ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2025, **17** (2), 447–460
doi:10.15330/cmp.17.2.447-460

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2025, Т.17, №2, С.447–460

Estimates of the characteristics of nonlinear approximation of classes of periodic functions of many variables

Pozharska K.V.[1,2,✉], Romanyuk A.S.[1], Yanchenko S.Ya.[1]

In the paper, we obtained exact order estimates of the best m-term trigonometric approximation and the best orthogonal trigonometric approximation of functions from the Nikol'skii-Besov $B^r_{p,\theta}(\mathbb{T}^d)$ and Sobolev $W^r_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ classes in the Lebesque subspaces $B_{q,1}(\mathbb{T}^d)$ for some relations between the parameters p and q. The received results yield that estimates of the considered approximation characteristics in the multivariate case, in contrast to the univariate, in the spaces $B_{q,1}(\mathbb{T}^d)$ and $L_q(\mathbb{T}^d)$ differ in order. Besides, for some parameter values the obtained exact order estimates of the best m-term trigonometric approximation and the best orthogonal trigonometric approximation in the spaces $B_{q,1}(\mathbb{T}^d)$ still remain unknown for the $L_q(\mathbb{T}^d)$-space.

Key words and phrases: Nikol'skii-Besov class, Sobolev class, best m-term trigonometric approximation, best orthogonal trigonometric approximation.

[1] Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine
[2] Chemnitz University of Technology, 39 Reichenhainer str., 09126, Chemnitz, Germany
✉ Corresponding author
E-mail: pozharska.k@gmail.com (Pozharska K.V.), romanyuk@imath.kiev.ua (Romanyuk A.S.), yan.sergiy@gmail.com (Yanchenko S.Ya.)

Introduction

In the paper, we investigate characteristics of nonlinear approximation of the Nikol'skii-Besov $B^r_{p,\theta}(\mathbb{T}^d)$ and Sobolev $W^r_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ classes in the space $B_{q,1}(\mathbb{T}^d)$, $1 < q < \infty$, for some relations between the parameters p and q. Note that the norm in the space $B_{q,1}(\mathbb{T}^d)$ is not "weaker" than the $L_q(\mathbb{T}^d)$-norm. A motivation for considering approximation characteristics in the spaces $B_{q,1}(\mathbb{T}^d)$, $q \in \{1,\infty\}$, in particular for the classes of multivariate functions, as indicated in the papers [6, 11, 13, 26–29, 32], was the fact that in certain important cases the questions on the exact orders of respective characteristics in the spaces $L_1(\mathbb{T}^d)$ and $L_\infty(\mathbb{T}^d)$ still remain open (see, e.g., [9]). In addition, some open questions remain also in the space $L_q(\mathbb{T}^d)$, $1 < q < \infty$. We will discuss that more detailed in comments to the obtained results.

The paper consists of three parts. In the first part we introduce necessary notation as well as definitions of the functional classes and spaces $B_{q,1}(\mathbb{T}^d)$. In the second part we define approximation characteristics under investigation and formulate auxiliary statements.

УДК 517.5
2020 *Mathematics Subject Classification:* 41A30, 41A46, 42A10.

This work was supported by a grant from the Simons Foundation (SFI-PD-Ukraine-00014586, A. Romanyuk, S. Yanchenko). K. Pozharska acknowledges support by the budget program of the NAS of Ukraine "Support for the development of priority areas of research" (KPKVK 6541230) and the German Research Foundation (DFG) with grant Ul403/4-1.

The third part of the paper in the main one. Here we obtain exact order estimates of the best m-term and the best orthogonal trigonometric approximations of the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ in the space $B_{q,1}(\mathbb{T}^d)$.

As a result of the research we show that in the multivariate case, in contrast to the univariate, estimates of the investigated approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ in the spaces $B_{q,1}(\mathbb{T}^d)$ and $L_q(\mathbb{T}^d)$ differ in order.

1 Definition of the functional classes and the spaces $B_{q,1}$

Let $\mathbb{R}^d$, $d \geq 1$, be an Euclidean space with elements $\boldsymbol{x} = (x_1, \ldots, x_d)$ and scalar product $(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + \ldots + x_dy_d$. By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d = \prod_{j=1}^{d}[0, 2\pi)$, $1 \leq p \leq \infty$, we denote the space of functions f, that are 2π-periodic in each variable and such that

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} = \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} = \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{T}^d} |f(\boldsymbol{x})| < \infty.$$

Remark 1. *If D is an arbitrary bounded region in $\mathbb{R}^d$, $d \in \mathbb{N}$, we similarly define the spaces $L_p(D)$, respectively, with the norm $\|f\|_{L_p(D)}$.*

In what follows, we consider the set of functions $L^0_p(\mathbb{T}^d)$ defined by

$$L^0_p(\mathbb{T}^d) := \left\{f\colon f \in L_p(\mathbb{T}^d),\ \int_0^{2\pi} f(\boldsymbol{x})dx_j = 0,\ j = \overline{1,d},\ \text{almost everywhere}\right\}.$$

Further we agree to use the simplified notation L_p and L^0_p in place of $L_p(\mathbb{T}^d)$ and $L^0_p(\mathbb{T}^d)$, respectively.

Let us define the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ and $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$, that are investigated in the paper. Here it will be convenient for us to use their definitions in terms of the so-called decomposition normalization (see, e.g., [2, 15]), in particular of the Vallée-Poussin-type for the parameter values $1 \leq p \leq \infty$ (see [15, Remark 2.1]).

Let $V_l(t)$, $t \in \mathbb{R}$, $l \in \mathbb{N}$, be the Vallée-Poussin kernel of the form

$$V_l(t) = 1 + 2\sum_{k=1}^{l} \cos kt + 2\sum_{k=l+1}^{2l-1} \left(1 - \frac{k-l}{l}\right) \cos kt,$$

where the third term is equal to zero for $l = 1$. We associate each vector $\boldsymbol{s} = (s_1, \ldots, s_d)$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$, with a polynomial $A_{\boldsymbol{s}}(\boldsymbol{x}) = \prod_{j=1}^{d}(V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j))$ and for $f \in L^0_p$, $1 \leq p \leq \infty$, we set $A_{\boldsymbol{s}}(f) := A_{\boldsymbol{s}}(f, \boldsymbol{x}) = (f * A_{\boldsymbol{s}})(\boldsymbol{x})$, where $*$ is the operation of convolution.

Then for $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $\boldsymbol{r} = (r_1, \ldots, r_d)$, $r_j > 0$, $j = \overline{1,d}$, the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ can be defines as follows

$$B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d) := \{f \in L^0_p\colon\ \|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \leq 1\},$$

where

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \asymp \left(\sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}(f)\|_p^\theta\right)^{1/\theta}$$

for $1 \le \theta < \infty$ and
$$\|f\|_{B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d)} \equiv \|f\|_{H^{\boldsymbol{r}}_p(\mathbb{T}^d)} \asymp \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})}\|A_{\boldsymbol{s}}(f)\|_p. \tag{1}$$

Here and in what follows, for positive quantities a and b we will use the notation $a \asymp b$ (the order equality), that means that there exist positive constants C_1 and C_2, which do not depend on one essential parameter in the quantities a and b, such that $C_1 a \le b$ (we write $a \ll b$, i.e. order inequality) and $C_2 a \ge b$ (we write $a \gg b$). The main results of the paper will be formulated in terms of the order relations. All of the constants C_i, $i = 1, 2, \ldots$, appearing in the paper, could depend only on the parameters from the definition of the class, metrics where we measure the approximation error, and from the dimension of the space $\mathbb{R}^d$. In some cases this dependence will be indicated in an explicit way, sometimes it will be clear from the context. For a finite set $\mathfrak{N}$ by $|\mathfrak{N}|$ we denote the number of its elements.

In the case $1 < p < \infty$, the norm definition of functions from the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ can be rewritten in other, equivalent form. Namely, in terms of dual "blocks" of the Fourier series of functions $f \in L^0_p$.

For the vectors $\boldsymbol{s} = (s_1, \ldots, s_d)$, $s_j \in \mathbb{N}$, $\boldsymbol{k} = (k_1, \ldots, k_d)$, $k_j \in \mathbb{Z}$, $j = \overline{1,d}$, we set
$$\rho(\boldsymbol{s}) := \left\{\boldsymbol{k} = (k_1, \ldots, k_d)\colon\ 2^{s_j-1} \le |k_j| < 2^{s_j},\ j = \overline{1,d}\right\}$$
and for $f \in L^0_p$ denote $\delta_{\boldsymbol{s}}(f) := \delta_{\boldsymbol{s}}(f, \boldsymbol{x}) = \sum_{\boldsymbol{k}\in\rho(\boldsymbol{s})} \widehat{f}(\boldsymbol{k}) e^{i(\boldsymbol{k},\boldsymbol{x})}$, where
$$\widehat{f}(\boldsymbol{k}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k},\boldsymbol{t})} d\boldsymbol{t}$$
are the Fourier coefficients of the function f.

Let $1 < p < \infty$, $1 \le \theta \le \infty$, $\boldsymbol{r} = (r_1, \ldots, r_d)$, $r_j > 0$, $j = \overline{1,d}$. Then the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ can be defined in the following way (see [15])
$$B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d) := \left\{f \in L^0_p\colon\ \|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \le 1\right\},$$
where
$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \asymp \left(\sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_p^\theta\right)^{1/\theta}$$
for $1 \le \theta < \infty$ and $\|f\|_{B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d)} \equiv \|f\|_{H^{\boldsymbol{r}}_p(\mathbb{T}^d)} \asymp \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(f)\|_p$.

Further we formulate a definition of the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$, that are also investigated in the paper.

Let $F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha})$ are multivariate analogs of the Bernoulli kernels, i.e.
$$F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}) = 2^d \sum_{\boldsymbol{k}} \prod_{j=1}^{d} k_j^{-r_j} \cos\left(k_j x_j - \frac{\alpha_j \pi}{2}\right), \quad r_j > 0, \quad \alpha_j \in \mathbb{R},$$
and we sum over the vectors $\boldsymbol{k} = (k_1, \ldots, k_d)$, such that $k_j > 0$, $j = \overline{1,d}$. Then by $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ we denote the class of functions f of the form
$$f(\boldsymbol{x}) = \varphi(\boldsymbol{x}) * F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}) = (2\pi)^{-d} \int_{\mathbb{T}^d} \varphi(\boldsymbol{y}) F_{\boldsymbol{r}}(\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{\alpha}) d\boldsymbol{y}, \quad \varphi \in L^0_p(\mathbb{T}^d), \quad \|\varphi\|_p \le 1.$$

The history of investigation of approximation characteristics of the classes $W^{\boldsymbol r}_{p,\boldsymbol\alpha}(\mathbb{T}^d)$, $H^{\boldsymbol r}_p(\mathbb{T}^d)$ and $B^{\boldsymbol r}_{p,\theta}(\mathbb{T}^d)$, $1 \le \theta < \infty$, can be found in the monographs [9, 20, 36, 39]. We recall that for the introduced classes the following embeddings hold:

$$B^{\boldsymbol r}_{p,p}(\mathbb{T}^d) \subset W^{\boldsymbol r}_{p,\boldsymbol\alpha}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,2}(\mathbb{T}^d), \quad 1 < p \le 2;$$
$$B^{\boldsymbol r}_{p,2}(\mathbb{T}^d) \subset W^{\boldsymbol r}_{p,\boldsymbol\alpha}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,p}(\mathbb{T}^d), \quad 2 \le p < \infty;$$
$$W^{\boldsymbol r}_{p,\boldsymbol\alpha}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,\infty}(\mathbb{T}^d) \equiv H^{\boldsymbol r}_p(\mathbb{T}^d), \quad 1 \le p \le \infty.$$

In particular, in the case $p = 2$ we have $W^{\boldsymbol r}_{2,\boldsymbol\alpha}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{2,2}(\mathbb{T}^d) \subset W^{\boldsymbol r}_{2,\boldsymbol\alpha}(\mathbb{T}^d)$.

Note that with the growth of the parameter θ, the classes $B^{\boldsymbol r}_{p,\theta}(\mathbb{T}^d)$ expand, i.e.

$$B^{\boldsymbol r}_{p,1}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,\theta_1}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,\theta_2}(\mathbb{T}^d) \subset B^{\boldsymbol r}_{p,\infty}(\mathbb{T}^d) \equiv H^{\boldsymbol r}_p(\mathbb{T}^d), \quad 1 \le \theta_1 < \theta_2 \le \infty.$$

Further, for simplicity, in place of $W^{\boldsymbol r}_{p,\boldsymbol\alpha}(\mathbb{T}^d)$, $H^{\boldsymbol r}_p(\mathbb{T}^d)$ and $B^{\boldsymbol r}_{p,\theta}(\mathbb{T}^d)$ we will use the notation $W^{\boldsymbol r}_{p,\boldsymbol\alpha}$, $H^{\boldsymbol r}_p$ and $B^{\boldsymbol r}_{p,\theta}$. This should not lead to misunderstanding because only these classes of functions are considered in the present paper.

In what follows, we assume that the coordinates of the vector $\boldsymbol r = (r_1, \dots, r_d)$ in the defined classes are arranged as $0 < r_1 = r_2 = \ldots = r_\nu < r_{\nu+1} \le \ldots \le r_d$. We associate the vector $\boldsymbol r \in \mathbb{R}^d$ with the vector $\boldsymbol\gamma \in \mathbb{R}^d$ with coordinates $\gamma_j = r_j/r_1$, $j = \overline{1,d}$, and the vector $\boldsymbol\gamma$ with the vector $\boldsymbol\gamma' \in \mathbb{R}^d$, where $\gamma'_j = \gamma_j$, $j = \overline{1,\nu}$, and $1 < \gamma'_j < \gamma_j$, $j = \overline{\nu+1,d}$.

We now define the norm $\|\cdot\|_{B_{q,1}(\mathbb{T}^d)}$, $1 \le q \le \infty$, in the subspaces $B_{q,1}(\mathbb{T}^d)$ (denoted in what follows by $B_{q,1}$) of functions $f \in L^0_q(\mathbb{T}^d)$. For trigonometric polynomials t with respect to the multiple trigonometric system $\{e^{i(\boldsymbol k,\boldsymbol x)}\}_{\boldsymbol k \in \mathbb{Z}^d}$, the norm $\|t\|_{B_{q,1}}$ is defined by the formula

$$\|t\|_{B_{q,1}} := \sum_{\boldsymbol s \in \mathbb{N}^d} \|A_{\boldsymbol s}(t)\|_q. \tag{2}$$

Similarly we define the norm $\|f\|_{B_{q,1}}$, $1 \le q \le \infty$, for any function $f \in L^0_q$ such that its series $\sum_{\boldsymbol s \in \mathbb{N}^d} \|A_{\boldsymbol s}(f)\|_q$ converges. In the case $1 < q < \infty$, we can rewrite this norm in an equivalent form $\|f\|_{B_{q,1}} \asymp \sum_{\boldsymbol s \in \mathbb{N}^d} \|\delta_{\boldsymbol s}(f)\|_q$.

Note that for $f \in B_{q,1}$, $1 \le q \le \infty$, the following relation holds

$$\|f\|_q \ll \|f\|_{B_{q,1}}. \tag{3}$$

Remark 2. *Such spaces and, respectively, norms, can be considered in a more general case, namely $B_{p,q}(\mathbb{T}^d)$, $1 \le p, q \le \infty$ (see, e.g., [14]).*

2 Approximation characteristics and auxiliary statements

Let $\mathscr{X}$ be a normed space with the norm $\|\cdot\|_{\mathscr{X}}$ and $\Theta_m := \{\boldsymbol k^1, \dots, \boldsymbol k^m\}$ be a set of vectors $\boldsymbol k^j \in \mathbb{Z}^d$, c_j be arbitrary complex numbers, $j = \overline{1,m}$.

We will consider trigonometric polynomials of the form

$$P(\Theta_m) := P(\Theta_m, \boldsymbol x) := \sum_{\boldsymbol k \in \Theta_m} c_{\boldsymbol k} e^{i(\boldsymbol k,\boldsymbol x)}$$

and for the function $f \in \mathscr{X}$ introduce the quantity

$$e_m(f)_{\mathscr{X}} := \inf_{c_{\boldsymbol k}} \inf_{\Theta_m} \|f - P(\Theta_m)\|_{\mathscr{X}}.$$

If $F \subset \mathscr{X}$ is a function class, then we set

$$e_m(F)_{\mathscr{X}} := \sup_{f \in F} e_m(f)_{\mathscr{X}}. \tag{4}$$

The approximation characteristic $e_m(F)_{\mathscr{X}}$ is called the best m-term trigonometric approximation of the class F in the space $\mathscr{X}$.

The quantity $e_m(f)_{L_2(\mathbb{T})}$ for functions of one variable in a more general case was introduced by S.B. Stechkin [34] when formulating the criterion of absolute convergence of orthogonal series. We note that the quantity (4) has a rich investigation history on different function classes. Let us mention several papers that are related to our research. For the introduced Sobolev $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ and Nikol'skii-Besov $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ classes in the spaces $\mathscr{X} = L_q(\mathbb{T}^d)$, $d \geq 1$, the estimates of the best m-term trigonometric approximations were obtained, in particular, in the papers [4,5,8,14,22,24,37,38], and in [25] the respective results were applied to estimate the best bilinear approximations. For isotropic Besov classes, the quantity (4) was studied in [7,33], for Nikol'skii-Besov- and Sobolev-type classes see, e.g., papers [1, 3, 16, 35], and for the Lizorkin-Triebel and Wiener classes see [12, 17]. A more detailed bibliography can be found in the monographs [9, 20, 36, 39]. We also note the recent papers [18, 30], where the quantity (4) was studied for isotropic Nikol'skii-Besov classes in the space $B_{q,1}$.

Let us define a close to $e_m(F)_{\mathscr{X}}$ approximation characteristics that is also investigated in the paper.

For the function $f \in \mathscr{X}$ let us denote $e^{\perp}_m(f)_{\mathscr{X}} := \inf_{\Theta_m} \|f - S_{\Theta_m}(f)\|_{\mathscr{X}}$, where

$$S_{\Theta_m}(f) := S_{\Theta_m}(f, \boldsymbol{x}) := \sum_{j=1}^{m} \widehat{f}(\boldsymbol{k}^j) e^{i(\boldsymbol{k}^j, \boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

and

$$\widehat{f}(\boldsymbol{k}^j) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k}^j, \boldsymbol{t})} d\boldsymbol{t}$$

are the Fourier coefficients of the function f that correspond to the set of vectors from Θ_m.

Respectively, for the function class $F \subset \mathscr{X}$ we set

$$e^{\perp}_m(F)_{\mathscr{X}} := \sup_{f \in F} e^{\perp}_m(f)_{\mathscr{X}}. \tag{5}$$

The quantity $e^{\perp}_M(F)_{\mathscr{X}}$ is called the best orthogonal trigonometric approximation of the class F in the space $\mathscr{X}$. The history of investigation of the quantity (5) for the classes $F = W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ and $F = B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ in the spaces $\mathscr{X} = L_q(\mathbb{T}^d)$ and $\mathscr{X} = B_{q,1}(\mathbb{T}^d)$, $1 < q \leq \infty$, is described in the papers [10, 21, 29, 32] and the monograph [20]. We note that a similar to $e^{\perp}_M(F)_{\mathscr{X}}$ quantity for the classes of non-periodic multivariate functions from the Nikol'skii-Besov classes in the space $\mathscr{X} = L_q(\mathbb{R}^d)$ was studied in the papers [40–42].

Immediately from the definitions of the quantities (4) and (5) the next relation

$$e_m(F)_{\mathscr{X}} \leq e^{\perp}_m(F)_{\mathscr{X}} \tag{6}$$

follows.

To formulate the known statements that we use in the proofs, we first introduce some more additional notation.

Let D be a bounded set in $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\Phi = \{\varphi_n(\boldsymbol{x})\}_{n=1}^{\infty}$ be a system of functions from $L_q(D)$, $1 \le q \le \infty$. For $f \in L_q(D)$ we set

$$e_m(f,\Phi)_{L_q(D)} := \inf_{\substack{\{n_j\}=\Lambda\in\mathbb{Z}_+,\ |\Lambda|=m \\ \{c_j\}\in\mathbb{R}^m}} \Big\| f - \sum_{j=1}^{m} c_j \varphi_{n_j} \Big\|_{L_q(D)},$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Further, if $K \subset L_q(D)$ is some class of functions, we define

$$e_m(K,\Phi)_{L_q(D)} := \sup_{f\in K} e_m(f,\Phi)_{L_q(D)}. \tag{7}$$

Remark 3. *In the case of trigonometric system $T := \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, we will write* (7) *as*

$$e_m(K,T)_{L_q(D)} = e_m(K,\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d})_{L_q(D)} := e_m(K)_q.$$

In what follows, for the vector $\boldsymbol{s} = (s_1,\dots,s_d)$ with even numbers $s_j \in \mathbb{N}$, $j = \overline{1,d}$, we denote $\rho^+(\boldsymbol{s}) := \big\{\boldsymbol{k} = (k_1,\dots,k_d)\colon 2^{s_j-1} \le k_j < 2^{s_j},\ k_j \in \mathbb{N},\ j = \overline{1,d}\big\}$ and for $n \in \mathbb{N}$ set

$$D_n := \big\{\boldsymbol{s} :\ (\boldsymbol{s},\boldsymbol{1}) = 2[n/2]\big\}, \qquad \mathcal{Y}_n := \bigcup_{\boldsymbol{s}\in D_n} \rho^+(\boldsymbol{s}),$$

where $[a]$ is the integer part of the number a.

Note that for the number of elements in the sets D_n and $\mathcal{Y}_n$ the following relations

$$|D_n| \asymp n^{d-1}, \qquad \mathcal{Y}_n \asymp 2^n n^{d-1}$$

hold.

Let $\mathcal{T}(Y_n)$ be a set of polynomials of the form

$$t(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|\in\mathcal{Y}_n} c_{\boldsymbol{k}} e^{i(\boldsymbol{k},\boldsymbol{x})},$$

where $|\boldsymbol{k}| = (|k_1|,\dots,|k_d|)$.

If $\mathscr{X}$ is a normed space with the norm $\|\cdot\|_{\mathscr{X}}$, by $\mathcal{T}(\mathcal{Y}_n)_{\mathscr{X}}$ we denote the unit ball in the space $\mathcal{T}(\mathcal{Y}_n)$.

In the introduced notation the following statement holds.

Theorem A ([14])**.** *There exists a constant $C_3(d) > 0$, such that for any set of functions $\Phi = \{\varphi_j\}_{j=1}^{l} \subset B_{1,1}$, $l \le C_4|\mathcal{Y}_n|$ the following estimate*

$$e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}},\Phi)_{B_{1,1}} \ge C_5 n^{d-1}, \quad C_5 = C_5(d,C_4) > 0,$$

holds for all $m \le C_3(d)|\mathcal{Y}_n|$.

Remark 4. *The norm in the space $B_{\infty,\infty}$ is defined as a modification of the relation* (2)*, by which we define the norm in the spaces $B_{q,1}$, namely $\|f\|_{B_{\infty,\infty}} := \sup_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(t)\|_\infty$.*

Lemma A ([36, p. 11])**.** *The following relation holds*

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-\beta(\boldsymbol{s},\boldsymbol{\gamma})} \asymp 2^{-\beta n} n^{\nu-1}, \quad \beta > 0.$$

Theorem B ([30]). *Let $d = 1$, $1 \le q \le p \le \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate $e_m(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp m^{-r_1}$ holds.*

Theorem C ([7,23]). *Let $d = 1$, $1 \le q \le p \le \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate $e_m(B^{r_1}_{p,\theta})_q \asymp m^{-r_1}$ holds.*

Theorem D ([22]). *Let $d \ge 2$, $1 < q \le p < \infty$, $p \ge 2$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate $e_m(B^{\boldsymbol{r}}_{p,\theta})_q \asymp m^{-r_1}(\log^{\nu-1} m)^{(r_1+1/2-1/\theta)_+}$ holds, where $a_+ = \max\{a, 0\}$.*

Further we formulate the known statements that concern the best orthogonal trigonometric approximations.

Theorem E ([30]). *Let $d = 1$, $1 \le p, q, \theta \le \infty$ and $(p, q) \notin \{(1, 1), (\infty, \infty)\}$. Then for $r_1 > (1/p - 1/q)_+$ the following estimate holds*

$$e^{\perp}_m(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp m^{-r_1+(1/p+1/q)_+}. \tag{8}$$

Remark 5. *Under the conditions of Theorem E, for the quantity $e^{\perp}_m(B^{r_1}_{p,\theta})_q$ the estimate (8) holds (see, e.g., [21]).*

Theorem F ([21]). *Let $d \ge 2$, $1 < q \le p < \infty$, $p \ge 2$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate $e^{\perp}_m(B^{\boldsymbol{r}}_{p,\theta})_q \asymp m^{-r_1}(\log^{\nu-1} m)^{(r_1+1/2-1/\theta)_+}$ holds.*

Theorem G ([21]). *Let $d \ge 2$, $1 < q < p \le 2$.*

a) If either $1 \le \theta < p$ and $r_1 \ge 1/\theta - 1/p$ or $\theta \ge p$ and $r_1 > 0$, then

$$m^{-r_1}(\log^{\nu-1} m)^{r_1+1/2-1/\theta} \ll e^{\perp}_m(B^{\boldsymbol{r}}_{p,\theta})_q \ll m^{-r_1}(\log^{\nu-1} m)^{r_1+1/p-1/\theta}.$$

b) If $1 \le \theta < p$ and $0 < r_1 < 1/\theta - 1/p$, then $e^{\perp}_m(B^{\boldsymbol{r}}_{p,\theta})_q \asymp m^{-r_1}$.

Theorem H ([21]). *Let $d \ge 2$, $1 < p < 2$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate $e^{\perp}_m(B^{\boldsymbol{r}}_{p,\theta})_p \asymp m^{-r_1}(\log^{\nu-1} m)^{(r_1+1/p-1/\theta)_+}$ holds.*

3 Main results

Theorem 1. *Let $d \ge 2$, $1 < q \le p \le \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_m(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1-1/\theta}. \tag{9}$$

Proof. We first establish the upper bound. Note that in view of the embedding $B^{\boldsymbol{r}}_{p,\theta} \subset B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < p$, it is sufficient to consider the case $p = q$.

So, let $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q \le \infty$, $1 \le \theta \le \infty$. As an approximation aggregate for function f we consider the polynomial $t_n := t_n(\boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} A_{\boldsymbol{s}}(f, \boldsymbol{x})$, where the number $n \in \mathbb{N}$ satisfies the condition $m \asymp 2^n n^{\nu-1}$.

Then defining $\gamma'(d) := \gamma'_1 + \ldots + \gamma'_d$, by the norm definition in the space $B_{q,1}$ and the properties of convolution, we can write

$$e_m(f)_{B_{q,1}} \ll \|f - t_n\|_{B_{q,1}} = \Big\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} A_{\boldsymbol{s}}(f) \Big\|_{B_{q,1}} = \sum_{\boldsymbol{s}\in\mathbb{N}^d} \Big\| A_{\boldsymbol{s}} * \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\boldsymbol{\gamma}')\ge n}} A_{\boldsymbol{s}'}(f) \Big\|_q$$

$$\le \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n-\gamma'(d)} \Big\| A_{\boldsymbol{s}} * \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty \le 1} A_{\boldsymbol{s}'}(f) \Big\|_q \le \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n-\gamma'(d)} \|A_{\boldsymbol{s}}\|_1 \Big\| \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty\le 1} A_{\boldsymbol{s}'}(f) \Big\|_q := J_1. \tag{10}$$

Further, since $\|A_s\|_1 \le C_6$, $C_6>0$ (see, e.g., [36, Ch. 2, §1]), we can continue estimating the quantity J_1 as follows

$$J_1 \ll \sum_{(s,\gamma')\ge n-\gamma'(d)} \Big\| \sum_{\|s-s'\|_\infty \le 1} A_{s'}(f)\Big\|_q \ll \sum_{(s,\gamma')\ge n-2\gamma'(d)} \|A_s(f)\|_q. \tag{11}$$

Let us consider first the case $1\le\theta<\infty$ and write (11) in the form

$$J_1 \ll \sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{(s,r)}\|A_s(f)\|_q 2^{-(s,r)} := J_2. \tag{12}$$

Using now for the expression J_2 Hölder's inequality with exponent θ (with corresponding modification of this inequality for $\theta=1$), and Lemma A, we get

$$\begin{aligned} J_2 &\le \Big(\sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{(s,r)\theta}\|A_s(f)\|_q^\theta\Big)^{1/\theta}\Big(\sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{-(s,r)\theta'}\Big)^{1/\theta'} \\ &\le \|f\|_{B^r_{q,\theta}}\Big(\sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{-(s,r)\theta'}\Big)^{1/\theta'} \le \Big(\sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{-(s,\gamma')r_1\theta'}\Big)^{1/\theta'} \\ &\asymp 2^{-nr_1}n^{(\nu-1)(1-1/\theta)}, \qquad 1/\theta+1/\theta'=1. \end{aligned} \tag{13}$$

So, combining (10), (12) with (13) and taking into account that $m\asymp 2^n n^{\nu-1}$, we get the estimate $e_m(f)_{B_{q,1}} \ll m^{-r_1}(\log^{\nu-1} m)^{r_1+1-1/\theta}$, $1\le\theta<\infty$.

Let $\theta=\infty$. For $f\in B^r_{q,\infty}$ from (1) we have that $\|A_s(f)\|_q \ll 2^{-(s,r)}$. Then in view of Lemma A and the fact that $m\asymp 2^n n^{\nu-1}$, we continue the relation (11) as follows

$$\begin{aligned} e_m(f)_{B_{q,1}} &\ll \sum_{(s,\gamma')\ge n-2\gamma'(d)} \|A_s(f)\|_q \ll \sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{-(s,r)} \\ &= \sum_{(s,\gamma')\ge n-2\gamma'(d)} 2^{-(s,\gamma)r_1} \asymp 2^{-nr_1}n^{\nu-1} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1}. \end{aligned}$$

The upper estimate is proved.

Moving to the lower estimate in (9) we note that it suffices to consider the case $p=\infty$, $1<q<\infty$ and $\nu=d$.

Let the number $n\in\mathbb{N}$ satisfies the relation $m\asymp 2^n n^{\nu-1}$ and $P_{\mathcal{Y}_n}$ denotes the operator of orthogonal projection on $\mathcal{T}(\mathcal{Y}_n)$. Then it is simple to show that the norm of the operator $P_{\mathcal{Y}_n}$, as an operator from $B_{q,1}$ to $B_{q,1}$ (notation $\|P_{\mathcal{Y}_n}\|_{B_{q,1}\to B_{q,1}}$), is bounded for $1<q<\infty$. We have

$$\begin{aligned} \|P_{\mathcal{Y}_n}\|_{B_{q,1}\to B_{q,1}} &= \sup_{\|f\|_{B_{q,1}}\le 1} \Big\| \sum_{|k|\in\mathcal{Y}_n} \widehat{f}(k)e^{i(k,x)}\Big\|_{B_{q,1}} \asymp \sup_{\|f\|_{B_{q,1}}\le 1} \sum_{s\in D_n} \|\delta_s(f)\|_q \\ &\le \sup_{\|f\|_{B_{q,1}}\le 1} \sum_{s\in\mathbb{N}^d} \|\delta_s(f)\|_q \le C_7, \quad C_7>0. \end{aligned} \tag{14}$$

Hence, in view of (14), for the trigonometric system $T=\{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$ we can write

$$e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}})_{B_{q,1}} = e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, T)_{B_{q,1}} \ge C_8 e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \{e^{i(k,x)}\}_{|k|\in\mathcal{Y}_n})_{B_{q,1}}. \tag{15}$$

In what follows, to use the relation (15) for estimating the quantity $e_m(B^r_{\infty,\theta})_{B_{q,1}}$ we consider two cases.

Case 1. Let $1 \leq \theta < \infty$. Then for the polynomial $t \in \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}$ we get

$$\|t\|_{B^{\boldsymbol{r}}_{\infty,\theta}} \asymp \left(\sum_{\boldsymbol{s} \in D_n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}(t)\|_\infty^\theta \right)^{1/\theta} \leq 2^{nr_1} \max_{\boldsymbol{s} \in D_n} \|A_{\boldsymbol{s}}(t)\|_\infty \left(\sum_{\boldsymbol{s} \in D_n} 1 \right)^{1/\theta} \ll 2^{nr_1} \|t\|_{B_{\infty,\infty}} n^{(d-1)/\theta}.$$

That yields existing of a constant $C_9(r,d,\theta) > 0$, such that it holds

$$C_9(r,d,\theta) 2^{-nr_1} n^{-(d-1)/\theta} \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset B^{\boldsymbol{r}}_{\infty,\theta} \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}. \tag{16}$$

Case 2. Let $\theta = \infty$. Then for $t \in \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}$ we get $\|t\|_{B^{\boldsymbol{r}}_{\infty,\theta}} \asymp 2^{nr_1} \max_{\boldsymbol{s} \in D_n} \|A_{\boldsymbol{s}}(t)\|_\infty \ll 2^{nr_1} \|t\|_{B_{\infty,\infty}}$ and conclude that with some constant $C_{10}(r,d) > 0$ the following embedding holds

$$C_{10}(r,d) 2^{-nr_1} \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset H^{\boldsymbol{r}}_\infty \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}. \tag{17}$$

Therefore, in view of (15)–(17) and Theorem A with respect to the trigonometric system $\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{|\boldsymbol{k}| \in \mathcal{Y}_n}$ we obtain

$$\begin{aligned} e_m(B^{\boldsymbol{r}}_{\infty,\theta})_{B_{q,1}} &\geq e_m(B^{\boldsymbol{r}}_{\infty,\theta} \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}})_{B_{q,1}} \\ &\gg 2^{-nr_1} n^{-(d-1)/\theta} e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{|\boldsymbol{k}| \in \mathcal{Y}_n})_{B_{1,1}} \\ &\gg 2^{-nr_1} n^{-(d-1)/\theta} n^{d-1} = 2^{-nr_1} n^{(d-1)(1-1/\theta)} \\ &\asymp m^{-r_1} (\log^{d-1} m)^{r_1+1-1/\theta}, \end{aligned} \tag{18}$$

where $1 < q < \infty, 1 \leq \theta \leq \infty$.

To conclude the proof we note that the lower estimate in (9) for $q = \infty$ follows from (18) and the inequality $\| \cdot \|_{B_{\infty,1}} \geq \| \cdot \|_{B_{q,1}}$, $1 < q < \infty$. Theorem 1 is proved. ☐

We now comment the obtained result.

First we note that in the univariate case the orders of the quantities $e_m(B^{r_1}_{p,\theta})_{B_{q,1}}$ and $e_m(B^{r_1}_{p,\theta})_q$, $1 \leq q \leq p \leq \infty$, are formulated in Theorems B and C. We see that the following relations

$$e_m(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp e_m(B^{r_1}_{p,\theta})_q \asymp m^{-r_1} \tag{19}$$

hold. We also note that in (19) the case $p = q = 1$ is included.

In the multivariate case ($d \geq 2$) the situation is different. So, comparing the results of Theorems 1 and D for corresponding values of the parameters p and q, we see that the orders of the quantities $e_m(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ and $e_m(B^{\boldsymbol{r}}_{p,\theta})_q$ coincide only for $\nu = 1$. Let us mention one more important issue that was actually a motivation for investigating the best m-term trigonometric approximations of the classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space $B_{q,1}$ for $d \geq 2$. We have in mind the fact that in Theorem 1 we obtained, in particular, estimates of the quantities $e_m(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ in the cases $1 < q \leq p \leq 2$ and $p = q = \infty$, where the orders of the respective approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space L_q still remain unknown (see, e.g., Theorem D, and also [9, Open problem 7.5]).

In the following statement we get the order of the quantity $e^{\perp}_m(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$.

Theorem 2. *Let $d \geq 2$, $1 < q \leq p \leq \infty$, $q \neq \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1-1/\theta}. \tag{20}$$

Proof. The lower estimate follows from Theorem 1 and the relation

$$e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \geq e_m(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1-1/\theta}.$$

Moving to proving the upper estimate in (20), we note that it suffices to get it for the case $1 < q = p < \infty$.

Hence, for the function $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < \infty$, as an approximation aggregate we use the polynomial $S_{\Theta_m}(f) := S_{\Theta_m}(f, \boldsymbol{x}) = \sum_{(\boldsymbol{s},\gamma')<n} \delta_{\boldsymbol{s}}(f, \boldsymbol{x})$, where the number $n \in \mathbb{N}$ satisfies the condition $m \asymp 2^n n^{\nu-1}$.

Therefore, we can write

$$\begin{aligned} e_m^{\perp}(f)_{B_{q,1}} &\leq \Big\| f - \sum_{(\boldsymbol{s},\gamma')<n} \delta_{\boldsymbol{s}}(f) \Big\|_{B_{q,1}} = \Big\| \sum_{(\boldsymbol{s},\gamma')\geq n} \delta_{\boldsymbol{s}}(f) \Big\|_{B_{q,1}} \\ &= \sum_{\boldsymbol{s}\in\mathbb{N}^d} \Big\| \delta_{\boldsymbol{s}}\Big(\sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\gamma')\geq n}} \delta_{\boldsymbol{s}'}(f) \Big) \Big\|_q \leq \sum_{(\boldsymbol{s},\gamma')\geq n} \|\delta_{\boldsymbol{s}}(f)\|_q := J_3. \end{aligned} \tag{21}$$

Further let us consider two cases.

Case 1. Let $1 \leq \theta < \infty$. Then, in view of Hölder's inequality with exponent θ (with corresponding modification of this inequality for $\theta = 1$) and Lemma A, we get

$$\begin{aligned} J_3 &\leq \Big(\sum_{(\boldsymbol{s},\gamma')\geq n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_q^{\theta} \Big)^{1/\theta} \Big(\sum_{(\boldsymbol{s},\gamma')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'} \Big)^{1/\theta'} \\ &\ll \|f\|_{B^{\boldsymbol{r}}_{q,\theta}} \Big(\sum_{(\boldsymbol{s},\gamma')\geq n} 2^{-(\boldsymbol{s},\gamma)r_1\theta'} \Big)^{1/\theta'} \ll 2^{-nr_1} n^{(\nu-1)(1-1/\theta)}. \end{aligned} \tag{22}$$

Case 2. Let $\theta = \infty$. In this case, taking into account that for $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < \infty$, it holds $\|\delta_{\boldsymbol{s}}(f)\|_q \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$, $\boldsymbol{s} \in \mathbb{N}^d$, and using Lemma A, we obtain

$$J_3 \ll \sum_{(\boldsymbol{s},\gamma')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{r})} \ll 2^{-nr_1} n^{\nu-1}. \tag{23}$$

Hence, combining (21)–(23) and noting that $m \asymp 2^n n^{\nu-1}$, we get the respective upper estimate of the quantity $e_m^{\perp}(B^{\boldsymbol{r}}_{q,\theta})_{B_{q,1}}$, and respectively

$$e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \ll m^{-r_1}(\log^{\nu-1} m)^{r_1+1-1/\theta}, \quad 1 < q \leq p \leq \infty, \quad q \neq \infty$$

Theorem 2 is proved. □

Let us comment the obtained result.

The result of Theorem 2 complements estimates of the quantity $e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$, $q \in \{1, \infty\}$, that were obtained in the papers [29,31]. Besides, in Theorem 2 we managed to expand the allowed parameter regions in comparison to the known by this time estimates of the quantity $e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_q$

(see Theorems F, G). Having this in mind, comparing the results of Theorems 2, F and G for mutual values of the parameters p and q we conclude that orders of the considered quantities in the spaces $B_{q,1}$ and L_q differ except the case $\nu = 1$. Here we note that in the univariate case the following relations hold (see [21,30]):

$$e_m^{\perp}(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp e_m^{\perp}(B^{r_1}_{p,\theta})_q \asymp m^{-r_1}.$$

To conclude the paper, let us obtain orders of the considered in Theorems 1, 2 approximation characteristics (4) and (5) for the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{q,1}$ for some relations between the parameters p and q.

Theorem 3. *Let $d \geq 2$, $1 < q \leq 2$, $q \leq p < \infty$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then for $r_1 > 0$ the following estimates hold*

$$e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp e_m^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1/2}. \tag{24}$$

Proof. We establish first the upper estimate of the quantity $e_m^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}}$ noting that it suffices to consider the case $1 < q \leq p \leq 2$. Then, in view of the fact that $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,2}$, $1 < p \leq 2$, and by the result of Theorem 2 for $\theta = 2$, we have

$$e_m^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \ll e_m^{\perp}(B^{\boldsymbol{r}}_{p,2})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1/2}. \tag{25}$$

What concerns the lower estimate in (24), it is sufficient to prove it for the quantity $e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}}$ for $1 < q \leq 2 \leq p < \infty$.

Hence, taking into account the embedding $B^{\boldsymbol{r}}_{p,2} \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $2 \leq p < \infty$, and using the estimate of Theorem 1 for $\theta = 2$, we get

$$e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \gg e_m(B^{\boldsymbol{r}}_{p,2})_{B_{q,1}} \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1/2}. \tag{26}$$

By (6) we have $e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \leq e_m^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}}$, and hence from the relations (25) and (26) we derive (24). Theorem 3 is proved. □

To complement Theorem 3, let us formulate the respective result for the univariate case.

Theorem 4. *Let $d = 1$, $1 < q \leq p < \infty$, $\alpha \in \mathbb{R}$. Then for $r_1 > 0$ the following estimates hold*

$$e_m(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp e_m^{\perp}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp m^{-r_1}. \tag{27}$$

Proof. The upper estimate of both of the quantities is a corollary from the obtained in [19] results (see Theorems 1′, 4′).

The lower estimate in (27) follows from the relation

$$e_m(W^{r_1}_{p,\alpha})_q \asymp m^{-r_1} \tag{28}$$

(see [9, Theorem 7.5.1]) and inequalities (3). Theorem 4 is proved. □

We now comment the results of Theorems 3, 4.

Comparing the estimates (27) and (28) we see that in the case $d = 1$ the respective approximation characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces $B_{q,1}$ and L_q coincide in order. A different situation is in the multivariate case ($d \geq 2$). For convenient comparison we formulate an analog of Theorem 3 in the space L_q.

Theorem I. *Let $d \geq 2$, $1 < q \leq p < \infty$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then for $r_1 > 0$ the following estimates hold*

$$e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp e^{\perp}_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1}. \quad (29)$$

Note that the upper estimate of both of the quantities in (29) is realized by approximation of functions $f \in W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space L_q by their step hyperbolic Fourier sums (see [9, Theorem 4.2.4])

$$S^{\gamma}_n(f) := S^{\gamma}_n(f, \boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \delta_{\boldsymbol{s}}(f, \boldsymbol{x}), \; m \asymp 2^n n^{\nu-1}.$$

The lower estimate of the quantity $e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q$ was obtained in the paper [14].

Hence, comparing (24) with (29) for $1 < q \leq 2$, $q \leq p < \infty$ we see that in the case $d \geq 2$ orders of the respective approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ coincide in the spaces $B_{q,1}$ and L_q only for $\nu = 1$.

References

[1] Akishev G.A. *Estimates of M-term approximations of functions of several variables in the Lorentz space by a constructive method*. Eurasian Math. J. 2024, **15** (2), 8–32.

[2] Amanov T.I. *Representation and imbedding theorems for function spaces $S^{(r)}_{p,\theta}B(\mathbb{R}_n)$ and $S^{(r)*}_{p,\theta}B$, $(0 \leq x_j \leq 2\pi;$ $j = 1, \ldots, n)$*. Tr. Mat. Inst. Steklova 1965, **77**, 5–34. (in Russian)

[3] Basarkhanov D.B. *Nonlinear approximations of classes of periodic functions of many variables*. Proc. Steklov Inst. Math. 2014, **284** (1), 2–31. doi:10.1134/S0081543814010027 (translation of Tr. Mat. Inst. Steklova 2014, **284**, 8–37. doi:10.1134/S0371968514010026 (in Russian))

[4] Belinskii E.S. *Approximation by a "floating" system of exponentials on classes of periodic functions with bounded mixed derivative*. In: Studies in the theory of functions of several real variables, Yaroslav State Univ., Yaroslavl, 1988, 16–33. (in Russian)

[5] Belinskii E.S. *Decomposition theorems and approximation by a "floating" system of exponentials*. Trans. Amer. Math. Soc. 1998, **350** (1), 43–53. doi:10.1090/S0002-9947-98-01556-6

[6] Belinsky E.S. *Estimates of entropy numbers and Gaussian measures for classes of functions with bounded mixed derivative*. J. Approx. Theory 1998, **93** (1), 114–127. doi:10.1006/jath.1997.3157

[7] DeVore R.A., Temlyakov V.N. *Nonlinear approximation by trigonometric sums*. J. Fourier Anal. Appl. 1995, **2** (1), 29–48. doi:10.1007/s00041-001-4021-8

[8] Dũng D. *Continuous algorithms in n-term approximation and non-linear width*. J. Approx. Theory 2000, **102** (2), 217–242. doi:10.1006/jath.1999.3399

[9] Dũng D., V. Temlyakov, Ullrich T. Hyperbolic Cross Approximation. Adv. Courses Math. Birkhäuser, CRM Barselona, 2018.

[10] Fedunyk-Yaremchuk O.V., Hembars'ka S.B. *Best orthogonal trigonometric approximations of the Nikol'skii-Besov-type classes of periodic functions of one and several variables*. Carpathian Math. Publ. 2022, **14** (1), 171–184. doi:10.15330/cmp.14.1.171-184

[11] Fedunyk-Yaremchuk O.V., Hembars'kyi M.V., Hembars'ka S. B. *Approximative characteristics of the Nikol'skii–Besov-type classes of periodic functions in the space $B_{\infty,1}$*. Carpathian Math. Publ. 2020, **12** (2), 376–391. doi:10.15330/cmp.12.2.376-391

[12] Hansen M., Sickel W. *Best m-term approximation and Sobolev-Besov spaces of dominating mixed smoothness — the case of compact embeddings*. Constr. Approx. 2012, **36** (1), 1–51. doi:10.1007/s00365-012-9161-3

[13] Hembarskyi M.V., Hembarska S.B. *Widths of the classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space $B_{1,1}$*. J. Math. Sci. (N.Y.) 2018, **235** (1), 35–45. doi:10.1007/s10958-018-4056-x (translation of Ukr. Mat. Visn. 2018, **15** (1), 43–57. (in Ukrainian))

[14] Kashin B.S., Temlyakov V.N. *On best m-term approximations and the entropy of sets in the space L^1*. Math. Notes 1994, **56** (5–6), 1137–1157. doi:10.1007/BF02274662 (translation of Mat. Zametki 1994, **56** (5), 57–86. (in Russian))

[15] Lizorkin P.I., Nikol'skii S.M. *Function spaces of mixed smoothness from the decomposition point of view*. Proc. Steklov Inst. Math. 1990, **187**, 163–184. (translation of Tr. Mat. Inst. Steklova 1989, **187**, 143–161. (in Russian))

[16] Moeller M., Stasyuk S., Ullrich T. *High-dimensional sparse trigonometric approximation in the uniform norm and consequences for sampling recovery*. arXiv:2407.15965 [math.NA]. doi:10.48550/arXiv.2407.15965

[17] Nguyen V.K., Nguyen V.D. *Best n-term approximation of diagonal operators and application to function spaces with mixed smoothness*. Anal. Math. 2022, **48** (4), 1127–1152. doi:10.1007/s10476-022-0169-z

[18] Pozharska K.V., Romanyuk A.S. *The best m-term trigonometric approximations of the classes of periodic functions of one and many variables in the space $B_{q,1}$*. Res. Math., 2024, **32** (2), 137–154. doi:10.15421/242425

[19] Pozharska K., Romanyuk A., Yanchenko S. *Best approximations for classes of periodic functions of many variables with bounded dominating mixed derivative*. Ukrainian Math. J. 2024, **76** (7), 1144–1162. doi:10.1007/s11253-024-02378-9 (translation of Ukraïn. Mat. Zh. 2024, **76** (7), 1007–1023. doi:10.3842/umzh.v76i7.8307 (in Ukrainian))

[20] Romanyuk A.S. Approximating Characteristics of the Classes of Periodic Functions of Many Variables. In: Samoilenko A.M. (Ed.) Proc. of the Institute of Mathematics of the National Academy of Sciences of Ukraine, 93. IM of NASU, Kyiv, 2012. (in Russian)

[21] Romanyuk A.S. *Approximation of classes of periodic functions in several variables*. Math. Notes 2002, **71** (1), 98–109. doi:10.1023/A:1013982425195 (translation of Mat. Zametki 2002, **71** (1), 109–121. doi:10.4213/mzm332 (in Russian))

[22] Romanyuk A.S. *Best M-term trigonometric approximations of Besov classes of periodic functions of several variables*. Izv. Math. 2003, **67** (2), 265–302. doi:10.1070/IM2003v067n02ABEH000427 (translation of Izv. Ross. Akad. Nauk Ser. Mat. 2003, **67** (2), 61–100. (in Russian))

[23] Romanyuk A.S. *Best trigonometric and bilinear approximations of classes of functions of several variables*. Math. Notes 2013, **94** (3), 379–391. doi:10.1134/S0001434613090095 (translation of Mat. Zametki 2013, **94** (3), 401–415. doi:10.4213/mzm8892 (in Russian))

[24] Romanyuk A.S. *Best trigonometric approximations for some classes of periodic functions of several variables in the uniform metric*. Math. Notes 2007, **82** (2), 216–228. doi:10.1134/S0001434607070279 (translation of Mat. Zametki 2007, **82** (2), 247–261. (in Russian))

[25] Romanyuk A.S. *Bilinear and trigonometric approximations of periodic functions of several variables of Besov classes $B^r_{p,q}$*. Izv. Math. 2006, **70** (2), 277–306. doi:10.1070/IM2006v070n02ABEH002313 (translation of Izv. Ross. Akad. Nauk Ser. Mat. 2006, **70** (2), 69–98. (in Russian))

[26] Romanyuk A.S. *Entropy numbers and widths for the classes $B^r_{p,\theta}$ of periodic functions of many variables*. Ukrainian Math. J. 2017, **68** (10), 1620–1636. doi:10.1007/s11253-017-1315-9 (translation of Ukraïn. Mat. Zh. 2016, **68** (10), 1403–1417. (in Russian))

[27] Romanyuk A.S., Romanyuk V.S. *Approximating characteristics of the classes of periodic multivariate functions in the space $B_{\infty,1}$*. Ukrainian Math. J. 2019, **71**(2), 308–321. doi:10.1007/s11253-019-01646-3 (translation of Ukraïn. Mat. Zh. 2019, **71** (2), 271–282. (in Ukrainian))

[28] Romanyuk A.S., Romanyuk V.S. *Approximative characteristics and properties of operators of the best approximation of classes of functions from the Sobolev and Nikol'skii-Besov spaces*. J. Math. Sci. (N.Y.) 2021, **252** (4), 508–525. doi:10.1007/s10958-020-05177-2 (translation of Ukr. Mat. Visn. 2020, **17** (3), 372–395 (in Ukrainian))

[29] Romanyuk A.S., Romanyuk V.S. *Estimation of some approximating characteristics of the classes of periodic functions of one and many variables*. Ukrainian Math. J. 2020, **71** (8), 1257–1272. doi:10.1007/s11253-019-01711-x (translation of Ukraïn. Mat. Zh. 2019, **71** (2), 1102–1115. (in Ukrainian))

[30] Romanyuk A.S., Romanyuk V.S., Pozharska K.V., Hembars'ka S.B. *Characteristics of linear and nonlinear approximation of isotropic classes of periodic multivariate functions*. Carpathian Math. Publ. 2023, **15** (1), 78–94. doi:10.15330/cmp.15.1.78-94

[31] Romanyuk A.S., Yanchenko S.Ya. *Approximation of the classes of periodic functions of one and many variables from the Nikol'skii-Besov and Sobolev spaces*. Ukrainian Math. J. 2022, **74** (6), 967–980 (2022). doi:10.1007/s11253-022-02110-5 (translation of Ukraïn. Mat. Zh. 2022, **74** (6), 857–868. doi:10.37863/umzh.v74i6.7141 (in Ukrainian))

[32] Romanyuk A.S., Yanchenko S.Ya. *Estimates of approximating characteristics and the properties of the operators of best approximation for the classes of periodic functions in the space $B_{1,1}$*. Ukrainian Math. J. 2022, **73** (8), 1278–1298. doi:10.1007/s11253-022-01990-x (translation of Ukraïn. Mat. Zh. 2021, **73** (8), 1102–1119. doi:10.37863/umzh.v73i8.6755 (in Ukrainian))

[33] Stasyuk S.A. *Best m-term trigonometric approximation of periodic functions of several variables from Nikol'skii-Besov classes for small smoothness*. J. Approx. Theory 2014, **177**, 1–16. doi:10.1016/j.jat.2013.09.006

[34] Stechkin S.B. *On absolute convergence of orthogonal series*. Dokl. Akad. Nauk SSSR 1955, **102** (2), 37–40. (in Russian)

[35] Shvai K.V. *The best m-term trigonometric approximations of the classes of periodic multivariate functions with bounded generalized derivative in the space L_q*. J. Math. Sci. (N.Y.) 2017, **222** (6), 750–761. doi:10.1007/s10958-017-3329-0 (translation of Ukr. Mat. Visn. 2016, **13** (3), 361–375. (in Ukrainian))

[36] Temlyakov V.N. *Approximation of functions with bounded mixed derivative*. Proc. Steklov Inst. Math. 1989, **178**, 1–121. (translation of Tr. Mat. Inst. Steklova 1986, **178**, 3–113. (in Russian))

[37] Temlyakov V.N. *Constructive sparse trigonometric approximation for functions with small mixed smoothness*. Constr. Approx. 2017, **45** (3), 467–495. doi:10.1007/s00365-016-9345-3

[38] Temlyakov V.N., Ullrich T. *Approximation of functions with small mixed smoothness in the uniform norm*. J. Approx. Theory 2022, **277**, 105718. doi:10.1016/j.jat.2022.105718

[39] Trigub R.M., Belinsky E.S. Fourier Analysis and Approximation of Functions. Kluwer Academic Publishers, Dordrecht, 2004.

[40] Yanchenko S.Ya. *Approximation of the classes $S^r_{p,\theta}B(\mathbb{R}^d)$ of functions of many variables by entire functions of a special form*. Ukrainian Math. J. 2011, **62** (8), 1307–1325. doi:10.1007/s11253-011-0431-1 (translation of Ukraïn. Mat. Zh. 2010, **62** (8), 1124–1138 (in Ukrainian))

[41] Yanchenko S.Ya. *Approximation of the Nikol'skii-Besov functional classes by entire functions of a special form*. Carpathian Math. Publ. 2020, **12** (1), 148–156. doi:10.15330/cmp.12.1.148-156

[42] Yanchenko S.Ya., Radchenko O.Ya. *Approximation characteristics of the isotropic Nikol'skii-Besov functional classes*. Carpathian Math. Publ. 2021, **13** (3), 851–861. doi:10.15330/cmp.13.3.851-861

Received 02.02.2025
Revised 07.08.2025

Пожарська К.В., Романюк А.С., Янченко С.Я. *Оцінки характеристик нелінійної апроксимації класів періодичних функцій багатьох змінних* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 447–460.

У роботі одержано точні за порядком оцінки найкращого m-членного тригонометричного наближення та найкращого ортогонального тригонометричного наближення функцій з класів Нікольського-Бєсова $B^r_{p,\theta}(\mathbb{T}^d)$ і Соболєва $W^r_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ у підпросторах Лебега $B_{q,1}(\mathbb{T}^d)$ для деяких співвідношень між параметрами p та q. При цьому встановлено, що одержані оцінки досліджуваних апроксимаційних характеристик у багатовимірному випадку, на противагу одновимірному, у просторах $B_{q,1}(\mathbb{T}^d)$ та $L_q(\mathbb{T}^d)$ відрізняються за порядком. Крім цього знайдені точні за порядком оцінки найкращого m-членного та ортогонального тригонометричних наближень у просторах $B_{q,1}(\mathbb{T}^d)$ для деяких значень параметрів, при яких вони залишаються невідомими у просторі $L_q(\mathbb{T}^d)$.

Ключові слова і фрази: клас Нікольського-Бєсова, клас Соболєва, найкраще m-членне тригонометричне наближення, найкраще ортогональне тригонометричне наближення.