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An application for dynamic programming via fixed point theorem using rational type

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In this study, we introduce and investigate the concept of generalized multivalued rational type F-contractions on spherically complete ultrametric spaces. Building upon the existing framework of fixed point theory, we establish novel fixed point theorems for such mappings and provide several corollaries that extend and unify known results in the literature. To highlight the applicability of our findings, we present an illustrative example that demonstrates the validity of the proposed approach. Furthermore, we explore an application to dynamic programming by formulating functional equations whose solutions can be obtained through the developed fixed point techniques. The results not only broaden the scope of contraction principles in non-Archimedean settings but also emphasize the utility of ultrametric structures in optimization and computational mathematics.

Key words and phrases: dynamic programming, fixed point, F-contraction, ultrametric space.

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Introduction

It is well known that the Banach fixed point theorem is important in both theoretical and computational aspects of mathematics. This theorem finds many practical applications due to its simple statement and proof. It provides the theoretical background for the successive approximation method widely used by É. Picard and J. Liouville. Banach's Contraction Principle says that, whenever (M, v) is complete, then any contraction selfmap of M has a unique fixed point. This fixed point result is one of the most powerful tools for many existence and uniqueness problems arising in mathematics. Because of its importance, Banach Contraction Principle has been extended and generalized in many ways (see, for instance, [3, 4, 6, 11, 14, 17, 18]). Among all these, an interesting generalization was given by D. Wardowski [20].

Theorem 1 ([20])**.** *Let (M, v) be a complete metric space and let $T : M \to M$ be an F-contraction. Then T has a unique fixed point in M.*

In this theorem, $F : (0, \infty) \to \mathbb{R}$ satisfies the following conditions:

(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, we have $F(\alpha) < F(\beta)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers we have

$$\lim_{n\to+\infty} \alpha_n = 0 \quad \Longleftrightarrow \quad \lim_{n\to+\infty} F(\alpha_n) = -\infty;$$

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(F3) there exists $k \in (0,1)$ such that

$$\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0.$$

By $\mathcal{F}$ we denote the set of all functions F. Many authors extend this result for different type mappings on different type metric space (see [1, 13, 15]). On the other hand, for the first time, in [19], A.C.M. Van Roovij introduced the concept of ultrametric space. Then L. Gajic [9] proved the following theorem.

Theorem 2 ([9])**.** *Let (M, v) be a spherically complete ultrametric space. If $T : M \to M$ is a mapping such that for all $x, y \in M$, $x \neq y$ we have*

$$v(Tx, Ty) < \max\{v(x,y), v(x,Tx), v(y,Ty)\},$$

then T has a unique fixed point.

Here, we must mention that L. Gajic [8] extended this result for multivalued maps.

Another important study in fixed point theory is to determine coincidence point results for single valued and set valued mappings. The analogue of such results in ultrametric spaces was obtained by K.P.R. Rao et. al. [16] for coincidence point theorems from single valued maps to set valued contractive maps.

Theorem 3. *Let (M, v) be an ultrametric space, and let $f, S, T : M \to M$ satisfy the following conditions:*

(i) $f(M)$ is spherically complete;

(ii) $v(Sx, Ty) < \max\{v(fx, fy), v(fx, Sx), v(fy, Ty)\}$ for all $x, y \in M$, $x \neq y$;

(iii) $fS = Sf$, $fT = Tf$, $ST = TS$;

(iv) $S(M) \subseteq f(M)$, $T(M) \subseteq f(M)$.

Then either $fw = Sw$ or $fw = Tw$ for some $w \in M$.

Q. Zhang and Y. Song (see [21]) gave some results for weak contraction on ultrametric space. On the other hand, in [10], give some fixed point results on ultrametric space using F-contraction. For other results for fixed point theory on ultrametric space you can see [2,7].

In this paper, we aim to obtain some fixed point theorems using rational type F-contraction for multivalued mapping on a spherically complete ultrametric space and give some corollaries and examples. Also, we give an application. But, firstly, we want to give some definitions about ultrametric space which are used for our main result.

Definition 1 ([9])**.** *Let (M, v) be a metric space. If the metric v satisfies the inequality*

$$v(x,y) \leq \max\{v(x,z), v(z,y)\} \quad \text{for all } x, y, z \in M,$$

then we say that v is an ultrametric on M and the pair (M, v) is an ultrametric space.

Example 1 ([9]). *Let v be a discrete metric on M $(M \neq \varnothing)$. Then v is an ultrametric on M.*

Example 2 ([9]). *Let $[x]$ be the entire part of x for $x \in \mathbb{R}$. For any $e \in \mathbb{R}\backslash\mathbb{Q}$ and for all $x, y \in \mathbb{Q}$*

$$v(x,y) = \inf\{2^{-n} : n \in \mathbb{Z}, [2^n(x-e)] = [2^n(y-e)]\}$$

is an ultrametric on $\mathbb{Q}$.

Remark 1 ([9]). *Every ultrametric space is a metric space but the converse need not be true. For example, $(\mathbb{R}, d)$ as a usual space is not an ultrametric space.*

Definition 2 ([19]). *Let (M, v) be a ultrametric space, $x \in M$ and $\varepsilon > 0$. Then*

$$\mathfrak{B}(x,\varepsilon) = \{y \in M : v(x,y) \leq \varepsilon\}$$

with $\mathfrak{B}(x,0) = \{x\}$ is called a ball.

Example 3. *Let v be a discrete metric and (M, v) be an ultrametric. Then for any $r < 1$ we have $\mathfrak{B}(x,r) = \{x\}$ and for any $r \geq 1$ we get $\mathfrak{B}(x,r) = M$.*

Remark 2 ([19]). *A well-known characteristic property of ultrametric spaces is the following:*

$$\text{if } x,y \in M,\ 0 \leq \varepsilon \leq r \text{ and } \mathfrak{B}(x,\varepsilon) \cap \mathfrak{B}(x,r) \neq \varnothing,\ \text{then } \mathfrak{B}(x,\varepsilon) \subset \mathfrak{B}(x,r).$$

Definition 3 ([19]). *Let (M, v) be an ultrametric space. If every shrinking collection of balls in M has a nonempty intersection, then an ultrametric space (M, v) is said to be spherically complete.*

Remark 3. *Every spherically complete ultrametric space is complete metric space. The conversely is usually not true.*

Definition 4 ([10]). *Let $CB(M)$ be the class of all nonempty closed bounded subsets of M. For $A, B \in CB(M)$ we have that*

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B),\ \sup_{y\in B} D(y,A)\right\}$$

is the Hausdorff metric induced by v, where $D(x, A) = \inf\{v(x,y) : y \in A\}$.

Definition 5 ([19]). *Let M be a nonempty set. A pseudo metric on M is a function*

$$v : M \times M \to \mathbb{R}^+ \cup \{0\}$$

satisfying for any $x, y, \omega \in M$ the following conditions:

(i) $v(x,y) = 0 \Longrightarrow x = y$;

(ii) $v(x,y) = v(y,x)$;

(iii) $v(x,y) \leq v(x,\omega) + v(\omega,y)$.

Definition 6 ([19]). *A pseudo metric on M is an ultrapseudo metric if in addition to conditions given in Definition 5 we add*

(iii') $v(x,y) \leq \max\{v(x,\omega), v(\omega,y)\}$

for any $x, y, \omega \in M$.

Proposition 1 ([19]). *A pseudo metric (M, v) is an ultrapseudo metric if and only if for any three points $x, y, \omega \in M$ the following conditions are satisfied: $v(\omega,y) \leq v(x,y) = v(x,\omega)$ or $v(x,\omega) \leq v(y,x) = v(y,\omega)$ or $v(x,y) \leq v(\omega,x) = v(\omega,y)$.*

1 Main Result

In this section, we prove a fixed point theorem in the sense of L. Gajic [8] and give an illustrative example. But first we define generalized multivalued rational type F-contraction and give some basic notations that we will use in the definition.

Let $\psi : [0, \infty)^5 \to [0, \infty)$ be such that:

(i) ψ is continuous and monotone nondecreasing in each coordinate;

(ii) $\psi(a, a, a, a, a) \leq a$ for all $a \geq 0$.

We denote the collection of such functions ψ by the symbol Ψ [5].

Let $\varphi : [0, \infty)^4 \to [0, \infty)$ be such that:

(i) φ is continuous and monotone nondecreasing in each coordinate;

(ii) $\varphi(x_1, x_2, x_3, x_4) = 0$ if $x_1 x_2 x_3 x_4 = 0$.

We denote the collection of such functions φ by the symbol Φ [5].

In the rest of the article, let 2^M_C be the class of all nonempty compact subsets of M.

Definition 7. *Let (M, v) be an ultrametric space and $T : M \to 2^M_C$ be a mapping. We say that T is a generalized multivalued rational type F-contraction on M if for all $x, y \in M$ with $x \neq y$ we have*

$$H(Tx, Ty) > 0 \implies \tau + F(H(Tx, Ty)) \leq F(M(x, y) + N(x, y)),$$

where $\tau > 0$, $F \in \mathcal{F}$, $\psi \in \Psi$ and $\varphi \in \Phi$ with

$$N(x, y) = \varphi(D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx))$$

and

$$M(x, y) = \psi\Big(v(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + v(x, y)}, \frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big).$$

Theorem 4. *Let (M, v) be a spherically complete ultrametric space and $T : M \to 2^M_C$ be a generalized multivalued rational type F-contraction. Then T has a fixed point.*

Proof. Let $\mathfrak{B}_\alpha = B(\alpha, D(\alpha, T\alpha))$ be a closed ball centered at α with radius $D(\alpha, T\alpha) = \inf_{s \in T\alpha} v(\alpha, s)$ and let $\mathcal{A}$ be the collection of these balls for all $\alpha \in M$. Let $\mathfrak{B}_\alpha \preceq \mathfrak{B}_\beta$ iff $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$. It is partial order on $\mathcal{A}$. Let $\mathcal{A}_1$ be a totally ordered subfamily of $\mathcal{A}$. Since M is spherically complete, we get

$$\bigcup_{\mathfrak{B}_\alpha \in \mathcal{A}_1} \mathfrak{B}_\alpha = B \neq \varnothing.$$

Let $\beta \in B$ and $\mathfrak{B}_\alpha \in \mathcal{A}_1$. Clearly, $\beta \in \mathfrak{B}_\alpha$ thus $v(\beta, \alpha) \leq D(\alpha, T\alpha)$. Since $T\alpha$ is a nonempty compact set, we can take $u \in T\alpha$ such that $v(\alpha, u) = D(\alpha, T\alpha)$. Then

$$\begin{aligned} D(\beta, T\beta) &\leq \inf_{c \in T\beta} v(\beta, c) \leq \max\{v(\beta, \alpha), v(\alpha, u), \inf_{c \in T\beta} v(u, c)\} \\ &\leq \max\{D(\alpha, T\alpha), H(T\alpha, T\beta)\} < \max\{D(\alpha, T\alpha), M(\alpha, \beta) + N(\alpha, \beta)\}, \end{aligned}$$

where

$$\begin{aligned}N(\alpha,\beta) &= \varphi(D(\alpha,T\alpha),D(\beta,T\beta),D(\alpha,T\beta),D(\beta,T\alpha))\\ &\leq \varphi(D(\alpha,T\alpha),H(T\alpha,T\beta),D(\alpha,T\beta),v(\beta,\beta)).\end{aligned}$$

Since $\varphi(x_1,x_2,x_3,x_4)=0$ if $x_1x_2x_3x_4=0$, we obtain $N(\alpha,\beta)=0$ and

$$\begin{aligned}M(\alpha,\beta) &= \psi\Big(v(\alpha,\beta),v(\alpha,T\alpha),v(\beta,T\beta),\frac{D(\alpha,T\alpha)D(\beta,T\beta)+D(\alpha,T\beta),D(\beta,T\alpha)}{1+v(\alpha,\beta)},\\ &\qquad \frac{D(\alpha,T\alpha)D(\beta,T\beta)+D(\alpha,T\beta),D(\beta,T\alpha)}{1+H(T\alpha,T\beta)}\Big)\\ &\leq \psi\Big(v(\alpha,\beta),v(\alpha,\beta),D(\beta,T\beta),\frac{v(\alpha,\beta)D(\beta,T\beta)+D(\alpha,T\beta)v(\beta,\beta)}{1+v(\alpha,\beta)},\\ &\qquad \frac{v(\alpha,\beta)D(\beta,T\beta)+D(\alpha,T\beta)v(\beta,\beta)}{1+D(\beta,T\beta)}\Big)\\ &\leq \psi\big(D(\alpha,T\alpha),D(\alpha,T\alpha),D(\beta,T\beta),D(\beta,T\beta),D(\alpha,T\alpha)\big).\end{aligned}$$

Assume that $D(T\alpha,T\alpha)<D(\beta,T\beta)$. Then, we get

$$\begin{aligned}D(\beta,T\beta) &\leq \inf_{c\in T\beta} v(\beta,c)\\ &< \max\{D(\alpha,T\alpha),\psi\big(D(\beta,T\beta),D(\beta,T\beta),D(\beta,T\beta),D(\beta,T\beta),D(\beta,T\beta)\big)\}\\ &< \max\{D(\beta,T\beta),D(\beta,T\beta)\}=D(\beta,T\beta),\end{aligned}$$

which is a contradiction, so $D(\beta,T\beta)<D(\alpha,T\alpha)$. Now, for any $x\in\mathfrak{B}_\beta$ we have

$$v(x,\beta)\leq D(\beta,T\beta)<D(\alpha,T\alpha),\quad v(x,\alpha)\leq\max\{v(x,\beta),v(\beta,\alpha)\}\leq D(\alpha,T\alpha),$$

so, $x\in\mathfrak{B}_\alpha$. From here $\mathfrak{B}_\beta\subseteq\mathfrak{B}_\alpha$ is hold for any $\mathfrak{B}_\alpha\in\mathcal{A}_1$. So, $\mathfrak{B}_\beta$ is upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. With Zorn's Lemma, there is a maximal element in $\mathcal{A}$, let $\mathfrak{B}_\wp$.

Now prove that $\wp\in T\wp$. Presume that $\wp\notin T\wp$, there exists $\mathrm{Im}\in T\wp$, $\mathrm{Im}\neq\wp$, such that $v(\wp,\mathrm{Im})=D(\wp,T\wp)$. Let us prove that $\mathfrak{B}_{\mathrm{Im}}\subseteq\mathfrak{B}_\wp$ and

$$\tau+F(D(\mathrm{Im},T\,\mathrm{Im}))\leq\tau+F(H(T\wp,T\,\mathrm{Im}))\leq F(M(\wp,\mathrm{Im})+N(\wp,\mathrm{Im})),$$

where

$$\begin{aligned}N(\wp,\mathrm{Im}) &= \varphi(D(\wp,T\wp),D(\mathrm{Im},T\,\mathrm{Im}),D(\wp,T\,\mathrm{Im}),D(\mathrm{Im},T\wp))\\ &\leq \varphi(d(\wp,\mathrm{Im}),D(\mathrm{Im},T\,\mathrm{Im}),D(\wp,T\,\mathrm{Im}),d(\mathrm{Im},\mathrm{Im})).\end{aligned}$$

Since $\varphi(x_1,x_2,x_3,x_4)=0$ if $x_1x_2x_3x_4=0$, we get $N(\wp,\mathrm{Im})=0$ and

$$\begin{aligned}M(\wp,\mathrm{Im}) &= \psi\Big(v(\wp,\mathrm{Im}),D(\wp,T\wp),D(\mathrm{Im},T\,\mathrm{Im}),\\ &\qquad \frac{D(\wp,T\wp)D(\mathrm{Im},T\,\mathrm{Im})+D(\wp,T\,\mathrm{Im}),D(\mathrm{Im},T\wp)}{1+v(\wp,\mathrm{Im})},\\ &\qquad \frac{D(\wp,T\wp)D(\mathrm{Im},T\,\mathrm{Im})+D(\wp,T\,\mathrm{Im}),D(\mathrm{Im},T\wp)}{1+H(T\wp,T\,\mathrm{Im})}\Big)\\ &\leq \psi\Big(v(\wp,\mathrm{Im}),v(\wp,\mathrm{Im}),D(\mathrm{Im},T\,\mathrm{Im}),\frac{v(\wp,\mathrm{Im})D(\mathrm{Im},T\,\mathrm{Im})}{v(\wp,\mathrm{Im})},\frac{v(\wp,\mathrm{Im})D(\mathrm{Im},T\,\mathrm{Im})}{D(\mathrm{Im},T\,\mathrm{Im})}\Big)\\ &= \psi(D(\wp,T\wp),D(\wp,T\wp),D(\mathrm{Im},T\,\mathrm{Im}),D(\mathrm{Im},T\,\mathrm{Im}),D(\wp,T\wp)).\end{aligned}$$

Suppose that $D(\wp, T\wp) < D(\mathrm{Im}, T\,\mathrm{Im})$. Then

$$\begin{aligned}\tau + F(D(\mathrm{Im}, T\,\mathrm{Im})) &\leq \tau + F(H(T\wp, T\,\mathrm{Im}))\\ &\leq F(\psi(D(\mathrm{Im}, T\,\mathrm{Im}), D(\mathrm{Im}, T\,\mathrm{Im}), D(\mathrm{Im}, T\,\mathrm{Im}), D(\mathrm{Im}, T\,\mathrm{Im}), D(\mathrm{Im}, T\,\mathrm{Im})))\\ &\leq F(D(\mathrm{Im}, T\,\mathrm{Im})),\end{aligned}$$

so, this is a contradiction and we get $D(\mathrm{Im}, T\,\mathrm{Im}) < D(\wp, T\wp)$. From here, we obtain

$$\tau + F(D(\mathrm{Im}, T\,\mathrm{Im})) \leq \tau + F(H(T\wp, T\,\mathrm{Im})) \leq F(D(\wp, T\wp)).$$

Thus,

$$F(D(\mathrm{Im}, T\,\mathrm{Im})) \leq F(H(T\wp, T\,\mathrm{Im})) \leq F(D(\wp, T\wp)) - \tau < F(D(\wp, T\wp)).$$

Using that F satisfies the property (F1), we get

$$D(\mathrm{Im}, T\,\mathrm{Im}) \leq H(T\wp, T\,\mathrm{Im}) \leq D(\wp, T\wp).$$

Now for any $y \in \mathfrak{B}_{\mathrm{Im}}$ we obtain

$$v(y, \mathrm{Im}) \leq D(\mathrm{Im}, T\,\mathrm{Im}) \leq D(\wp, T\wp)$$

and then

$$v(y, \wp) \leq \max\{v(y, \mathrm{Im}), v(\mathrm{Im}, \wp)\} \leq D(\wp, T\wp),$$

which implies that $y \in \mathfrak{B}_{\wp}$, so $\mathfrak{B}_{\mathrm{Im}} \subseteq \mathfrak{B}_{\wp}$. On the other hand

$$v(\wp, \mathrm{Im}) = D(\wp, T\wp) > D(\mathrm{Im}, T\,\mathrm{Im}),$$

therefore, $\wp \notin \mathfrak{B}_{\mathrm{Im}}$, so $\mathfrak{B}_{\mathrm{Im}} \subsetneq \mathfrak{B}_{\wp}$. This contradicts the fact that $\mathfrak{B}_{\wp}$ is the maximal element. So, T has a fixed point, that is $\wp \in T\wp$. This completes the proof. □

Corollary 1. *Let (M, v) be a spherically complete ultrametric space and $T : M \to 2^M_C$ be a mapping such that for all $x, y \in M$ with $x \neq y$ we have*

$$H(Tx, Ty) > 0 \implies H(Tx, Ty) < M(x, y) + N(x, y),$$

where $\psi \in \Psi$ and $\varphi \in \Phi$ with

$$N(x, y) = \varphi\big(D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\big)$$

and

$$\begin{aligned}M(x, y) = \psi\Big(v(x, y), D(x, Tx), D(y, Ty), &\frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + v(x, y)},\\ &\frac{D(x, Tx)D(y, Ty) + D(x, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big).\end{aligned}$$

Then T has a fixed point.

Corollary 2. *Let (M, v) be a spherically complete ultrametric space and $T : M \to 2^M_C$ be a mapping such that $H(Tx, Ty) < v(x, y)$ for all $x, y \in M$ with $x \neq y$. Then T has a fixed point.*

Corollary 3. *Let* (M, v) *be a spherically complete ultrametric space and* $T : M \to M$ *be a mapping such that* $v(Tx, Ty) < v(x, y)$ *for all* $x, y \in M$ *with* $x \neq y$*. Then* T *has a unique fixed point.*

Example 4. *Take* $M = \{\omega, 1, 2, 3, \ldots\}$*. Define* $v : M \times M \to [0, +\infty)$ *by*

$$\begin{cases} v(x,y) = 0 \;\; \text{if and only if} \;\; x = y, \\ v(\omega, y) = \dfrac{1}{y}, \\ v(x,y) = \max\left\{\dfrac{1}{y}, \dfrac{1}{x}\right\}. \end{cases}$$

Then, (M, v) *is a spherically complete ultrametric space. Define* $T : M \to 2^M_C$ *by*

$$\begin{cases} T(\omega) = \{\omega\}, \\ T(x) = \{2x\}. \end{cases}$$

Let $\psi \in \Psi$ *with* $\psi(x_1, x_2, x_3, x_4, x_5) = x_1$ *and* $\varphi \in \Phi$ *with* $\varphi(x_1, x_2, x_3, x_4) = 0$*. Then all conditions of Theorem 4 are satisfied for* $F(\alpha) = \alpha + \ln \alpha$ *and* $\tau = 1/2$*.*

To see this, we consider the following cases. First, observe that $x, y \in \mathbb{N}$ *and*

$$H(Tx, Ty) > 0 \iff (y \geq 1 \text{ and } x = \omega) \text{ or } (x > y \geq 1).$$

Case 1. For $y \geq 1$ *and* $x = \omega$ *, we have*

$$\frac{H(Tx,Ty)}{M(x,y)} e^{H(Tx,Ty) - M(x,y)} = \frac{\frac{1}{2y}}{\frac{1}{y}} e^{\frac{1}{2y} - \frac{1}{y}} < \frac{1}{2} e^{-\frac{y}{2}} < e^{-\frac{1}{2}}. \tag{1}$$

Case 2. For $x > y \geq 1$*, we obtain the same as in* (1)*.*

Also, T *has a fixed point as* $\omega \in T\omega$*.*

2 Applications

In this section, we discuss our results on finding a common solution to functional equations that are used in dynamic programming. The study of dynamic programming divide into two parts that are state space and decision space. In this theory firstly, the problem of dynamic programming is transformed into functional equations:

$$g(x) = \max_{y \in U} \{N'(x,y) + J(x, y, g(\eta(x,y)))\} \;\; \text{for} \;\; x \in A, \tag{2}$$

$$g'(x) = \max_{y \in U} \{N'(x,y) + K(x, y, g(\eta(x,y)))\} \;\; \text{for} \;\; x \in A, \tag{3}$$

where W and V are Banach spaces such that $A \subseteq W$ and $U \subseteq V$ and,

$$\eta : A \times U \to A, \quad N' : A \times U \to \mathbb{R}, \quad J, K : A \times U \times \mathbb{R} \to \mathbb{R}$$

are bounded. Assume A and U are state space and decision space, respectively. Assume $B(A)$ denotes a set of all bounded real-valued maps on A and $CB(A)$ denotes a subset of all closed and bounded classes on A. Let $h \in CB(A)$ and

$$\|h\| = \max_{x \in A} |h(x)|.$$

Then, $(CB(A), \|\cdot\|)$ is a Banach space and v is the Hausdorff metric defined as

$$H(h,k) = \max_{x\in A} |h(x) - k(x)|. \tag{4}$$

By Proposition 1, $H(h,k)$ is an ultrapseudo metric.

Suppose the following conditions hold:

(i) $N^{'}$, J and K are bounded;

(ii) for $x \in A$ and $h \in CB(A)$, define $P, Q : CB(A) \to CB(A)$ by

$$Ph(x) = \max_{y\in U}\{N'(x,y) + J(x,y,h(\eta(x,y)))\} \text{ for } x \in A,$$
$$Qh(x) = \max_{y\in U}\{N'(x,y) + K(x,y,h(\eta(x,y)))\} \text{ for } x \in A;$$

note, that P and Q are well-defined, because of item (i);

(iii) for all $h,k \in CB(A)$, $(x,y) \in A \times U$, $t \in A$ and $T : CB(A) \to CB(A)$ we have

$$|J(x,y,h(t)) - K(x,y,k(t))| < M(h,k) + N(h,k),$$

where

$$N(h,k) = \varphi\big(D(h,Th), D(k,Tk), D(h,Tk), D(k,Th)\big)$$

and

$$M(h,k) = \psi\Big(v(h,k), D(h,Th), D(k,Tk), \frac{D(h,Th)D(k,Tk) + D(h,Tk)D(k,Th)}{1+v(h,k)}, \frac{D(h,Th)D(k,Tk) + D(h,Tk)D(k,Th)}{1+H(h,k)}\Big).$$

Lemma 1 ([12]). *Let $(CB(A), \|\cdot\|)$ be a Banach space and v be the Hausdorff metric defined by*

$$H(J,h) = \|J-h\| = \max_{a\in A} |J(a) - h(a)|, \quad J,h \in CB(A).$$

Then $(CB(A), \|\cdot\|)$ is a complete metric space. Since $H(J,h)$ is an ultrapseudo metric by Proposition 1, hence $(CB(A), H)$ is a spherically complete ultrametric space.

Theorem 5. *Let the conditions (i)–(iii) hold. Then functional equations* (2) *and* (3) *have a bounded solution.*

Proof. We know by Lemma 1, that $(CB(A), H)$ is a spherically complete ultrametric space, v is stated by (4), and (i) say that $P, Q : CB(A) \to CB(A)$. Choose any positive number λ and $h_1, h_2 \in CB(A)$. Take $x \in A$ and $y_1, y_2 \in U$, such that

$$Ph_1 < N'(x,y_2) + J(x,y_c,h_c(\eta(x,y_2))) + \lambda, \tag{5}$$
$$Qh_2 < N^{'}(x,y_1) + K(x,y_c,h_c(\eta(x,y_1))) + \lambda, \tag{6}$$
$$Ph_1 \geq N^{'}(x,y_2) + J(x,y_2,h_1(\eta(x,y_2))), \tag{7}$$
$$Qh_2 \geq N^{'}(x,y_1) + K(x,y_1,h_2(\eta(x,y_1))). \tag{8}$$

Then using (5) and (8), we obtain

$$\begin{aligned} Ph_1(x) - Qh_2(x) &< J(x, y_1, h_1(\eta(x, y_2))) - K(x, y_1, h_2(\eta(x, y_1))) + \lambda \\ &\leq |J(x, y_1, h_1(\eta(x, y_2))) - K(x, y_1, h_2(\eta(x, y_1)))| + \lambda \\ &< M(h_1(x), h_2(x)) + N(h_1(x), h_2(x)) + \lambda. \end{aligned} \tag{9}$$

Similarly, by (6) and (7), we get

$$Qh_2(x) - Ph_1(x) < M(h_1(x), h_2(x)) + N(h_1(x), h_2(x)) + \lambda. \tag{10}$$

Combining inequalities (9) and (10), we obtain

$$|Ph_1(x) - Qh_2(x)| < M(h_1(x), h_2(x)) + N(h_1(x), h_2(x)) + \lambda$$

for all $\lambda > 0$. Hence $H(Ph_1, Qh_2) < M(h_1, h_2) + N(h_1, h_2)$ for each $x \in A$.

As every condition of Corallary 1 is fulfilled, therefore T has a bounded solution of the equations (2) and (3). This completes the proof. □

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Озкапу А.С., Акар О. *Застосування до динамічного програмування теореми про нерухому точку із використанням раціонального типу* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 19–28.

У цьому дослідженні ми вводимо та досліджуємо концепцію узагальнених багатозначних раціонального типу F-стиснень на сферично повних ультраметричних просторах. Спираючись на існуючі рамки теорії нерухомої точки, ми встановлюємо нові теореми про нерухому точку для таких відображень та пропонуємо кілька наслідків, які розширюють та уніфікують відомі результати з літератури. Щоб підкреслити застосовність наших висновків, ми представляємо ілюстративний приклад, який демонструє обґрунтованість запропонованого підходу. Крім того, ми досліджуємо застосування до динамічного програмування шляхом формулювання функціональних рівнянь, розв'язки яких можна отримати за допомогою розроблених методів нерухомої точки. Результати не лише розширюють область застосування принципу стиснення у неархімедових умовах, але й підкреслюють корисність ультраметричних структур в оптимізації та обчислювальній математиці.

Ключові слова і фрази: динамічне програмування, нерухома точка, F-стиснення, ультраметричний простір.