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Topological isomorphism of a countably generated algebra of entire functions on ℓ_∞ and the algebra of symmetric entire functions on $L^2_\infty[0,1]$

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In the paper, we establish some general results on countably generated algebras of entire functions of bounded type on a complex Banach space. Specifically, we study the Fréchet subalgebra $H_{b\mathbf{P}}(X)$ of the algebra of all entire functions of bounded type $H_b(X)$, generated by a countable set $\mathbf{P}$ of continuous algebraically independent complex-valued homogeneous polynomials on a complex Banach space X such that some finite number of elements of the set $\mathbf{P}$ can share the same degree of homogeneity. We investigate the form of elements of this subalgebra. Furthermore, we show that every linear multiplicative functional, acting from $H_{b\mathbf{P}}(X)$ to $\mathbb{C}$, is completely determined by its values on the elements of $\mathbf{P}$. We also establish an upper estimate for the value of such a functional on an arbitrary n-homogeneous polynomial in $H_{b\mathbf{P}}(X)$. We apply these results to some specific algebras.

Let $L_\infty[0,1]$ be the complex Banach space of all complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$. Let $L^2_\infty[0,1]$ be the Cartesian square of $L_\infty[0,1]$. We consider the Fréchet algebra $H_{bs}(L^2_\infty[0,1])$ of all entire symmetric functions of bounded type on $L^2_\infty[0,1]$. In the paper, we construct a countably generated Fréchet subalgebra of the Fréchet algebra $H_b(\ell_\infty)$, which is topologically isomorphic to $H_{bs}(L^2_\infty[0,1])$, where ℓ_∞ is the complex Banach space of all bounded sequences of complex numbers. Namely, let $\mathcal{I} = (I_{11}, I_{12}, I_{21}, I_{22}, I_{23}, \ldots, I_{n1}, I_{n2}, \ldots, I_{n,n+1}, \ldots)$, where $I_{11}(x) = x_1$, $I_{12}(x) = x_2$, $I_{21}(x) = x_3^2$, $I_{22}(x) = x_4^2$, $I_{23}(x) = x_5^2$, $I_{31}(x) = x_6^3$, $I_{32}(x) = x_7^3$, $I_{33}(x) = x_8^3$, $I_{34}(x) = x_9^3, \ldots$ for $x = (x_1, x_2, \ldots) \in \ell_\infty$. We denote by $H_{b\mathcal{I}}(\ell_\infty)$ the Fréchet subalgebra of the algebra $H_b(\ell_\infty)$, generated by the sequence of polynomials $\mathcal{I}$. We construct a topological isomorphism between the algebras $H_{b\mathcal{I}}(\ell_\infty)$ and $H_{bs}(L^2_\infty[0,1])$.

Results of the paper can be used for investigations of the algebras of symmetric analytic functions on Banach spaces.

Key words and phrases: n-homogeneous polynomial, symmetric function, analytic function, spectrum of algebra.

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Introduction

The problem of the description of the spectrum of the Fréchet algebra $H_b(X)$ of entire functions of bounded type on a complex Banach space X is one of the most important problems in nonlinear functional analysis and remains unsolved in the general case (see [1, 27]). However, the description of the spectrum is simplified for certain subalgebras of the algebra $H_b(X)$. Im-

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portant examples of such subalgebras include the subalgebras of symmetric elements of the algebra $H_b(X)$ in the case when X has a symmetric structure. For many of such algebras explicit descriptions of spectra have been obtained (see [3–5, 24]). This is primarily due to the fact that these algebras are finitely or countably generated, that is, the corresponding dense subalgebras of symmetric continuous polynomials have finite or countable algebraic bases. In particular, the algebraic bases of the algebras of continuous polynomials on the spaces ℓ_p, where $p \in [1, +\infty)$, and $L_\infty[0,1]$ are the sequences of homogeneous polynomials with exactly one polynomial of each degree of homogeneity (see [4,6,14]). Such types of algebras have been investigated in the general case in [7, 8, 15, 18]. However, in the case of algebras of symmetric polynomials on Cartesian powers of certain Banach spaces (see [19–23, 25, 26]), or in the case of algebras of so-called block-symmetric polynomials (see [2,9–12,28]) the algebraic bases may contain multiple polynomials sharing the same degree of homogeneity. In this paper, we investigate the general case of such algebras. Section 1 presents the fundamental definitions and preliminary results from the theory of polynomials, the theory of symmetric analytic functions, and the theory of Fréchet algebras of analytic functions on Banach spaces. The main results of the paper are presented in Section 2. Specifically, in Subsection 2.1 we study the Fréchet subalgebra $H_{b\mathbf{P}}(X)$ of the Fréchet algebra $H_b(X)$, generated by a countable set

$$\mathbf{P} = \left(P_{11}, P_{12}, \ldots, P_{1m_1}, P_{21}, P_{22}, \ldots, P_{2m_2}, \ldots, P_{n1}, P_{n2}, \ldots, P_{nm_n}, \ldots\right)$$

of continuous algebraically independent complex-valued polynomials on a complex Banach space X, such that $P_{n1}, P_{n2}, \ldots, P_{nm_n}$, where $m_n \in \mathbb{N}$, are n-homogeneous polynomials for every $n \in \mathbb{N}$. We prove that each term of the Taylor series expansion of entire function that belongs to the subalgebra $H_{b\mathbf{P}}(X)$ can be uniquely represented as an algebraic combination of the elements of $\mathbf{P}$. Furthermore, we show that every linear multiplicative functional, acting from $H_{b\mathbf{P}}(X)$ to $\mathbb{C}$, is completely determined by its values on the elements of $\mathbf{P}$. We also establish an upper estimate for the value of such a functional on an arbitrary n-homogeneous polynomial in $H_{b\mathbf{P}}(X)$. In Subsection 2.2 we consider a particular case of the sequence of polynomials $\mathbf{P}$ on the space ℓ_∞. Namely, let

$$\mathcal{I} = \left(I_{11}, I_{12}, I_{21}, I_{22}, I_{23}, \ldots, I_{n1}, I_{n2}, \ldots, I_{n,n+1}, \ldots\right),$$

where

$$\begin{aligned} &I_{11}(x) = x_1, \quad I_{12}(x) = x_2, \\ &I_{21}(x) = x_3^2, \quad I_{22}(x) = x_4^2, \quad I_{23}(x) = x_5^2, \\ &I_{31}(x) = x_6^3, \quad I_{32}(x) = x_7^3, \quad I_{33}(x) = x_8^3, \quad I_{34}(x) = x_9^3, \ \ldots \end{aligned}$$

for $x = (x_1, x_2, \ldots) \in \ell_\infty$. We denote by $H_{b\mathcal{I}}(\ell_\infty)$ the Fréchet subalgebra of the algebra $H_b(\ell_\infty)$, generated by the sequence of polynomials $\mathcal{I}$. We investigate some properties of the algebra $H_{b\mathcal{I}}(\ell_\infty)$, in particular, we describe the spectrum of this algebra. In Subsection 2.3 we construct a topological isomorphism between the Fréchet algebra of all entire symmetric functions of bounded type on $L_\infty^2[0,1]$ and the Fréchet algebra $H_{b\mathcal{I}}(\ell_\infty)$.

1 Preliminaries

Let us denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{Z}_+$ the set of all nonnegative integers and by $\mathbb{Q}^+$ the set of all nonnegative rational numbers.

Polynomials on Banach spaces. Let X be a complex Banach space. Let a mapping $P : X \to \mathbb{C}$ be such that there exist $n \in \mathbb{N}$ and some n-linear form $A_P : X^n \to \mathbb{C}$ such that $P(x) = A_P(\underbrace{x, \ldots, x}_{n})$ for every $x \in X$. Then the mapping P is called an *n-homogeneous polynomial*.

A mapping $P : X \to \mathbb{C}$ is said to be a *polynomial of degree at most n*, where $n \in \mathbb{Z}_+$, if it can be represented as

$$P = P_0 + P_1 + \ldots + P_n,$$

where P_j is a j-homogeneous polynomial acting from X to $\mathbb{C}$ for every $j = 1, \ldots, n$ and P_0 is a constant mapping acting from X to $\mathbb{C}$.

For each polynomial $P : X \to \mathbb{C}$ we shall set

$$\|P\| = \sup\{|P(x)| : x \in X, \|x\| \le 1\}. \tag{1}$$

It is known that a polynomial $P : X \to \mathbb{C}$ is continuous if and only if $\|P\| < \infty$.

Polynomials $P_1, P_2, \ldots$ that act from X to $\mathbb{C}$ are called *algebraically independent polynomials* when the following condition is satisfied: for every $n \in \mathbb{N}$ and for every polynomial $q : \mathbb{C}^n \to \mathbb{C}$, if

$$q\big(P_1(x), P_2(x), \ldots, P_n(x)\big) = 0$$

for every $x \in X$, then $q \equiv 0$.

A polynomial $P : X \to \mathbb{C}$ is called an *algebraic combination* of elements of the set $\mathbb{P} = \{P_1, P_2, \ldots\}$ if there exist $n \in \mathbb{N}$ and a polynomial $q : \mathbb{C}^n \to \mathbb{C}$ such that

$$P(x) = q\big(P_1(x), \ldots, P_n(x)\big)$$

for every $x \in X$.

Let A be some subalgebra of the algebra of all continuous polynomials on X. A finite or countable subset $G \subset A$ is called an *algebraic basis* of the algebra A if the elements from the set G are algebraically independent and every element of the algebra A can be represented as an algebraic combination of the elements of G.

Spectrum of an algebra. Let $A(T)$ be a topological algebra of some functions on a nonempty set T over a field $\mathbb{C}$. A nontrivial continuous linear multiplicative functional $\varphi : A \to \mathbb{C}$ is called a character of the algebra A. The set of all characters of the algebra A is called the *spectrum* of the algebra A. For $x \in T$ let $\delta_x : A(T) \to \mathbb{C}$ be defined by $\delta_x(f) = f(x)$, where $f \in A(T)$. The mapping δ_x is called a *point-evaluation functional*. Note that δ_x is linear and multiplicative.

The algebra $H_b(X)$. Let X be a complex Banach space. Let $H_b(X)$ be the Fréchet algebra of all entire functions $f : X \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. Let

$$\|f\|_r = \sup_{\|x\| \le r} \big|f(x)\big|. \tag{2}$$

The topology of $H_b(X)$ can be generated by an arbitrary set of norms $\big\{\| \cdot \|_r : r \in \Gamma\big\}$, where Γ is any unbounded subset of $(0, +\infty)$.

The algebra $H(\mathbb{C})$. Note that every entire function on $\mathbb{C}$ is a function of bounded type. Therefore, $H_b(\mathbb{C}) = H(\mathbb{C})$. Thus, $H(\mathbb{C})$ is the Fréchet algebra of all entire functions $f : \mathbb{C} \to \mathbb{C}$ with the topology of uniform convergence on bounded sets.

The Cartesian square of $L_\infty[0,1]$. Let $L_\infty[0,1]$ be the complex Banach space of all Lebesgue measurable essentially bounded functions $y : [0,1] \to \mathbb{C}$ with norm

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\in[0,1]} \big|y(t)\big|.$$

Let $L_\infty^2[0,1]$ be the Cartesian square of $L_\infty[0,1]$ with norm

$$\|y\|_{\infty,2} = \max\big\{\|y_1\|_\infty, \|y_2\|_\infty\big\},$$

where $y = (y_1, y_2) \in L_\infty^2[0,1]$.

Symmetric functions on the space $L_\infty^2[0,1]$. A function $f : L_\infty^2[0,1] \to \mathbb{C}$ is called symmetric if

$$f(y \circ \sigma) = f(y)$$

for every $y = (y_1, y_2) \in L_\infty^2[0,1]$ and for every bijection $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure, i.e.

$$\mu\big(\sigma(A)\big) = \mu\big(\sigma^{-1}(A)\big) = \mu(A)$$

for every Lebesgue measurable set $A \subset [0,1]$, where μ is the Lebesgue measure and

$$y \circ \sigma = (y_1 \circ \sigma, y_2 \circ \sigma).$$

Let the mapping $R_{n-k,k} : L_\infty^2[0,1] \to \mathbb{C}$ be defined by the formula

$$R_{n-k,k}(y) = \int_{[0,1]} \big(y_1(t)\big)^{n-k} \big(y_2(t)\big)^k \, dt \tag{3}$$

for every $y = (y_1, y_2) \in L_\infty^2[0,1]$, where $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Note that $R_{n-k,k}$ is a continuous symmetric n-homogeneous polynomial and $\|R_{n-k,k}\| = 1$ for all $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$.

Let us consider the sequence of polynomials

$$\mathcal{R} = \big(R_{10}, R_{01}, R_{20}, R_{11}, R_{02}, \ldots, R_{n0}, R_{n-1,1}, \ldots, R_{0n}, \ldots\big)$$

on the space $L_\infty^2[0,1]$. By [21, Corollary 3] the polynomials from the sequence $\mathcal{R}$ form an algebraic basis for the algebra of all symmetric continuous polynomials on $L_\infty^2[0,1]$.

We shall use the following particular result of [24, Theorem 3].

Theorem 1. *There exists $K > 0$ such that for every sequence of complex numbers*

$$c = (c_{10}, c_{01}, c_{20}, c_{11}, c_{02}, \ldots, c_{n0}, c_{n-1,1}, \ldots, c_{0n}, \ldots)$$

such that

$$\sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} |c_{n-k,k}|^{1/n} < +\infty$$

there exists $y_c \in L_\infty^2[0,1]$ such that

$$R_{n-k,k}(y_c) = c_{n-k,k}$$

for all $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$ and

$$\|y_c\|_{\infty,2} \le K \sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} |c_{n-k,k}|^{1/n},$$

where polynomials $R_{n-k,k}$ are defined by (3).

The algebra of all entire symmetric functions of bounded type on the space $L^2_\infty[0,1]$ and its spectrum. Let $H_{bs}(L^2_\infty[0,1])$ be the subalgebra of the algebra $H_b(L^2_\infty[0,1])$ that consists of all entire symmetric functions of bounded type $f : L^2_\infty[0,1] \to \mathbb{C}$.

From [24, p. 17] it follows that every function $g \in H_{bs}(L^2_\infty[0,1])$ can be uniquely represented as

$$g(y) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n\left(l_{n0}+l_{n-1,1}+\ldots+l_{0n}\right)=n} \alpha_{l_{10}\ldots l_{0n}} \times \prod_{\substack{m=1,\ldots,n\\ k=0,\ldots,m}} \left(R_{m-k,k}(y)\right)^{l_{m-k,k}} \tag{4}$$

for every $y \in L^2_\infty[0,1]$, where $\alpha_{l_{10}\ldots l_{0n}} \in \mathbb{C}$ and $l_{10}, \ldots, l_{0n} \in \mathbb{Z}_+$. Here we assume that $0^0 = 1$. We denote by M_{bs} the spectrum of the algebra $H_{bs}(L^2_\infty[0,1])$. For every character $\psi \in M_{bs}$, by (4), we have that

$$\psi(g) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n\left(l_{n0}+l_{n-1,1}+\ldots+l_{0n}\right)=n} \alpha_{l_{10}\ldots l_{0n}} \times \prod_{\substack{m=1,\ldots,n\\ k=0,\ldots,m}} \Big(\psi\big(R_{m-k,k}\big)\Big)^{l_{m-k,k}},$$

since ψ is continuous, linear and multiplicative, and $\psi(1) = 1$. Thus, every character $\psi \in M_{bs}$ is completely defined by its values on the polynomials $R_{n-k,k}$, where $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Hence, we can uniquely identify every character $\psi \in M_{bs}$ with the sequence

$$\nu(\psi) = \Big(\psi(R_{10}), \psi(R_{01}), \psi(R_{20}), \psi(R_{11}), \psi(R_{02}), \ldots, \psi(R_{n0}), \psi(R_{n-1,1}), \ldots, \psi(R_{0n}), \ldots\Big). \tag{5}$$

Therefore, the following lemma is true.

Lemma 1. *From the equality $\nu(\varphi) = \nu(\psi)$ it follows that $\varphi = \psi$ for arbitrary $\varphi, \psi \in M_{bs}$.*

In [24, Theorem 4, p. 18] it is proved that for every character $\psi \in M_{bs}$ we have

$$\sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} \sqrt[n]{\left|\psi(R_{n-k,k})\right|} < +\infty. \tag{6}$$

Let us consider the following set of sequences of complex numbers

$$\Delta = \Big\{c = (c_{10}, c_{01}, \ldots, c_{n0}, c_{n-1,1}, \ldots, c_{0n}, \ldots) \in \mathbb{C}^\infty : \sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} |c_{n-k,k}|^{1/n} < +\infty\Big\}. \tag{7}$$

The inequality (6) implies the following result.

Corollary 1. *The sequence $\nu(\psi)$, defined by* (5)*, belongs to the set Δ, defined by* (7)*.*

Let us define the mapping $\nu : M_{bs} \to \Delta$ by

$$\nu : \psi \mapsto \nu(\psi), \tag{8}$$

where $\nu(\psi)$ is defined by (5).

By (8), where we set $\psi = \delta_{y_c}$, it follows that

$$\nu(\delta_{y_c}) = \left\{\delta_{y_c}(R_{n-k,k})\right\}_{n=1}^{\infty} = \left\{R_{n-k,k}(y_c)\right\}_{n=1}^{\infty},$$

where $k \in \{0, \ldots, n\}$ and the polynomials $R_{n-k,k}$ are defined by (3). Thus, Theorem 1 implies the following result.

Corollary 2. *There exists $K > 0$ such that for every $c = \{c_{n-k,k}\}_{n=1}^{\infty} \in \Delta$, where $k \in \{0, \ldots, n\}$, there exists $y_c \in L_\infty^2[0,1]$ such that*

$$\nu(\delta_{y_c}) = c$$

and

$$\|y_c\|_{\infty,2} \le K \sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} |c_{n-k,k}|^{1/n}.$$

Theorem 2. *The mapping ν, defined by* (8)*, is a bijection.*

Proof. From Lemma 1 it follows that the mapping ν is injective. From Corollary 2 it follows that the mapping ν is surjective. □

Thus, the mapping ν is a bijection between the spectrum M_{bs} of the algebra $H_{bs}(L_\infty^2[0,1])$ and the set of sequences of complex numbers Δ, defined by (7).

In [24, Theorem 5, p. 18] it is proved that every $\psi \in M_{bs}$ is a point-evaluation functional.

2 Main results

2.1 On the algebra $H_{b\mathbf{P}}(X)$

Let X be a complex Banach space. Let

$$\mathbf{P} = \left(P_{11}, P_{12}, \ldots, P_{1m_1}, P_{21}, P_{22}, \ldots, P_{2m_2}, \ldots, P_{n1}, P_{n2}, \ldots, P_{nm_n}, \ldots\right) \tag{9}$$

be a sequence of continuous algebraically independent complex-valued polynomials on the space X such that $P_{n1}, P_{n2}, \ldots, P_{nm_n}$, where $m_n \in \mathbb{N}$, are n-homogeneous polynomials for every $n \in \mathbb{N}$.

Let us denote by $P_{\mathbf{P}}(X)$ the algebra of all polynomials which are algebraic combinations of elements of the set $\mathbf{P}$. Let us denote by $H_{b\mathbf{P}}(X)$ the closure of the algebra $P_{\mathbf{P}}(X)$ in the metric of the algebra $H_b(X)$. It is easy to check that the algebra $H_{b\mathbf{P}}(X)$ is a Fréchet subalgebra of the Fréchet algebra $H_b(X)$.

Theorem 3. *Each term of the Taylor series of a function $f \in H_{b\mathbf{P}}(X)$ can be uniquely represented as an algebraic combination of elements of the set $\mathbf{P}$. Consequently,*

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{k_{11}+k_{12}+\ldots+k_{1m_1}+2(k_{21}+\ldots+k_{2m_2})+\ldots+n(k_{n1}+\ldots+k_{nm_n})=n} \alpha_{k_{11},\ldots,k_{nm_n}} \times \prod_{\substack{i=1,\ldots,n\\ j=1,\ldots,m_i}} \left(P_{ij}(x)\right)^{k_{ij}}, \tag{10}$$

where $x \in X$, $\alpha_{k_{11},\ldots,k_{nm_n}} \in \mathbb{C}$ and $k_{11}, \ldots, k_{nm_n} \in \mathbb{Z}_+$.

Proof. Let $f \in H_{b\mathbf{P}}(X)$ and $\alpha_0 = f(0)$. For any $n \in \mathbb{N}$ let the polynomial f_n be the nth term of the Taylor series of the function f. Let us prove that the polynomial f_n can be uniquely represented as an algebraic combination of the polynomials $P_{11}, \ldots, P_{nm_n}$.

Denote by $\mathcal{P}_{\mathbf{P}}({}^nX)$ the space of all n-homogeneous polynomials which are algebraic combinations of the polynomials $P_{11}, \ldots, P_{nm_n}$.

Note that the set of polynomials of the form

$$P_{11}^{k_{11}} \cdot \ldots \cdot P_{1m_1}^{k_{1m_1}} \cdot \ldots \cdot P_{n1}^{k_{n1}} \cdot \ldots \cdot P_{nm_n}^{k_{nm_n}},$$

where $k_{11}, \ldots, k_{nm_n}$ are nonnegative integers such that

$$k_{11} + k_{12} + \ldots + k_{1m_1} + 2(k_{21} + k_{22} + \ldots + k_{2m_2}) + \ldots + n(k_{n1} + \ldots + k_{nm_n}) = n,$$

is a Hamel basis of the space $\mathcal{P}_{\mathbf{P}}({}^nX)$. Since there is a finite number of such polynomials, the space $\mathcal{P}_{\mathbf{P}}({}^nX)$ is finite dimensional. Therefore the space $\mathcal{P}_{\mathbf{P}}({}^nX)$ is complete with respect to each norm. In particular, the space $\mathcal{P}_{\mathbf{P}}({}^nX)$ is complete with respect to the norm $\|\cdot\|_1$.

Since the algebra $H_{b\mathbf{P}}(X)$ is the closure of the algebra $\mathcal{P}_{\mathbf{P}}(X)$, there exists a sequence $\{a_l\}_{l=1}^{\infty} \subset \mathcal{P}_{\mathbf{P}}(X)$ that converges to the function f with respect to the metric of the algebra $H_b(X)$. Let a_{ln} be the nth term of the Taylor series of the polynomial a_l. Note that $a_{ln} \in \mathcal{P}_{\mathbf{P}}({}^nX)$ for every $l \in \mathbb{N}$.

Let us prove that the sequence $\{a_{ln}\}_{l=1}^{\infty}$ converges to the polynomial f_n with respect to the norm $\|\cdot\|_1$.

According to the Cauchy integral formula with $r = 1$, we have

$$f_n(x) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta x)}{\zeta^{n+1}}\, d\zeta$$

and

$$a_{ln}(x) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{a_l(\zeta x)}{\zeta^{n+1}}\, d\zeta.$$

Therefore

$$|f_n(x) - a_{ln}(x)| = \left| \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta x) - a_l(\zeta x)}{\zeta^{n+1}}\, d\zeta \right| \leq \frac{1}{2\pi} \int_{|\zeta|=1} |f(\zeta x) - a_l(\zeta x)|\, d\zeta. \tag{11}$$

If we take $x \in X$ such that $\|x\| \leq 1$ and $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, we have $\|\zeta x\| \leq 1$. Therefore,

$$|f(\zeta x) - a_l(\zeta x)| \leq \|f - a_l\|_1. \tag{12}$$

By (11) and (12), it follows that

$$\|f_n - a_{ln}\|_1 = \sup_{\|x\| \leq 1} |f_n(x) - a_{ln}(x)| \leq \frac{1}{2\pi} \|f - a_l\|_1 \int_{|\zeta|=1} d\zeta = \|f - a_l\|_1.$$

Since $a_l \to f$, it follows that $\|f - a_l\|_1 \to 0$ as $l \to \infty$. Therefore $\|f_n - a_{ln}\|_1 \to 0$ as $l \to \infty$. Thus $a_{ln} \to f_n$ as $l \to \infty$ with respect to the norm $\|\cdot\|_1$.

Since the space $\mathcal{P}_{\mathbf{P}}({}^nX)$ is complete with respect to the norm $\|\cdot\|_1$, we have $f_n \in \mathcal{P}_{\mathbf{P}}({}^nX)$.

Therefore

$$f_n(x) = \sum_{k_{11}+k_{12}+\ldots+k_{1m_1}+2(k_{21}+\ldots+k_{2m_2})+\ldots+n(k_{n1}+\ldots+k_{nm_n})=n} \alpha_{k_{11},\ldots,k_{nm_n}} \prod_{\substack{i=1,\ldots,n\\ j=1,\ldots,m_i}} \big(P_{ij}(x)\big)^{k_{ij}},$$

where $x \in X$, $\alpha_{k_{11},\ldots,k_{nm_n}} \in \mathbb{C}$ and $k_{11}, \ldots, k_{nm_n} \in \mathbb{Z}_+$.

This representation is unique, since the polynomials $P_{11}, \ldots, P_{nm_n}$ are algebraically independent. □

Let us denote by $M_{b\mathbf{P}}$ the spectrum of the algebra $H_{b\mathbf{P}}(X)$. According to Theorem 3 every function $f \in H_{b\mathbf{P}}(X)$ can be uniquely represented in the form (10). Therefore for every character $\varphi \in M_{b\mathbf{P}}$ we have

$$\varphi(f) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{k_{11}+k_{12}+\ldots+k_{1m_1}+2(k_{21}+\ldots+k_{2m_2})+\ldots+n(k_{n1}+\ldots+k_{nm_n})=n} \alpha_{k_{11},\ldots,k_{nm_n}} \prod_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_i}} \big(\varphi(P_{ij})\big)^{k_{ij}},$$

since φ is continuous, linear and multiplicative, and $\varphi(1) = 1$. Thus we see that every character $\varphi \in M_{b\mathbf{P}}$ is completely defined by its values on the polynomials P_{ij}, where $i \in \mathbb{N}$ and $j \in \{1, \ldots, m_i\}$ for every $i \in \mathbb{N}$. Hence, every character $\varphi \in M_{b\mathbf{P}}$ can be uniquely identified with the sequence

$$\big(\varphi(P_{11}), \varphi(P_{12}), \ldots, \varphi(P_{1m_1}), \varphi(P_{21}), \varphi(P_{22}), \ldots, \varphi(P_{2m_2}), \ldots, \varphi(P_{n1}), \varphi(P_{n2}), \ldots, \varphi(P_{nm_n}), \ldots\big).$$

Let $H_b^{(0)}(X)$ be an arbitrary subalgebra of the algebra $H_b(X)$ such that for every function that belongs to the subalgebra, all the terms of its Taylor series also belong to this subalgebra.

For every linear continuous functional $\varphi \in \big(H_b^{(0)}(X)\big)^*$ there exists $r \in \mathbb{Q}^+$ such that the functional φ is continuous with respect to the norm $\|\cdot\|_r$, where $\big(H_b^{(0)}(X)\big)^*$ is the space of all continuous linear functionals on the algebra $H_b^{(0)}(X)$.

Let us define the radius function on the space $\big(H_b^{(0)}(X)\big)^*$ similar to [1, p. 53]. We define the *radius function* R on $\big(H_b^{(0)}(X)\big)^*$ by declaring $R(\varphi)$ to be the infimum of all $r \in \mathbb{Q}^+$ such that φ is continuous with respect to the norm $\|\cdot\|_r$. Thus

$$0 \le R(\varphi) < \infty.$$

Theorem 4. *For every character $\varphi \in M_{b\mathbf{P}}$ there exists $r \in \mathbb{Q}^+$ such that the estimate*

$$|\varphi(Q)| \le r^n \|Q\|_1$$

holds for every n-homogeneous polynomial Q that belongs to the algebra $H_{b\mathbf{P}}(X)$.

Proof. Let $\varphi \in M_{b\mathbf{P}}$ and $R(\varphi)$ be the radius function of the character φ. Since φ is continuous, it follows that $0 \le R(\varphi) < \infty$.

Let $r > R(\varphi)$. Then, since the norm of each nonzero continuous complex-valued homomorphism is equal to 1, the estimate

$$|\varphi(Q)| \le \|Q\|_r$$

holds for every n-homogeneous polynomial Q that belongs to the algebra $H_{b\mathbf{P}}(X)$.

Thus

$$|\varphi(Q)| \le \sup_{\|x\| \le r} |Q(x)|. \tag{13}$$

Substituting $x = ry$ into the right-hand side of (13) and taking into account that Q is an n-homogeneous polynomial, we obtain

$$\sup_{\|x\|\le r} |Q(x)| = \sup_{\|ry\|\le r} |Q(ry)| = \sup_{r\|y\|\le r} r^n |Q(y)| = r^n \sup_{\|y\|\le 1} |Q(y)| = r^n \|Q\|_1. \tag{14}$$

By (13) and (14), we obtain

$$|\varphi(Q)| \le r^n \|Q\|_1.$$

□

2.2 The algebra $H_{b\mathcal{I}}(\ell_\infty)$ and its spectrum

In this subsection, we consider a particular case of the sequence of polynomials **P**, defined by (9), on the space $X = \ell_\infty$. Let us first establish some auxiliary results.

Let $A = \{(n,k) \in \mathbb{N} \times \mathbb{Z}_+ : n \geq k\}$. Let us define the mapping $\tau : A \to \mathbb{N}$ by the formula

$$\tau(n,k) = \frac{(n+2)(n-1)}{2} + k + 1. \tag{15}$$

It can be checked that the mapping τ, defined by (15), is a bijection. Consequently, for every $x = (x_1, x_2, \ldots) \in \ell_\infty$ we have

$$\sup_{j\in\mathbb{N}} |x_j| = \sup_{\substack{n\in\mathbb{N} \\ k=0,\ldots,n}} |x_{\tau(n,k)}|,$$

that is,

$$\|x\| = \sup_{\substack{n\in\mathbb{N} \\ k=0,\ldots,n}} |x_{\tau(n,k)}|. \tag{16}$$

Let $I_{n,k+1} : \ell_\infty \to \mathbb{C}$ be defined by

$$I_{n,k+1}(x) = x^n_{\tau(n,k)} \tag{17}$$

for $x = (x_1, x_2, \ldots) \in \ell_\infty$, where $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$ and $\tau(n,k)$ is defined by (15). For example,

$$I_{11}(x) = x_1, \quad I_{12}(x) = x_2, \quad I_{21}(x) = x_3^2, \quad I_{22}(x) = x_4^2, \quad I_{23}(x) = x_5^2$$

for $x = (x_1, x_2, \ldots) \in \ell_\infty$. It is clear that $I_{n,k+1}$ is an n-homogeneous polynomial. For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$, by (1) and (17), we get

$$\|I_{n,k+1}\| = \sup_{\|x\|\leq 1} |I_{n,k+1}(x)| = \sup_{\|x\|\leq 1} |x^n_{\tau(n,k)}| = 1, \tag{18}$$

and so $I_{n,k+1}$ is continuous.

Let

$$\mathcal{I} = \big(I_{11}, I_{12}, I_{21}, I_{22}, I_{23}, \ldots, I_{n1}, I_{n2}, \ldots, I_{n,n+1}, \ldots\big). \tag{19}$$

Since the mapping τ is injective, it follows that the elements of the sequence (19) are algebraically independent.

Then, according to Theorem 3, every function $f \in H_{b\mathcal{I}}(\ell_\infty)$ can be uniquely represented in the form

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \times \prod_{\substack{m=1,\ldots,n \\ k=0,\ldots,m}} \big(I_{m,k+1}(x)\big)^{l_{m,k+1}}, \tag{20}$$

where $x \in \ell_\infty$, $\alpha_{l_{11},\ldots,l_{n,n+1}} \in \mathbb{C}$ and $l_{11}, \ldots, l_{n,n+1} \in \mathbb{Z}_+$.

Let

$$f_0 = \alpha_0, \tag{21}$$

where $\alpha_0 \in \mathbb{C}$, and

$$f_n = \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=1,\ldots,n\\ k=0,\ldots,m}} \left(I_{m,k+1}\right)^{l_{m,k+1}} \tag{22}$$

for every $n \in \mathbb{N}$.

Then f can also be uniquely represented in the form

$$f = \sum_{n=0}^{\infty} f_n.$$

Let us denote by $M_{b\mathcal{I}}$ the spectrum of the algebra $H_{b\mathcal{I}}(\ell_\infty)$. Then, from the results of the previous section, we have that every character $\varphi \in M_{b\mathcal{I}}$ can be uniquely identified with the sequence

$$\eta(\varphi) = \big(\varphi(I_{11}), \varphi(I_{12}), \varphi(I_{21}), \varphi(I_{22}), \varphi(I_{23}), \ldots, \varphi(I_{n1}), \varphi(I_{n2}), \ldots, \varphi(I_{n,n+1}), \ldots\big). \tag{23}$$

Lemma 2. *From the equality $\eta(\varphi) = \eta(\psi)$ it follows that $\varphi = \psi$ for arbitrary $\varphi, \psi \in M_{b\mathcal{I}}$.*

Proposition 1. *For every character $\varphi \in M_{b\mathcal{I}}$ there exists $r \in \mathbb{Q}^+$ such that the estimate*

$$|\varphi(I_{n,k+1})| \leq r^n$$

holds for every polynomial $I_{n,k+1}$, where $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$.

Proof. Theorem 4 and the equality (18) imply the statement of the proposition. □

Corollary 3. *The sequence $\eta(\varphi)$, defined by* (23)*, belongs to the set Δ, defined by* (7)*.*

Let us define the mapping $\eta : M_{b\mathcal{I}} \to \Delta$ by

$$\eta : \varphi \mapsto \eta(\varphi), \tag{24}$$

where $\eta(\varphi)$ is defined by (23).

Theorem 5. *For every $c \in \Delta$ there exists $x \in \ell_\infty$ such that $\eta(\delta_x) = c$.*

Proof. Let $c = (c_{10}, c_{01}, c_{20}, c_{11}, c_{02}, \ldots, c_{n0}, c_{n-1,1}, \ldots, c_{0n}, \ldots) \in \Delta$, where Δ is defined by (7). Let

$$x = (c_{10}, c_{01}, \sqrt{c_{20}}, \sqrt{c_{11}}, \sqrt{c_{02}}, \ldots, \sqrt[n]{c_{n0}}, \sqrt[n]{c_{n-1,1}}, \ldots, \sqrt[n]{c_{0n}}, \ldots),$$

where $\sqrt[n]{z} = \sqrt[n]{|z|} \exp\left(\frac{i \arg z}{n}\right)$ for $z \in \mathbb{C}$.

Since $c \in \Delta$, it follows that

$$\|x\| = \sup_{\substack{n\in\mathbb{N}\\ k=0,\ldots,n}} |c_{n-k,k}|^{1/n} < +\infty.$$

Therefore $x \in \ell_\infty$.

For all $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$, by (17), the following equality

$$\delta_x(I_{n,k+1}) = I_{n,k+1}(x) = x^n_{\tau(n,k)} = \left(\sqrt[n]{c_{n-k,k}}\right)^n = c_{n-k,k} \tag{25}$$

holds. Then from (23) and (25) it follows that $\eta(\delta_x) = c$. □

Theorem 6. *The mapping η, defined by* (24)*, is a bijection.*

Proof. From Lemma 2 it follows that the mapping η is injective. From Theorem 5 it follows that the mapping η is surjective.

Thus, η is a bijection between the spectrum $M_{b\mathcal{I}}$ of the algebra $H_{b\mathcal{I}}(\ell_\infty)$ and the set of all sequences of complex numbers Δ, defined by (7). □

Theorem 7. *The spectrum $M_{b\mathcal{I}}$ of the algebra $H_{b\mathcal{I}}(\ell_\infty)$ coincides with the set of all point-evaluation functionals at points of the space ℓ_∞.*

Proof. Let $\varphi \in M_{b\mathcal{I}}$. Let us define $c \in \Delta$ by

$$c = \eta(\varphi), \tag{26}$$

where Δ is defined by (7). By Theorem 5, there exists $x \in \ell_\infty$ such that

$$\eta(\delta_x) = c. \tag{27}$$

From (26) and (27) it follows that $\eta(\varphi) = \eta(\delta_x)$. Then from Lemma 2 it follows that $\varphi = \delta_x$. □

2.3 On topological isomorphism of the algebras $H_{b\mathcal{I}}(\ell_\infty)$ and $H_{bs}(L^2_\infty[0,1])$

Lemma 3. *For every $y \in L^2_\infty[0,1]$ there exists $x \in \ell_\infty$ such that*

$$\delta_y(R_{n-k,k}) = \delta_x(I_{n,k+1}) \tag{28}$$

and

$$\|x\| \le \|y\|_{\infty,2}, \tag{29}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) *and* (17) *respectively for every $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$.*

Proof. Let $y \in L^2_\infty[0,1]$. Let us consider the sequence of complex numbers

$$\zeta_y = \big(R_{10}(y), R_{01}(y), R_{20}(y), R_{11}(y), R_{02}(y), \ldots, R_{n0}(y), \ldots, R_{n-k,k}(y), \ldots, R_{0n}(y), \ldots\big). \tag{30}$$

By (6), where we set $\psi = \delta_y$, we have that $\zeta_y \in \Delta$, where Δ is defined by (7). Then by Theorem 5, where we set $c = \zeta_y$, there exists $x = (x_1, x_2, \ldots) \in \ell_\infty$ such that

$$\eta(\delta_x) = \zeta_y, \tag{31}$$

where the mapping η is defined by (24).

By (23), where we set $\varphi = \delta_x$, we have

$$\eta(\delta_x) = \big\{\delta_x(I_{n,k+1})\big\}_{n=1}^{\infty}, \tag{32}$$

where $k = 0, \ldots, n$.

Then, by (31), (32) and (30), it follows that

$$\big\{\delta_x(I_{n,k+1})\big\}_{n=1}^{\infty} = \{R_{n-k,k}(y)\}_{n=1}^{\infty}, \tag{33}$$

where $k = 0, \ldots, n$.

Since for every $n \in \mathbb{N}$ and $k = 0, \ldots, n$ we have $\delta_y(R_{n-k,k}) = R_{n-k,k}(y)$, from (33) we get the desired equality (28).

Next, let us prove the inequality (29). By (17), we have

$$\left|x^n_{\tau(n,k)}\right| = \left|I_{n,k+1}(x)\right| = \left|\delta_x(I_{n,k+1})\right|. \tag{34}$$

By (28), we get

$$\left|\delta_x(I_{n,k+1})\right| = \left|\delta_y(R_{n-k,k})\right| = \left|R_{n-k,k}(y)\right|. \tag{35}$$

Since $R_{n-k,k}$ is an n-homogeneous polynomial and $\|R_{n-k,k}\| = 1$, we have

$$\left|R_{n-k,k}(y)\right| \le \|R_{n-k,k}\| \|y\|^n_{\infty,2} = \|y\|^n_{\infty,2}. \tag{36}$$

Therefore, from (34), (35) and (36) it follows that

$$\left|x^n_{\tau(n,k)}\right| \le \|y\|^n_{\infty,2}.$$

Then

$$\left|x_{\tau(n,k)}\right| \le \|y\|_{\infty,2}.$$

Therefore, by (16), we have the desired inequality (29). □

Lemma 4. *For every $x \in \ell_\infty$ there exists $y \in L^2_\infty[0,1]$ such that*

$$\delta_x(I_{n,k+1}) = \delta_y(R_{n-k,k}) \tag{37}$$

for every $n \in \mathbb{N}$ and $k \in \overline{0,n}$ and

$$\|y\|_{\infty,2} \le K\|x\|, \tag{38}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) and (17) respectively, and K is given by Corollary 2.

Proof. Let $x \in \ell_\infty$. Consider the sequence $\eta(\delta_x)$, defined by (23), where we set $\varphi = \delta_x$. By Corollary 3, the sequence $\eta(\delta_x)$ belongs to the set Δ, defined by (7). By Corollary 2, where we set $c = \eta(\delta_x)$, there exists $y \in L^2_\infty[0,1]$ such that $\nu(\delta_y) = \eta(\delta_x)$. Then, by (8), where we set $\psi = \delta_y$, and by (24), where we set $\varphi = \delta_x$, we have the desired equality (37).

Let us prove the inequality (38). By Corollary 2 and by the equalities (17), (16), we have that there exists $K > 0$ such that

$$\|y\|_{\infty,2} \le K \sup_{\substack{n\in\mathbb{N}\\ k=\overline{0,n}}} \sqrt[n]{\left|\delta_x(I_{n,k+1})\right|} = K \sup_{\substack{n\in\mathbb{N}\\ k=\overline{0,n}}} \sqrt[n]{\left|I_{n,k+1}(x)\right|} = K \sup_{\substack{n\in\mathbb{N}\\ k=\overline{0,n}}} |x_{n,k}| = K\|x\|.$$

□

Let us define the mapping $\Theta : H_{bs}(L^2_\infty[0,1]) \to H_{b\mathcal{I}}(\ell_\infty)$ by

$$\Theta(g) = \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n\left(l_{n0}+l_{n-1,1}+\ldots+l_{0n}\right)=n} \alpha_{l_{10}\ldots l_{0n}} \times \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}\right)^{l_{m-k,k}}, \tag{39}$$

where $g \in H_{bs}(L^2_\infty[0,1])$ is of the form (4) and the polynomials $I_{m,k+1}$ are defined by (17).

Theorem 8. *The mapping Θ, defined by* (39)*, is a topological isomorphism between the algebras $H_{bs}(L^2_\infty[0,1])$ and $H_{b\mathcal{I}}(\ell_\infty)$.*

Proof. Let us show that the mapping Θ is well defined. Let $g \in H_{bs}(L^2_\infty[0,1])$. By (4), g has a Taylor series representation

$$g = \sum_{n=0}^{\infty} g_n, \tag{40}$$

where

$$g_0 = \alpha_0, \qquad g_n = \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(R_{m-k,k}\right)^{l_{m-k,k}} \tag{41}$$

for $n \in \mathbb{N}$, where $\alpha_0 \in \mathbb{C}, \alpha_{l_{10},\ldots,l_{0n}} \in \mathbb{C}$ and $l_{10},\ldots,l_{0n} \in \mathbb{Z}_+$. Let $x \in \ell_\infty$. By Lemma 4, there exists $y \in L^2_\infty[0,1]$ such that the equality (37) and the inequality (38) hold. Then by (39), (37) and by the linearity, continuity and multiplicativity of the character $\delta_y \in M_{bs}$, we have that

$$\begin{aligned}
\Theta(g)(x) &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}(x)\right)^{l_{m,k+1}} \\
&= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \Big(\delta_x(I_{m,k+1})\Big)^{l_{m,k+1}} \\
&= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \Big(\delta_y(R_{m-k,k})\Big)^{l_{m,k+1}} \\
&= \delta_y\left(\alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{11}+l_{12}+2(l_{21}+l_{22}+l_{23})+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(R_{m-k,k}\right)^{l_{m,k+1}}\right) \\
&= \delta_y(g).
\end{aligned} \tag{42}$$

Therefore, since $g \in H_{bs}(L^2_\infty[0,1])$ and $\delta_y \in M_{bs}$, it follows that $\delta_y(g)$ is well defined and therefore the function $\Theta(g)$ is also well defined for every $x \in \ell_\infty$.

Next, let us prove that $\Theta(g) \in H_{b\mathcal{I}}(\ell_\infty)$ for every $g \in H_{bs}(L^2_\infty[0,1])$. Let $g \in H_{bs}(L^2_\infty[0,1])$. Then g is of the form (40). According to [13, Section 8, p. 61, Proposition 8.6 and Theorem 8.7] and taking into account that every function of bounded type is locally bounded, in order to prove that $\Theta(g) \in H_{b\mathcal{I}}(\ell_\infty)$ it is sufficient to show that $\Theta(g)$ is a G-holomorphic function of bounded type.

Firstly, let us show that the function $\Theta(g)$ is G-holomorphic. According to the definition of a G-holomorphic function (see [13, Section 8, p. 58, Definition 8.1]) we need to prove that for all $b, c \in \ell_\infty$ the function $h : \mathbb{C} \to \mathbb{C}$, defined by

$$h(\mu) = \Theta(g)(b + \mu c), \tag{43}$$

where $\mu \in \mathbb{C}$, is entire on the set $\mathbb{C}$, that is, $h \in H(\mathbb{C})$. Let $b = (b_1, b_2, \ldots) \in \ell_\infty$ and $c = (c_1, c_2, \ldots) \in \ell_\infty$. Let h be defined by (43). For any $n \in \mathbb{N}$, let

$$h_n(\mu) = \sum_{l_{11}+l_{12}+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}(b+\mu c)\right)^{l_{m,k+1}}, \tag{44}$$

where $\mu \in \mathbb{C}$. By (44) and (17), we get

$$h_n(\mu) = \sum_{l_{11}+l_{12}+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(b_{\tau((m,k))} + \mu c_{\tau((m,k))}\right)^{ml_{m,k+1}}.$$

So, h_n is a polynomial of μ. Therefore, $h_n \in H(\mathbb{C})$.

Let $p \in \mathbb{N}$, $\alpha_0 \in \mathbb{C}$ and

$$S_p(\mu) = \alpha_0 + \sum_{n=1}^{p} h_n(\mu), \tag{45}$$

where $\mu \in \mathbb{C}$. It is clear that $S_p \in H(\mathbb{C})$ since $h_n \in H(\mathbb{C})$.

For $\mu \in \mathbb{C}$ by Lemma 4, where we set $x = b + \mu c$, there exists $y_\mu \in L^2_\infty[0,1]$ such that

$$\delta_{b+\mu c}(I_{n,k+1}) = \delta_{y_\mu}(R_{n-k,k}) \tag{46}$$

for every $n \in \mathbb{N}$ and $k = \overline{0,n}$ and

$$\|y_\mu\|_{\infty,2} \le K\|b+\mu c\|, \tag{47}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) and (17) respectively, and K is given by Corollary 2. From the properties of the norm it follows that

$$\|b+\mu c\| \le \|b\| + |\mu|\|c\|. \tag{48}$$

Therefore, from (47) and (48) it follows that

$$\|y_\mu\|_{\infty,2} \le K\big(\|b\| + |\mu|\|c\|\big). \tag{49}$$

Thus, by (43), (46) and (42), we obtain

$$h(\mu) = \Theta(g)(b+\mu c) = \delta_{y_\mu}(g). \tag{50}$$

By (45) and (44), we get

$$S_p(\mu) = \alpha_0 + \sum_{n=1}^{p} \sum_{l_{11}+l_{12}+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}(b+\mu c)\right)^{l_{m,k+1}}. \tag{51}$$

By (39), we have

$$\begin{aligned}\alpha_0 + \sum_{n=1}^{p} \sum_{l_{11}+l_{12}+\ldots+n(l_{n1}+\ldots+l_{n,n+1})=n} \alpha_{l_{11},\ldots,l_{n,n+1}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}(b+\mu c)\right)^{l_{m,k+1}} \\ = \Theta\Big(\sum_{n=0}^{p} g_n\Big)(b+\mu c),\end{aligned} \tag{52}$$

where g_n is defined by (41) for every $n \in \mathbb{N} \cup \{0\}$. By Lemma 4, where we set $x = b + \mu c$, there exists $y_\mu \in L^2_\infty[0,1]$ such that (46) holds. Then by (42), we obtain

$$\Theta\Big(\sum_{n=0}^{p} g_n\Big)(b+\mu c) = \delta_{y_\mu}\Big(\sum_{n=0}^{p} g_n\Big). \tag{53}$$

Thus, by (51), (52) and (53), we have that

$$S_p(\mu) = \Theta\Big(\sum_{n=0}^{p} g_n\Big)(b+\mu c) = \delta_{y_\mu}\Big(\sum_{n=0}^{p} g_n\Big). \tag{54}$$

Let $l \in \mathbb{N}$. By (2), (43), (54), (50) and (49), we have

$$\begin{aligned}
\|h - S_p\|_l &= \sup_{\{\mu\in\mathbb{C}\,:\,|\mu|\le l\}} |h(\mu) - S_p(\mu)| \\
&\le \sup_{\{\omega\in\ell_\infty\,:\,\|\omega\|\le\|b\|+l\|c\|\}} \Big|\Theta(g)(\omega) - \Theta\Big(\sum_{n=0}^{p} g_n\Big)(\omega)\Big| \\
&\le \sup_{\{z\in L^2_\infty[0,1]\,:\,\|z\|_{\infty,2}\le K(\|b\|+l\|c\|)\}} \Big|\delta_z(g) - \delta_z\Big(\sum_{n=0}^{p} g_n\Big)\Big| \\
&= \sup_{\{z\in L^2_\infty[0,1]\,:\,\|z\|_{\infty,2}\le K(\|b\|+l\|c\|)\}} \Big|g(z) - \sum_{n=0}^{p} g_n(z)\Big| \\
&= \Big\|g - \sum_{n=0}^{p} g_n\Big\|_{K(\|b\|+l\|c\|)} \to 0 \ \text{ as } \ p\to\infty,
\end{aligned}$$

since $g \in H_{bs}(L^2_\infty[0,1])$. Therefore by [17, Chapter 2, p.53, Theorem 5.2], we conclude that $h \in H(\mathbb{C})$. So, the mapping $\Theta(g)$ is G-holomorphic.

Now let us show that $\Theta(g)$ is bounded on bounded sets, that is, $\Theta(g)$ is a function of bounded type. Consider a ball $B(0,r) \subset \ell_\infty$, where $r > 0$. Prove that the function $\Theta(g)$ is bounded on $B(0,r)$, that is, $\|\Theta(g)\|_r < +\infty$. By (2), we obtain

$$\|\Theta(g)\|_r = \sup_{\|x\|\le r} \big|\Theta(g)(x)\big|. \tag{55}$$

By Lemma 4, for every $x \in \ell_\infty$ there exists $y_x \in L^2_\infty[0,1]$ such that

$$\delta_x(I_{n,k+1}) = \delta_{y_x}(R_{n-k,k}) \tag{56}$$

for every $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$ and

$$\|y_x\|_{\infty,2} \le K\|x\|, \tag{57}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) and (17) respectively, and K is given by Corollary 2. Since $\|x\| \le r$, from (57) it follows that

$$\|y_x\|_{\infty,2} \le Kr. \tag{58}$$

Then by (55), (56), (42) and (58), we have

$$\begin{aligned}
\|\Theta(g)\|_r = \sup_{\|x\|\le r} \big|\Theta(g)(x)\big| &\le \sup_{\{y\in L^2_\infty[0,1]:\,\|y\|_{\infty,2}\le Kr\}} \big|\delta_y(g)\big| \\
&= \sup_{\{y\in L^2_\infty[0,1]:\,\|y\|_{\infty,2}\le Kr\}} \big|g(y)\big| = \|g\|_{Kr} < +\infty
\end{aligned}$$

since $g \in H_{bs}(L^2_\infty[0,1])$ is the function of bounded type. We have thus proved that $\Theta(g)$ is bounded on an arbitrary ball of the space ℓ_∞. Therefore, $\Theta(g)$ is bounded on every bounded set of ℓ_∞ and so, $\Theta(g)$ is a function of bounded type. Thus, we have proved that $\Theta(g) \in H_{b\mathcal{I}}(\ell_\infty)$.

It can be checked that the given mapping Θ preserves the operations of the algebras $H_{bs}(L^2_\infty[0,1])$ and $H_{b\mathcal{I}}(\ell_\infty)$.

Next let us show that Θ is continuous. It is sufficient to prove that there exist $C > 0$ and $r_2 > 0$ such that for all $r_1 > 0$ and $g \in H_{bs}(L^2_\infty[0,1])$ the inequality

$$\|\Theta(g)\|_{r_1} \leq C \cdot \|g\|_{r_2}$$

holds. Let us give an estimate of $\|\Theta(g)\|_{r_1}$. By Lemma 4, for every $x \in \ell_\infty$ there exists $y \in L^2_\infty[0,1]$ such that (37) and (38) hold. Then by (2) and (42), we have

$$\begin{aligned}\|\Theta(g)\|_{r_1} = \sup_{\{x\in\ell_\infty : \|x\|\leq r_1\}} |\Theta(g)(x)| &\leq \sup_{\{y\in L^2_\infty[0,1]:\|y\|_{\infty,2}\leq Kr_1\}} |\delta_y(g)| \\ &= \sup_{\{y\in L^2_\infty[0,1]:\|y\|_{\infty,2}\leq Kr_1\}} |g(y)| = \|g\|_{Kr_1}.\end{aligned}$$

Thus $C = 1$ and $r_2 = Kr_1$ and so, the mapping Θ is continuous.

Let us prove that Θ is bijective. Firstly, let us show that for all $g_1, g_2 \in H_{bs}(L^2_\infty[0,1])$ whenever $\Theta(g_1) = \Theta(g_2)$, then $g_1 = g_2$. Let us consider a function $g \in H_{bs}(L^2_\infty[0,1])$ such that $g = g_1 - g_2$ and g has a Taylor series representation (40), where g_n is defined by (41). By assumption $\Theta(g_1) = \Theta(g_2)$, therefore

$$\Theta(g) = \Theta(g_1 - g_2) = \Theta(g_1) - \Theta(g_2) = 0.$$

Let $r > 0$. By the Cauchy estimate, we obtain $\|\Theta(g_n)\|_r \leq \|\Theta(g)\|_r$ for every $n \in \mathbb{N} \cup \{0\}$. Then $\|\Theta(g_n)\|_r = 0$, since $\Theta(g) = 0$. Therefore $\Theta(g_n) = 0$ for every $n \in \mathbb{N} \cup \{0\}$. And so, by (41), we get

$$\Theta(g_0) = \Theta(\alpha_0) = \alpha_0 = 0, \tag{59}$$

and by (41) and (39), we obtain

$$\begin{aligned}\Theta(g_n) &= \Theta\Big(\sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(R_{m-k,k}\right)^{l_{m-k,k}}\Big) \\ &= \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \Big(\Theta\big(R_{m-k,k}\big)\Big)^{l_{m-k,k}} \\ &= \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(I_{m,k+1}\right)^{l_{m-k,k}} = 0\end{aligned}$$

for every $n \in \mathbb{N}$. Since the polynomials from the sequence $\mathcal{I}$, defined by (19), are algebraically independent, from the last equality it follows that

$$\alpha_{l_{10},\ldots,l_{0n}} = 0.$$

Therefore for every $n \in \mathbb{N}$ we have

$$g_n = \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(R_{m-k,k}\right)^{l_{m-k,k}} = 0$$

and by (41), (59), we get

$$g_0 = \alpha_0 = 0.$$

Thus,

$$g_1 - g_2 = g = \sum_{n=0}^{\infty} g_n = 0.$$

So, $g_1 = g_2$ and the mapping Θ is injective.

Next let us show that the mapping Θ is surjective, that is for every function $f \in H_{b\mathcal{I}}(\ell_\infty)$ there exists a function $g \in H_{bs}(L_\infty^2[0,1])$ such that $\Theta(g) = f$. Let $f \in H_{b\mathcal{I}}(\ell_\infty)$ be an arbitrary chosen function. Then f is of the form (20). Let us consider the function

$$g = \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(R_{m-k,k}\big)^{l_{m-k,k}}, \tag{60}$$

where the polynomials $R_{m-k,k}$ are defined by (3), $l_{10},\ldots,l_{0n} \in \mathbb{Z}_+$, α_0 and $\alpha_{l_{10}\ldots l_{0n}}$ are given by (20). Note that g can be represented in the form

$$g = \sum_{n=0}^{\infty} g_n, \tag{61}$$

where g_n is defined by (41) for every $n \in \mathbb{N} \cup \{0\}$. Let us show that g is well defined on the set $L_\infty^2[0,1]$. Let $y \in L_\infty^2[0,1]$. By Lemma 3, there exists $x \in \ell_\infty$ such that (28) and (29) hold. Then by (60), we have

$$\begin{aligned} g(y) &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(R_{m-k,k}(y)\big)^{l_{m-k,k}} \\ &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(\delta_x(I_{m,k+1})\big)^{l_{m-k,k}} \\ &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(I_{m,k+1}(x)\big)^{l_{m-k,k}} \\ &= f(x). \end{aligned} \tag{62}$$

Since $x \in \ell_\infty$ and $f \in H_{b\mathcal{I}}(\ell_\infty)$, it follows that f is well defined on the element x and so, the given function g is well defined on the set $L_\infty^2[0,1]$.

Next let us prove that $g \in H_{bs}(L_\infty^2[0,1])$. Firstly, let us prove that g is G-holomorphic. According to the definition of a G-holomorphic function (see [13, Section 8, p. 58, Definition 8.1]) we need to prove that for all $b, c \in L_\infty^2[0,1]$ the function $h : \mathbb{C} \to \mathbb{C}$, defined by

$$h(\mu) = g(b + \mu c), \tag{63}$$

where $\mu \in \mathbb{C}$, is entire on the set $\mathbb{C}$, that is, $h \in H(\mathbb{C})$. Let $b = (b_1, b_2) \in L_\infty^2[0,1]$ and $c = (c_1, c_2) \in L_\infty^2[0,1]$. Let h be defined by (63). For any $p \in \mathbb{N}$, let

$$S_p(\mu) = \sum_{n=0}^{p} g_n, \tag{64}$$

where g_n is defined by (61). Then by (61), we have

$$S_p(\mu) = \alpha_0 + \sum_{n=1}^{p} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \times \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(R_{m-k,k}(b+\mu c)\big)^{l_{m-k,k}}, \tag{65}$$

where $\mu \in \mathbb{C}$, $l_{10}, \ldots, l_{0n} \in \mathbb{Z}_+$, $\alpha_0, \alpha_{l_{10},\ldots,l_{0n}}$ are given by (60). By (65) and (3), we have

$$S_p(\mu) = \alpha_0 + \sum_{n=1}^{p} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \times \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \left(\int_{[0,1]} \big(b_1(t)+\mu c_1(t)\big)^{m-k} \big(b_2(t)+\mu c_2(t)\big)^{k} dt \right)^{l_{m-k,k}}.$$

So, S_p is a polynomial of μ. Therefore, $S_p \in H(\mathbb{C})$.

For $\mu \in \mathbb{C}$ by Lemma 3, where we set $y = b + \mu c$, there exists $x_\mu \in \ell_\infty$ such that

$$\delta_{b+\mu c}(R_{n-k,k}) = \delta_{x_\mu}(I_{n,k+1}) \tag{66}$$

for every $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$, and

$$\|x_\mu\| \leq \|b+\mu c\|_{\infty,2}, \tag{67}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) and (17) respectively. By (48), we have

$$\|b+\mu c\|_{\infty,2} \leq \|b\|_{\infty,2} + |\mu|\|c\|_{\infty,2}. \tag{68}$$

Therefore by (67) and (68), we get

$$\|x_\mu\| \leq \|b\|_{\infty,2} + |\mu|\|c\|_{\infty,2}.$$

Thus, by (65) and (66), we obtain

$$S_p(\mu) = \alpha_0 + \sum_{n=1}^{p} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(I_{m,k+1}(x_\mu)\big)^{l_{m-k,k}}. \tag{69}$$

By (21) and (22), we have

$$\alpha_0 + \sum_{n=1}^{p} \sum_{l_{10}+l_{01}+2(l_{20}+l_{11}+l_{02})+\ldots+n(l_{n0}+\ldots+l_{0n})=n} \alpha_{l_{10},\ldots,l_{0n}} \prod_{\substack{m=\overline{1,n}\\ k=\overline{0,m}}} \big(I_{m,k+1}\big)^{l_{m-k,k}} = \sum_{n=0}^{p} f_n. \tag{70}$$

Therefore, by (69) and (70), we obtain

$$S_p(\mu) = \sum_{n=0}^{p} f_n(x_\mu).$$

Then by (2), (63), (64), (29) and (62), we have

$$\begin{aligned}\|h - S_p\|_l &= \sup_{\{\mu\in\mathbb{C}\,:\,|\mu|\le l\}} |h(\mu) - S_p(\mu)| \\ &\le \sup_{\{\omega\in L^2_\infty[0,1]\,:\,\|\omega\|_{\infty,2}\le\|b\|_{\infty,2}+l\|c\|_{\infty,2}\}} \Big|g(\omega) - \sum_{n=0}^{p} g_n(\omega)\Big| \\ &\le \sup_{\{x\in\ell_\infty\,:\,\|x\|\le\|b\|_{\infty,2}+l\|c\|_{\infty,2}\}} \Big|f(x) - \sum_{n=0}^{p} f_n(x)\Big| \\ &= \Big\|f - \sum_{n=0}^{p} f_n\Big\|_{\|b\|_{\infty,2}+l\|c\|_{\infty,2}} \to 0 \text{ as } p\to\infty,\end{aligned}$$

since $f \in H_{b\mathcal{I}}(\ell_\infty)$. Therefore by [17, Chapter 2, p.53, Theorem 5.2], we conclude that $h \in H(\mathbb{C})$. And so, g is G-holomorphic.

Secondly, let us show that g is a function of bounded type, that is, g is bounded on bounded sets of the space $L^2_\infty[0,1]$. Consider a ball $B(0,r) \subset L^2_\infty[0,1]$, where $r > 0$. Prove that the function g is bounded on $B(0,r)$, that is, $\|g\|_r < +\infty$. By (2), we get

$$\|g\|_r = \sup_{\|y\|_{\infty,2}\le r} \big|g(y)\big|. \tag{71}$$

By Lemma 3, for every $y \in L^2_\infty[0,1]$ there exists $x_y \in \ell_\infty$ such that

$$\delta_y(R_{n-k,k}) = \delta_{x_y}(I_{n,k+1}) \tag{72}$$

for every $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$ and

$$\|x_y\| \le \|y\|_{\infty,2}, \tag{73}$$

where the polynomials $R_{n-k,k}$ and $I_{n,k+1}$ are defined by (3) and (17) respectively. Then by (71), (72), (62) and (73), we obtain

$$\|g\|_r = \sup_{\|y\|_{\infty,2}\le r} \big|g(y)\big| \le \sup_{\{x\in\ell_\infty\,:\,\|x\|\le r\}} |f(x)| = \|f\|_r < +\infty$$

since $f \in H_{b\mathcal{I}}(\ell_\infty)$ is the function of bounded type. We have thus proved that g is bounded on an arbitrary ball of the space $L^2_\infty[0,1]$. Therefore, g is bounded on every bounded set of $L^2_\infty[0,1]$ and so, g is a function of bounded type. We have proved that $g \in H_{bs}(L^2_\infty[0,1])$. Thus the function g is desired and the mapping Θ is surjective.

Finally, we have proved that the mapping $\Theta : H_{bs}(L^2_\infty[0,1]) \to H_{b\mathcal{I}}(\ell_\infty)$ is well defined, continuous, bijective and preserves the operations of the algebras $H_{bs}(L^2_\infty[0,1])$ and $H_{b\mathcal{I}}(\ell_\infty)$. Then according to [16, Chapter 2, Corollaries 2.12] the inverse mapping

$$\Theta^{-1} : H_{b\mathcal{I}}(\ell_\infty) \to H_{bs}(L^2_\infty[0,1])$$

is also continuous. Thus the given mapping Θ is an isomorphism between the algebras $H_{bs}(L^2_\infty[0,1])$ and $H_{b\mathcal{I}}(\ell_\infty)$. ☐

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Василишин С.І. *Топологічний ізоморфізм зліченно-породженої алгебри цілих функцій на* ℓ_∞ *та алгебри симетричних цілих функцій на* $L^2_\infty[0,1]$ // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 78–98.

У даній статті отримано деякі загальні результати про зліченно-породжені алгебри цілих функцій обмеженого типу на комплексному банаховому просторі. Зокрема, досліджено підалгебру Фреше $H_{b\mathbf{P}}(X)$ алгебри Фреше всіх цілих функцій обмеженого типу $H_b(X)$, породжену зліченною множиною $\mathbf{P}$ неперервних алгебраїчно незалежних комплекснозначних однорідних поліномів на комплексному банаховому просторі X, такою, що деяка скінченна кількість елементів множини $\mathbf{P}$ може мати однаковий степінь однорідності. Встановлено вигляд елементів цієї підалгебри. Крім того, показано, що кожний лінійний мультиплікативний функціонал, який діє з алгебри $H_{b\mathbf{P}}(X)$ у простір $\mathbb{C}$, цілком визначений своїми значеннями на елементах множини $\mathbf{P}$. Також встановлено оцінку зверху для значення такого функціонала на довільному n-однорідному поліномі, який належить алгебрі $H_{b\mathbf{P}}(X)$. Ці результати використано для деяких алгебр.

Нехай $L_\infty[0,1]$ — комплексний банаховий простір усіх комплекснозначних вимірних за Лебегом суттєво обмежених функцій на відрізку $[0,1]$. Нехай $L^2_\infty[0,1]$ є декартовим квадратом простору $L_\infty[0,1]$. Розглянуто алгебру Фреше $H_{bs}(L^2_\infty[0,1])$ усіх цілих симетричних функцій обмеженого типу на просторі $L^2_\infty[0,1]$. У роботі побудовано зліченно-породжену підалгебру Фреше алгебри $H_b(\ell_\infty)$, яка є топологічно ізоморфною до $H_{bs}(L^2_\infty[0,1])$, де ℓ_∞ є комплексним банаховим простором усіх обмежених послідовностей комплексних чисел. А саме, нехай $\mathcal{I} = (I_{11}, I_{12}, I_{21}, I_{22}, I_{23}, \ldots, I_{n1}, I_{n2}, \ldots, I_{n,n+1}, \ldots)$, де $I_{11}(x) = x_1$, $I_{12}(x) = x_2$, $I_{21}(x) = x_3^2$, $I_{22}(x) = x_4^2$, $I_{23}(x) = x_5^2$, $I_{31}(x) = x_6^3$, $I_{32}(x) = x_7^3$, $I_{33}(x) = x_8^3$, $I_{34}(x) = x_9^3, \ldots$, де $x = (x_1, x_2, \ldots) \in \ell_\infty$. Позначимо через $H_{b\mathcal{I}}(\ell_\infty)$ підалгебру Фреше алгебри $H_b(\ell_\infty)$, породжену послідовністю поліномів $\mathcal{I}$. У роботі побудовано топологічний ізоморфізм між алгебрами $H_{b\mathcal{I}}(\ell_\infty)$ та $H_{bs}(L^2_\infty[0,1])$.

Результати статті можуть бути використані для досліджень алгебр симетричних аналітичних функцій на банахових просторах.

Ключові слова і фрази: n-однорідний поліном, симетрична функція, аналітична функція, спектр алгебри.