'$. And therefore, for every $y \in L^{(\tau)}$, by (117), we get $\iota(g_1(\iota^{-1}(y))) \in L^{(\tau)}$, i.e. $g_2(y) \in L^{(\tau)}$. So, g_2 acts from $L^{(\tau)}$ to $L^{(\tau)}$. Since $g_1 \in G_{p_1,\dots,p_n}$, by (4) and (5), there exists a bijection $\sigma \in \Xi_{[0;1]}$ such that for every $x = (x_1, \dots, x_n) \in L'$ we have $g_1(x) = (x_1 \circ \sigma, \dots, x_n \circ \sigma)$. Then, for every $y = (y_1, \dots, y_n) \in L^{(\tau)}$, we obtain

$$g_1(\iota^{-1}(y)) = (y_{\tau^{-1}(1)} \circ \sigma, \dots, y_{\tau^{-1}(n)} \circ \sigma).$$

Therefore, by (121), we conclude

$$\begin{aligned} g_2(y) = \iota(g_1(\iota^{-1}(y))) &= \iota((y_{\tau^{-1}(1)} \circ \sigma, \dots, y_{\tau^{-1}(n)} \circ \sigma)) \\ &= (y_{\tau(\tau^{-1}(1))} \circ \sigma, \dots, y_{\tau(\tau^{-1}(n))} \circ \sigma) = (y_1 \circ \sigma, \dots, y_n \circ \sigma) \end{aligned}$$

for every $y \in L^{(\tau)}$. So, we represented g_2 in the form (4). Consequently, by (5), $g_2 \in G_{p_{\tau(1)},\dots,p_{\tau(n)}}$.

Let us verify Condition *2)* of Theorem 1, which in our case will take the following form: for every $y \in L^{(\tau)}$ and $g_2 \in G_{p_{\tau(1)}\dots;p_{\tau(n)}}$ there exists $g_1 \in G_{p_1,\dots,p_n}$ such that $\iota^{-1}(g_2(y)) = g_1(\iota^{-1}(y))$. Let $g_2 \in G_{p_{\tau(1)}\dots;p_{\tau(n)}}$. Let us define $g_1 : L' \to L'$ by

$$g_1(x) = \iota^{-1}(g_2(\iota(x))), \tag{122}$$

where $x \in L'$. Then for every $y \in L^{(\tau)}$ we obtain

$$g_1(\iota^{-1}(y)) = \iota^{-1}(g_2(\iota(\iota^{-1}(y)))) = \iota^{-1}(g_2(y)).$$

Let us show that $g_1 \in G_{p_1,\dots,p_n}$. For every $x \in L'$, by (117), $\iota(x) \in L^{(\tau)}$. Therefore, by the definition of g_2, we get $g_2(\iota(x)) \in L^{(\tau)}$. And therefore, for every $x \in L'$, by (117), we obtain $\iota^{-1}(g_2(\iota(x))) \in L'$, i.e. $g_1(x) \in L'$. So, g_1 acts from L' to L'. Since $g_2 \in G_{p_{\tau(1)},\dots,p_{\tau(n)}}$, by (4) and (5), there exists a bijection $\sigma \in \Xi_{[0;1]}$ such that $g_2(y) = (y_1 \circ \sigma, \dots, y_n \circ \sigma)$ for every $y = (y_1, \dots, y_n) \in L^{(\tau)}$. Then for every $x = (x_1, \dots, x_n) \in L'$ we obtain

$$g_2(\iota(x)) = (x_{\tau(1)} \circ \sigma, \dots, x_{\tau(n)} \circ \sigma).$$

Therefore, by (122), we conclude

$$\begin{aligned} g_1(x) = \iota^{-1}(g_2(\iota(x))) &= \iota^{-1}((x_{\tau(1)} \circ \sigma, \dots, x_{\tau(n)} \circ \sigma)) \\ &= (x_{\tau^{-1}(\tau(1))} \circ \sigma, \dots, x_{\tau^{-1}(\tau(n))} \circ \sigma) = (x_1 \circ \sigma, \dots, x_n \circ \sigma) \end{aligned}$$

for every $x \in L'$. So, we represented g_1 in the form (4). Consequently, by (5), $g_1 \in G_{p_1,\dots,p_n}$.

Thus, both conditions are satisfied. Then, by Theorem 1, Conditions a), b), and c) of the Theorem are satisfied. □

Lemma 15. *Let the spaces L' and $L^{(\tau)}$ be defined by* (116). *Let I be the mapping given by item a) of Theorem 10. Then, $I\left(R_\alpha^{(L^{(\tau)})}\right) = R_{\phi^{-1}(\alpha)}^{(L')}$ for every $\alpha \in \aleph_{L'}$, where $\aleph_{L'}$ is defined by* (7)*, ϕ is defined by* (118) *and $R_\alpha^{(L^{(\tau)})}, R_{\phi^{-1}(\alpha)}^{(L')}$ are defined by* (8).

Proof. Let $\alpha = (k_1, \dots, k_n)$. Note that

$$\begin{aligned} I(R_\alpha^{(L^{(\tau)})})(x) = R_\alpha^{(L^{(\tau)})}(\iota(x)) &= R_{(k_1,\dots,k_n)}^{(L^{(\tau)})}((x_{\tau(1)}, \dots, x_{\tau(n)})) \\ &= \int_0^1 x_{\tau(1)}^{k_1}(t) \cdot \ldots \cdot x_{\tau(n)}^{k_n}(t)\, dt = R_{\phi^{-1}(\alpha)}^{(L')}(x) \end{aligned}$$

for every $x \in L'$. This completes the proof. □

Aforementioned results lead us to the following conclusion.

Theorem 11. *Let* $p_1, \ldots, p_n \in [1; +\infty)$. *Let* $L' = L_{p_1} \times \cdots \times L_{p_n}$. *The set of polynomials* $\{R_\alpha^{(L')} : \alpha \in \aleph_{L'}\}$, *where* $R_\alpha^{(L')}$ *is defined by* (8) *and* $\aleph_{L'}$ *is defined by* (7), *is an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on* L'.

Proof. Let $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that $p_{\tau(1)} \geq \cdots \geq p_{\tau(n)}$. Set

$$L^{(\tau)} = L_{p_{\tau(1)}} \times L_{p_{\tau(2)}} \times \cdots \times L_{p_{\tau(n)}}.$$

Let us define ι and ϕ by (117) and (118), respectively, using this permutation τ.

By Theorem 9, $\mathcal{P}_{G_{p_{\tau(1)},\ldots,p_{\tau(n)}}}(L^{(\tau)})$ has the algebraic basis

$$B = \{R_\alpha^{(L^{(\tau)})} : \alpha \in \aleph_{L^{(\tau)}}\},$$

where $\aleph_{L^{(\tau)}}$ is defined by (7). Let us substitute $\tau, L', L^{(\tau)}, B$ into Theorem 10. Then, by item c) of Theorem 10, $I(B)$ is an algebraic basis in $\mathcal{P}_{G_{p_1,\ldots,p_n}}(L')$. For every $\alpha \in \aleph_{L^{(\tau)}}$ we have

$$I(R_\alpha^{(L^{(\tau)})}) = R_{\phi^{-1}(\alpha)}^{(L')}$$

by Lemma 15. So, $I(B) = \{R_{\phi^{-1}(\alpha)}^{(L')} : \alpha \in \aleph_{L^{(\tau)}}\}$. Consequently, this set is an algebraic basis in $\mathcal{P}_{G_{p_1,\ldots,p_n}}(L')$. By Lemma 14, we obtain

$$\{R_{\phi^{-1}(\alpha)}^{(L')} : \alpha \in \aleph_{L^{(\tau)}}\} = \{R_\alpha^{(L')} : \alpha \in \aleph_{L'}\}.$$

Consequently, $\{R_\alpha^{(L')} : \alpha \in \aleph_{L'}\}$ is an algebraic basis in $\mathcal{P}_{G_{p_1,\ldots,p_n}}(L')$. This completes the proof. □

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Дану роботу присвячено вивченню комплекснозначних неперервних симетричних поліномів на декартових добутках комплексних банахових просторів інтегровних за Лебегом функцій. Позначимо L_p, де $p \in [1;+\infty)$, комплексний банахів простір всіх комплекснозначних функцій на відрізку $[0;1]$, p-ті степені абсолютних значень яких є інтегровними за Лебегом. Позначимо $\Xi_{[0;1]}$ множину всіх бієкцій $\sigma : [0;1] \to [0;1]$ таких, що σ і σ^{-1} є вимірними і зберігають міру Лебега, тобто $\mu(\sigma(E)) = \mu(\sigma^{-1}(E)) = \mu(E)$ для кожної вимірної за Лебегом множини $E \subset [0;1]$, де μ — міра Лебега. Функцію f на декартовому добутку $L_{p_1} \times \cdots \times L_{p_n}$, де $p_1, \ldots, p_n \in [1;+\infty)$, називають симетричною, якщо $f((x_1 \circ \sigma, \ldots, x_n \circ \sigma)) = f((x_1, \ldots, x_n))$ для кожних $\sigma \in \Xi_{[0;1]}$ і $(x_1, \ldots, x_n) \in L_{p_1} \times \ldots \times L_{p_n}$. В роботі побудовано алгебраїчний базис алгебри всіх комплекснозначних неперервних симетричних поліномів на просторі $L_{p_1} \times \ldots \times L_{p_n}$. Також побудовано деякі ізоморфізми алгебр Фреше комплекснозначних цілих симетричних функцій обмеженого типу на просторі $L_{p_1} \times \ldots \times L_{p_n}$.

Ключові слова і фрази: поліном, ціла функція, симетрична функція, алгебраїчний базис, банахів простір інтегровних за Лебегом функцій.