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Strictly convex abelian metric groups are normed spaces

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We prove that every strictly convex abelian metric group has a canonical structure of a normed space over the field of real numbers. We deduce this fact from the $\mathbb{R}$-normability of strictly convex metric groups. Moreover, we prove that a strictly convex (more generaly, $\mathbb{R}$-normable) metric group is a finite-dimensional normed space if and only if it is locally compact if and only if it is (compactly) finite-dimensional. Also we prove that every strictly convex metric space is geodesic.

Key words and phrases: strictly convex metric space, normed space, abelian metric group, geodesic metric space, finite-dimensional normed space, locally compact metric group, $\mathbb{R}$-normable metric group.

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1 Introduction

The strict convexity is a fundamental geometric property of Banach spaces, ensuring uniqueness of best approximations, playing a central role in duality theory, and underpinning applications in optimization, approximation theory, and fixed point theory (cf. [3,4,8,9]). In spite of the fact that the strict convexity usually is defined for normed or Banach spaces, it is a purely metric property and can be defined without involving the linear or convex structure.

Definition 1. *A metric space (X,d) is defined to be strictly convex if for any points $x,y \in X$ and any positive real numbers a,b with $a+b=d(x,y)$, the intersection $B[x,a]\cap B[y,b]$ is a singleton.*

Here we denote by $B[x,a] := \{z \in X : d(x,z) \le a\}$ the closed ball of radius a centered at a point x of the metric space (X,d).

In this paper, we study strictly convex metric groups and prove that every strictly convex metric abelian group $(G,+,0,d)$ possesses a canonical structure of a normed space. The latter means that it admits a unique function

$$\cdot\, : \mathbb{R}\times G \ni (t,x)\mapsto t\cdot x\in G,$$

that turns G into a linear space over the field of real numbers and also turns the function

$$\|\cdot\| : G\ni x\mapsto \|x\| := d(x,0)\in\mathbb{R}$$

into a norm on the linear space $(G,+,0,\cdot)$.

This structure result is applied in the paper [1] devoted to studying plastic metric spaces and groups (a metric space is *plastic* if every its non-expanding permutation is an isometry).

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2 The Main Result

Our study will mainly consider the strict convexity in metric groups. Let us recall the definition of this mathematical structure.

Definition 2. *A group is an algebraic structure $(G, +, 0)$, consisting of a set G, a binary operation $+ : G \times G \to G$ and an identity element $0 \in G$, satisying the following axioms:*

(1) $\forall x, y, z \in G \quad (x + y) + z = x + (y + z)$ *(associativity),*

(2) $\forall x \in G \quad x + 0 = x = 0 + x$ *(identity),*

(3) $\forall x \in G \; \exists y \in G \quad x + y = 0 = y + x$ *(inverse).*

If, in addition, $x + y = y + x$ for all $x, y \in G$, then $(G, +, 0)$ is called an abelian group.

Definition 3. *A metric group is a group $(G, +, 0)$ equipped with a metric $d : G \times G \to \mathbb{R}$, which is translation invariant in the sense that $d(x + c, y + c) = d(x, y) = d(c + x, c + y)$ for all elements $x, y, c \in G$. The metric d can be recovered from the norm $\| \cdot \| : G \ni x \mapsto d(x, 0) \in \mathbb{R}$ generated by this metric.*

The main result of this paper is the following $\mathbb{R}$-normability theorem.

Theorem 1. *Every strictly convex metric group $(G, +, 0, d)$ admits a unique scalar multiplication operation $\cdot : \mathbb{R} \times G \ni (t, x) \mapsto tx \in G$, such that*

(1) $\forall x \in G \quad 1x = x$ (identity),

(2) $\forall t, v \in \mathbb{R} \;\; \forall x \in G \quad (tv)x = t(vx)$ (associativity),

(3) $\forall t, v \in \mathbb{R} \;\; \forall x \in G \quad (t + v)x = tx + vx$ (distributivity),

(4) $\forall t \in \mathbb{R} \;\; \forall x, y \in G \quad x + y = y + x \Rightarrow t(x + y) = tx + ty$ (codistributivity),

and the function $\| \cdot \| : G \ni x \mapsto \|x\| := d(x, 0) \in \mathbb{R}$ satisfies the usual axioms of a norm:

(5) $\forall x \in G \quad \|x\| = 0 \iff x = 0$ (non-degeneracy),

(6) $\forall x, y \in G \quad \|x + y\| \le \|x\| + \|y\|$ (triangle inequality),

(7) $\forall\, t \in \mathbb{R} \;\; \forall x \in G \quad \|tx\| = |t| \cdot \|x\|$ (homogeneity).

Theorem 1 will be proved in Section 7 after some preliminary work made in Sections 3–6.

Now we discuss some implications of this theorem. A metric group G admitting a binary operation $\cdot : \mathbb{R} \times G \to G$ that has properties (1)–(7) of Theorem 1 will be called $\mathbb{R}$-*normable*. In this terminology, Theorem 1 says that every strictly convex metric group is $\mathbb{R}$-normable. If a strictly convex metric group G is abelian, then the property (4) of Theorem 1, called codistributivity[1], becomes the standard distributivity of multiplication over vector addition, and then the metric group G endowed with the scalar multiplication becomes a normed space over the field of real numbers. Therefore, $\mathbb{R}$-normable abelian metric groups are underlying additive groups of normed spaces (all normed spaces considered in this paper are over the field of real numbers). In this case, we will say that such abelian groups are normed spaces. Therefore, in commutative case, Theorem 1 implies the following corollary.

Corollary 1. *Every strictly convex metric abelian group is a normed space.*

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[1] coined by removing "mmutative" from the commutative *distributivity*

We do not know whether Corollary 1 generalizes to nonabelian groups, which motivates the following open problem.

Problem 1. *Is every strictly convex metric group abelian? Is every* $\mathbb{R}$*-normable metric group abelian?*

In this paper, we provide an affirmative answer to this problem for strictly convex (more generally, $\mathbb{R}$-normable) metric groups which are locally compact or (compactly) finite-dimensional. A metric group G is *locally compact* if every point of G has a neighborhood with compact closure. A metric space X is called *compactly finite-dimensional* if all compact subets in X are finite-dimensional. By dimension we will here and henceforth mean the small inductive dimension of a topological space.

Theorem 2. *Every locally compact* $\mathbb{R}$*-normable metric group is abelian and hence is a finite-dimensional normed space.*

Theorem 2 will be proved in Section 8. We will deduce from Theorem 2 another partial answer to Problem 1 that yields also a characterization of finite-dimensional normed spaces among $\mathbb{R}$-normable metric groups.

Theorem 3. *For an* $\mathbb{R}$*-normable metric group* G*, the following conditions are equivalent:*

(1) *G is locally compact;*
(2) *G is finite-dimensional;*
(3) *G is compactly finite-dimensional.*

The equivalent conditions (1)–(3) *imply that G is abelian.*

We expect that Problem 1 has negative answer, so can be reformulated in the following "constructive" fashion.

Problem 2. *Study structural properties of (strictly convex)* $\mathbb{R}$*-normable metric groups.*

In fact, the $\mathbb{R}$-normability established in Theorem 1 is one of such structural properties. It implies that strictly convex metric groups have no small subgroups and hence are close to being Lie groups. Let us recall that a topological group *has no small subgroups* if some neighborhood of the identity contains no nontrivial subgroups. By the classical characterization of A.M. Gleason [7], Lie groups are exactly locally compact topological groups without small subgroups. The homogeneity of the norm implies that $\mathbb{R}$-normable metric groups cannot have small subgroups. This (trivial) fact, combined with the (nontrivial) Theorem 1 implies the following (nonobvious) property of strictly convex metric groups.

Corollary 2. *Strictly convex metric groups do not have small subgroups.*

3 Strictly convex metric spaces are geodesic

Definition 4. *Let $(X,d),(Y,\rho)$ be metric spaces. A map $f : X \to Y$ is called an* isometry *if $\rho(f(x),f(y)) = d(x,y)$ for all points $x,y \in X$.*

Definition 5. *A metric space (X,d) is called* geodesic *if for all $x,y \in X$ there exists a unique isometry $\gamma : [0,d(x,y)] \subseteq \mathbb{R} \to X$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$.*

The main result of this section is the following (probably known) proposition.

Proposition 1. *Every strictly convex metric space* (X, d) *is geodesic.*

Proof. Take arbitrary points $x, y \in X$ and let $c := d(x, y)$. Consider the function $\gamma : [0, c] \to X$ that assigns to each real number $a \in [0, c]$ the unique point in the intersection of the closed balls $B[x, a], B[y, c-a]$ in the strictly convex metric space X. The triangle inequality ensures that $d(x, \gamma(a)) = a$ and $d(y, \gamma(a)) = c - a$. This implies that $\gamma(0) = x$ and $\gamma(c) = y$. We claim that the map γ is an isometry.

Take any two real numbers $a, b \in [0, c]$. Without loss of generality we can assume that $a \leq b$. Consider the closed balls $B[x, a]$ and $B[\gamma(b), b-a]$. Since X is strictly convex and $a + (b - a) = b = d(x, \gamma(b))$, the balls intersect at exactly one point $z \in X$. Using the triangle inequality, we obtain

$$c = d(x, y) \leq d(x, z) + d(z, y) \leq a + d(z, y)$$

and hence $c - a \leq d(z, y)$. Using triangle inequality again in the different setting, we obtain

$$d(z, y) \leq d(z, \gamma(b)) + d(\gamma(b), y) \leq (b - a) + (c - b) = c - a.$$

Combining the results together, we conclude that $d(z, y) = c - a$. Then the point z belongs to the intersection of closed balls $B[x, a]$ and $B[y, c - a]$. Since this intersection is the singleton $\{\gamma(a)\}$, the point z coincides with $\gamma(a)$. Hence $d(\gamma(a), \gamma(b)) = d(z, \gamma(b)) = b - a$. This proves that γ is an isometry.

Finally, let us prove that γ is a unique isometry from $[0, c]$ to X such that $\gamma(0) = x, \gamma(c) = y$. Assume that $f : [0, c] \to X$ is another isometry with $f(0) = x$ and $f(c) = y$. Then for an arbitrary real number $a \in [0, c]$ we have $d(x, f(a)) = d(f(0), f(a)) = d(0, a) = a$ and similarly $d(f(a), y) = c - a$. Hence $f(a)$ belongs to the intersection of closed balls $B[x, a]$ and $B[y, c - a]$. Since this intersection contains exactly one point $\gamma(a)$, we conclude that $f(a) = \gamma(a)$. Therefore γ is a unique isometry with wanted properties, and as a result, the metric space (X, d) is geodesic. ☐

4 2-divisibility in strictly convex metric groups

In this section we use the existence of midponts in strictly convex metric groups in order to prove that such groups are uniquely 2-divisible.

Definition 6. *Let* (X, d) *be a strictly convex metric space. For two points* $x, y \in X$ *the midpoint between points* x *and* y *is the unique point of the singleton* $B\left[x, \frac{d(x,y)}{2}\right] \cap B\left[y, \frac{d(x,y)}{2}\right]$.

Lemma 1. *Let* $(G, +, 0, d)$ *be a strictly convex metric group. For all points* $x, y \in G$ *the midpoint* $z \in G$ *between* x *and* y *satisfies the algebraic identity* $z = x - z + y$.

Proof. Let us consider the real number $c := \frac{1}{2}d(x, y) = d(z, x) = d(z, y)$ and the element $z' := x - z + y$ of the group G. Since the metric d is translation invariant, we obtain $d(z', x) = d(x - z + y, x) = d(y, z) = c$ and similarly $d(z', y) = d(x, z) = c$. Since G is strictly convex, z' must coincide with z. Hence, $z = x - z + y$. ☐

Definition 7. *An additive group* $(G, +, 0)$ *is called (uniquely)* 2*-divisible if for every* $x \in G$ *there exist a (unique) element* $y \in G$ *such that* $y + y = x$.

The main result of this section is the following proposition.

Proposition 2. *Every strictly convex metric group is uniquely* 2*-divisible.*

Proof. Let $(G,+,0,d)$ be a strictly convex metric group. First we prove that the group G has no elements of order 2.

Claim 1. *An element $x \in G$ equals zero if and only if $x+x=0$.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that $x+x=0$. Let z be the midpoint between 0 and x. Lemma 1 ensures that $z=0-z+x$ and hence $z+z=x$. Consider the element $y:=z+x$ and observe that $z+y=z+(z+x)=(z+z)+x=x+x=0$. Then $d(0,y)=d(z,z+y)=d(z,0)=\frac{1}{2}d(0,x)$ and $d(y,x)=d(y+x,x+x)=d(z,0)=\frac{1}{2}d(0,x)$, which implies that y is a midpoint between 0 and x. The uniqueness of the midpoint ensures that $z=y=z+x$ and $x=0$. □

Now we are able to prove that the group G is uniquely 2-divisible. Given any element $x\in G$, consider the midpoint z between 0 and x. Lemma 1 implies $z+z=x$. Assume that $y\in G$ is another element with $y+y=x$. Consider the element $z':=y+z-y$ of the group G. The translation invariance of the metric d ensures that $d(0,z')=d(0,y+z-y)$ $=d(-y+y,z)=d(0,z)=\frac{1}{2}d(0,x)$ and $d(z',x)=d(y+z-y,x)=d(z,-y+x+y)$ $=d(z,-y+y+y+y)=d(z,y+y)=d(z,x)=\frac{1}{2}d(0,x)$, which means that z' is the midpoint between 0 and x. The uniqueness of a midpoint between 0 and x guarantees that $z=z':=y+z-y$, which implies $y+z=z+y$. Then we obtain the following equalities $(y-z)+(y-z)=(y+y)-(z+z)=x-x=0$ and $y-z=0$, by Claim 1. Therefore, $y=z$ and z is a unique element of G such that $z+z=x$, witnessing that the group G is uniquely 2-divisible. □

5 Every strictly convex metric group is a $\mathbb{Z}[\frac{1}{2}]$-comodule

Let R be a commutative unital ring. A metric group G admitting a binary operation $\cdot : R\times G\to G$ that has properties (1)–(4) of Theorem 1 will be called an *R-comodule*. In this terminology, every commutative R-comodule is an R-module. In this section, we prove that every strictly convex metric group is a comodule over the ring $\mathbb{Z}[\frac{1}{2}]=\{\frac{m}{2^n}:m\in\mathbb{Z},n\in\mathbb{N}\}$ of dyadic fractions.

Proposition 3. *Every uniquely* 2*-divisible additive group $(G,+,0)$ is a $\mathbb{Z}[\frac{1}{2}]$-comodule.*

Proof. Define a binary operation $\cdot:\mathbb{Z}\times G\ni(n,x)\mapsto nx\in G$ by the following recursive formulas: $0\cdot x=0$, $(n+1)\cdot x=n\cdot x+x$, and $-(n+1)x=-nx-x$ for all $n\in\mathbb{N}\cup\{0\}$. A straightforward verification of properties (1)–(4) ensures that with the defined operation the following claim holds.

Claim 2. *Every additive group is a $\mathbb{Z}$-comodule.*

The unique 2-divisibility of the group G ensures that we can extend the $\cdot$ operation by setting $\frac{1}{2}x$ to be the unique element with $2(\frac{1}{2}x)=x$ and putting $\frac{1}{2^n}x=\frac{1}{2}(\frac{1}{2^{n-1}}x)$ for all $n\in\mathbb{N}$. Then the following claim holds by induction.

Claim 3. *$\frac{1}{2^n}x$ is the unique element with $2^n(\frac{1}{2^n}x)=x$ for all $x\in G$ and $n\in\mathbb{N}$.*

Claim 4. *If for $x, y \in G$ we have $x + y = y + x$, then $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}y + \frac{1}{2}x$.*

Proof. Since $x + y = y + x$, we obtain $x + y - x = y$, which implies that $2(x + \frac{1}{2}y - x) = y$. Then the identity $x + \frac{1}{2}y - x = \frac{1}{2}y$ must hold by the unique 2-divisibility of G. This gives $\frac{1}{2}y + x - \frac{1}{2}y = x$, which implies $\frac{1}{2}y + \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}x$ by the same argument. Then we get $\frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}y + \frac{1}{2}x$, which completes the proof. ☐

Claim 4 ensures the following claim holds by induction.

Claim 5. *If for $x, y \in G$ we have $x + y = y + x$, then $\frac{1}{2^n}x + \frac{1}{2^n}y = \frac{1}{2^n}y + \frac{1}{2^n}x$ for all $n \in \mathbb{N}$.*

Claim 6. *$\frac{1}{2^n}(x + y) = \frac{1}{2^n}x + \frac{1}{2^n}y$ for all $x, y \in G$ with $x + y = y + x$ and $n \in \mathbb{N}$.*

Proof. Claims 2, 3, and 5 ensure that $2^n(\frac{1}{2^n}x + \frac{1}{2^n}y) = 2^n(\frac{1}{2^n}x) + 2^n(\frac{1}{2^n}y) = x + y$. ☐

Claim 7. *$\frac{1}{2^n}(\frac{1}{2^m}x) = \frac{1}{2^{n+m}}x$ for all $x \in G$ and $n, m \in \mathbb{N}$.*

Proof. Claims 2 and 3 ensure that $2^{n+m}(\frac{1}{2^n}(\frac{1}{2^m}x)) = 2^m(\frac{1}{2^m}x) = x$. ☐

Finally, let us further extend the multiplication operation to $\cdot : \mathbb{Z}[\frac{1}{2}] \times G \to G$ by setting $\frac{m}{2^n}x = \frac{1}{2^n}(mx)$ for all $m \in \mathbb{Z}, n \in \mathbb{N}$. The following assertion implies that this operation is well-defined.

Claim 8. *$\frac{1}{2^n}(mx) = \frac{m}{2^n}x = m(\frac{1}{2^n}x)$ for all $x \in G$ and $\frac{m}{2^n} \in \mathbb{Z}[\frac{1}{2}]$.*

Proof. Statement holds by induction on m using codistributivity proven in Claim 6. ☐

Now we are able to prove that $(G, +, 0, \cdot)$ satisfies properties (1)–(4) of the $\mathbb{Z}[\frac{1}{2}]$-comodule.

(1) For every $x \in G$ the equality $1x = x$ holds by the definition of $\cdot$ operation.

(2) For every $x \in G$ and $\frac{m}{2^n}, \frac{p}{2^k} \in \mathbb{Z}[\frac{1}{2}]$ the equality

$$\frac{m}{2^n}\Big(\frac{p}{2^k}x\Big) = m\Big(\frac{1}{2^n}\Big(p\Big(\frac{1}{2^k}x\Big)\Big)\Big) = m\Big(p\Big(\frac{1}{2^n}\Big(\frac{1}{2^k}x\Big)\Big)\Big) = mp\Big(\frac{1}{2^{n+m}}x\Big) = \frac{mp}{2^{n+m}}x$$

holds by Claims 2, 7, 8, and the definition of $\cdot$ operation.

(3) For every $x \in G$ and $\frac{m}{2^n}, \frac{p}{2^k} \in \mathbb{Z}[\frac{1}{2}]$ we have

$$2^{n+k}\Big(\frac{m}{2^n}x + \frac{p}{2^k}x\Big) = 2^k\Big(2^n\Big(\frac{m}{2^n}x\Big)\Big) + 2^n\Big(2^k\Big(\frac{p}{2^k}x\Big)\Big) = m2^kx + p2^nx = (m2^k + p2^n)x$$

by Claims 2 and 3. This implies

$$\frac{m}{2^n}x + \frac{p}{2^k}x = \frac{1}{2^{n+k}}\Big((m2^k + p2^n)x\Big) = \frac{m2^k + p2^n}{2^{n+k}}x = \Big(\frac{m}{2^n} + \frac{p}{2^k}\Big)x$$

by Claim 3 and the definition of $\cdot$ operation.

(4) For every $x, y \in G$ with $x + y = y + x$ and $\frac{m}{2^n} \in \mathbb{Z}[\frac{1}{2}]$ the equality

$$\frac{m}{2^n}\Big(x + y\Big) = \frac{1}{2^n}\Big(m\Big(x + y\Big)\Big) = \frac{1}{2^n}\Big(mx + my\Big) = \frac{1}{2^n}mx + \frac{1}{2^n}my = \frac{m}{2^n}x + \frac{m}{2^n}y$$

holds by Claims 2, 6, the definition of $\cdot$ operation, and the fact that $mx + my = my + mx$ if $x + y = y + x$. ☐

Propositions 2 and 3 imply the following corollary, which is the main result of this section.

Corollary 3. *Every strictly convex metric group is a $\mathbb{Z}[\frac{1}{2}]$-comodule.*

6 The norm is $\mathbb{Z}[\frac{1}{2}]$-homogeneous

In this section we prove that the norm of a strictly convex metric group is compatible with the $\mathbb{Z}[\frac{1}{2}]$-comodule structure.

Proposition 4. *For every stricly convex metric group* $(G, +, 0, d)$*, the norm*

$$\|\cdot\| : G \ni x \mapsto \|x\| := d(x, 0) \in \mathbb{R}$$

has the following properties:

(1) $\forall x \in G \quad \|x\| = 0 \iff x = 0,$

(2) $\forall x, y \in G \quad \|x + y\| \leq \|x\| + \|y\|,$

(3) $\forall\, t \in \mathbb{Z}[\frac{1}{2}] \quad \forall x \in G \quad \|tx\| = |t| \cdot \|x\|.$

Proof. Let $(G, +, 0, d)$ be a strictly convex metric group and $\|\cdot\|$ be its norm. The first two properties follow immediately from the axioms of a translation invariant metric. The proof of the third property is more complicated and is preceded by four lemmas.

Lemma 2. $\|2x\| = 2\|x\|$ *for every* $x \in G$.

Proof. By Lemma 1 and Proposition 2, the point x is a midpoint between 0 and $2x = x + x$, which implies $2\|x\| = 2d(0, x) = d(0, 2x) = \|2x\|$. □

Lemma 3. $\|2^n x\| = 2^n\|x\|$ *for all* $x \in G$ *and* $n \in \mathbb{Z}$.

Proof. Fix any $x \in G$. By induction, Lemma 2 implies $\|2^n x\| = 2^n\|x\|$ for all $n \in \mathbb{N} \cup \{0\}$. Then for every $n \in \mathbb{N} \cup \{0\}$, we have $\|x\| = \|\frac{2^n}{2^n}x\| = 2^n\|\frac{1}{2^n}x\|$ and hence $\|2^{-n}x\| = 2^{-n}\|x\|$. Therefore, $\|2^n x\| = 2^n\|x\|$ for all $n \in \mathbb{Z}$. □

Lemma 4. $\|nx\| = n\|x\|$ *for all* $x \in G$ *and* $n \in \mathbb{N}$.

Proof. From the triangle inequality it follows that $\|nx\| \leq n\|x\|$. Assuming to the contrary that $\|nx\| \neq n\|x\|$, we conclude $\|nx\| < n\|x\|$. Choose any $k \in \mathbb{N}$ with $n \leq 2^k$. Lemma 3, the $\mathbb{Z}$-comodule structure of the group G, and the triangle inequality ensure that

$$2^k\|x\| = \|2^k x\| = \|nx + (2^k - n)x\| \leq \|nx\| + \|(2^k - n)x\| < n\|x\| + (2^k - n)\|x\| = 2^k\|x\|,$$

which is a contradiction. Hence, our initial assumption is wrong and $\|nx\| = n\|x\|$. □

Lemma 5. $\|nx\| = |n| \cdot \|x\|$ *for all* $x \in G$ *and* $n \in \mathbb{Z}$.

Proof. If $n \geq 0$, then the equality $\|nx\| = n \cdot \|x\| = |n| \cdot \|x\|$ follows from Lemma 4.

If $n = -1$, then

$$\|nx\| = \|-x\| = d(0, -x) = d(x, -x + x) = d(x, 0) = d(0, x) = \|x\| = |-1| \cdot \|x\| = |n| \cdot \|x\|,$$

by the translation invariance of the metric d.

If $n < 0$, then $\|nx\| = \||n| \cdot (-x)\| = |n| \cdot \|-x\| = |n| \cdot \|x\|$, by the two preceding cases. □

Taking into account that G is a $\mathbb{Z}[\frac{1}{2}]$-comodule, we can apply Lemmas 3, 5 and conlcude that

$$\left\|\frac{n}{2^k}x\right\| = \|n(2^{-k}x)\| = |n| \cdot \|2^{-k}x\| = |n| \cdot (2^{-k}\|x\|) = \frac{|n|}{2^k} \cdot \|x\|$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $x \in G$, completing the proof of the third property of Proposition 4. □

7 Proof of Theorem 1

Now we are able to present the proof of Theorem 1. Let $(G,+,0,d)$ be a strictly convex metric group and $\|\cdot\|:G\ni x\mapsto\|x\|:=d(x,0)\in\mathbb{R}$ be its norm. By Proposition 3, G admits a binary operation $\cdot:\mathbb{Z}[\frac12]\times G\ni(t,x)\mapsto tx\in G$, turning G into a $\mathbb{Z}[\frac12]$-comodule such that $\|tx\|=|t|\cdot\|x\|$ for all $t\in\mathbb{Z}[\frac12]$ and $x\in G$. This binary operation will be called the *dyadic multiplication* on G.

Now we are going to extend the dyadic multiplication to the operation

$$*:\mathbb{R}\times G\ni(t,x)\mapsto t*x\in G,$$

which is defined as follows. For every $t\in\mathbb{R}_+:=\{x\in\mathbb{R}:x\ge0\}$ and $x\in G$, let $t*x$ be the unique point of the intersection $B[0,t\|x\|]\cap B[rx,(r-t)\|x\|]$, where $r\in\mathbb{Z}[\frac12]$ is any dyadic fraction with $r\ge t$. Let us show that $t*x$ does not depend on the choice of r.

Given any dyadic fractions $r,r'\in\mathbb{Z}[\frac12]\cap[t,\infty)$, we have to show that the singletons

$$\{z\}:=B[0,t\|x\|]\cap B[rx,(r-t)\|x\|]\quad\text{and}\quad\{z'\}:=B[0,t\|x\|]\cap B[r'x,(r'-t)\|x\|]$$

are equal. We lose no generality assuming that $r\le r'$. Observe that $d(0,z)=t\|x\|=d(0,z')$. Using the triangle inequality and Proposition 4, we obtain

$$d(z,r'x)\le d(z,rx)+d(rx,r'x)=(r-t)\|x\|+(r'-r)\|x\|=(r'-t)\|x\|=d(z',r'x).$$

On the other hand, we have

$$r'\|x\|=d(0,r'x)\le d(0,z)+d(z,r'x)=t\|x\|+d(z,r'x)$$

and hence, $d(z,r'x)\ge(r'-t)\|x\|=d(z',r'x)$.

Combining the two inequalities yields $d(z,r'x)=d(z',r'x)$. Since G is strictly convex, $z=z'$, witnessing that $B[0,t\|x\|]\cap B[rx,(r-t)\|x\|]=B[0,t\|x\|]\cap B[r'x,(r'-t)\|x\|]$ and the element $t*x$ is well-defined.

For a negative real number t, put $t*x:=|t|*(-x)$.

Thus we have defined the binary operation $*:\mathbb{R}\times G\ni(t,x)\mapsto t*x\in G$, which will be called the *metric multiplication* on G.

If $t\in\mathbb{Z}[\frac12]\cap[0,\infty)$, then for every $x\in G$, we can take $r=t$ and conclude that $t*x\in B[0,t\|x\|]\cap B[tx,(t-t)\|x\|]=\{tx\}$ and $(-t)*x=t*(-x)=t(-x)=(-t)x$, witnessing that the metric multiplication $*:\mathbb{R}\times G\to G$ extends the dyadic multiplication $\cdot:\mathbb{Z}[\frac12]\times G\to G$.

It remains to check that the metric multiplication has the properties (1)–(7) of Theorem 1. The properties (5) and (6) follow from (1) and (2) of Proposition 4. The other five properties will be derived from the following lemmas.

Lemma 6. *For all $t,v\in\mathbb{R}_+$ and $x\in G$ we have $\|t*x-v*x\|=d(t*x,v*x)=|t-v|\cdot\|x\|$.*

Proof. Choose any dyadic fraction $r\in\mathbb{Z}[\frac12]$ such that $r\ge\max\{t,v\}$. Property (3) of Proposition 4 ensures that $\|rx\|=r\|x\|$. By Proposition 1, the metric space (G,d) is geodesic and hence there exists an isometry $\gamma:[0,\|rx\|]\to G$ such that $\gamma(0)=0$ and $\gamma(\|rx\|)=rx$. The definition of the element $t*x$, the equality $\|rx\|-t\|x\|=(r-t)\|x\|$, and the strict convexity of the metric space (G,d) ensure that both elements $t*x$ and $\gamma(t\|x\|)$ belong to the singleton

$B[0, t\|x\|] \cap B[rx, (r-t)\|x\|]$ and hence $t * x = \gamma(t\|x\|)$. By the same reason, $v * x = \gamma(v\|x\|)$. Since γ is an isometry, we get

$$\|t * x - v * x\| = d(t * x, v * x) = d\big(\gamma(t\|x\|), \gamma(v\|x\|)\big) = \big|t\|x\| - v\|x\|\big| = |t - v| \cdot \|x\|.$$

☐

The following lemma establishes the property (7) of Theorem 1.

Lemma 7. *For all* $t \in \mathbb{R}$ *and* $x \in G$ *we have* $\|t * x\| = |t| \cdot \|x\|$.

Proof. If $t \geq 0$, then $\|t * x\| = \|t * x - 0 * x\| = |t - 0| \cdot \|x\| = |t| \cdot \|x\|$, by Lemma 6. If $t \leq 0$, then $\|t * x\| = \||t| * (-x)\| = |t| \cdot \|-x\| = |t| \cdot \|x\|$, by the "positive" case and the definition of the multiplication by a negative real number. ☐

Lemma 8. *For all* $t, v \in \mathbb{R}$ *and* $x \in G$ *we have* $\|t * x - v * x\| \leq |t - v| \cdot \|x\|$.

Proof. If $t, v \geq 0$, then $\|t * x - v * x\| = |t - v| \cdot \|x\|$, by Lemma 6. If $t, v \leq 0$, then

$$\|t * x - v * x\| = \||t| * (-x) - |v| * (-x)\| = \big||t| - |v|\big| \cdot \|-x\| = |t - v| \cdot \|x\|$$

by Lemma 6 and the definition of the multiplication by negative real numbers. If $t \cdot v < 0$, then

$$\|t * x - v * x\| = d(t * x, v * x) \leq d(t * x, 0) + d(0, v * x) = |t| \cdot \|x\| + |v| \cdot \|x\| = |t - v| \cdot \|x\|,$$

by Lemma 7 and the triangle inequality. ☐

Lemma 8 implies the following assertion.

Lemma 9. *For every* $x \in G$*, the function* $*_x : \mathbb{R} \ni t \mapsto t * x \in G$ *is continuous.*

Lemma 10. $-(t * x) = t * (-x)$ *for all* $t \in \mathbb{R}$ *and* $x \in G$.

Proof. For every $x \in G$, the continuity of the functions $*_x$, $*_{-x}$, and $G \ni g \mapsto -g \in G$ implies that the set $\{t \in R : -(tx) = t(-x)\}$ is closed. Since this set contains the dense subset $\mathbb{Z}[\frac{1}{2}]$ of dyadic fractions, we conclude that it is equal to $\mathbb{R}$, witnessing that $-(tx) = t(-x)$ for all $t \in \mathbb{R}$. ☐

Now we are able to prove the properties (1)–(4) of Theorem 1.

(1) For every $x \in X$ the equality $1 * x = 1 \cdot x = x$ holds, because 1 is a dyadic fraction.

(2) Given any $x \in G$, we should prove that $t * (v * x) = (tv) * x$ for all real numebrs $t, v \in \mathbb{R}$. This equality is trivially true if $x = 0$. So, assume that $x \neq 0$, which implies $\|x\| > 0$.

First we consider the case $t, v \geq 0$. Choose any dyadic fraction $r \in \mathbb{Z}[\frac{1}{2}]$ such that $r > \max\{t, v, t \cdot v\}$. Lemma 7 ensures that $\|t * (v * x)\| = |t| \cdot \|v * x\| = |t| \cdot |v| \cdot \|x\| = |tv| \cdot \|x\|$. We claim that $\|t * (v * x) - rx\| = |tv - r| \cdot \|x\|$. To derive a contradiction, assume that $\|t * (v * x) - rx\| \neq |tv - r| \cdot \|x\|$ and consider the positive real number

$$\varepsilon := \frac{1}{\|x\|} \cdot \big|\|t * (v * x) - rx\| - |tv - r| \cdot \|x\|\big|.$$

Choose any dyadic fractions $t', v' \in \mathbb{Z}[\frac{1}{2}]$ such that $t'v' < r$ and the real numbers

$$|t - t'| \cdot \|v * x\|, \quad |t'| \cdot |v - v'| \cdot \|x\| \qquad \text{and} \qquad |tv - t'v'| \cdot \|x\|,$$

all are smaller than $\frac{\varepsilon}{3}\|x\|$. Applying Lemma 8 and the triangle inequality, we conclude that

$$\begin{aligned}\|t * (v * x) - t' * (v' * x)\| &\le \|t * (v * x) - t' * (v * x)\| + \|t' * (v * x) - t' * (v' * x)\|\\ &\le |t - t'| \cdot \|v * x\| + \|t' \cdot (v * x - v' * x)\|\\ &< \tfrac{1}{3}\varepsilon\|x\| + |t'| \cdot |v - v'| \cdot \|x\| < \tfrac{1}{3}\varepsilon\|x\| + \tfrac{1}{3}\varepsilon\|x\| = \tfrac{2}{3}\varepsilon\|x\|.\end{aligned}$$

Since G is a $\mathbb{Z}[\frac{1}{2}]$-module and the metric multiplication extends the dyadic multiplication, $t' * (v' * x) = t'(v'x) = (t'v')x$ and

$$\|t' * (v' * x) - rx\| = \|(t'v')x - rx\| = \|(t'v' - r)x\| = |t'v' - r| \cdot \|x\| = (r - t'v') \cdot \|x\|,$$

according to Lemma 7. The above equality and the triangle inequality imply

$$\begin{aligned}\big|\|t * (v * x) - rx\| - (r - tv)\|x\|\big| &\le \|t * (v * x) - t' * (v' * x)\|\\ &\quad + \big|\|t' * (v' * x) - rx\| - (r - tv)\|x\|\big|\\ &< \tfrac{2}{3}\varepsilon\|x\| + \big|(r - t'v')\|x\| - (r - tv)\|x\|\big|\\ &= \tfrac{2}{3}\varepsilon\|x\| + |tv - t'v'| \cdot \|x\|\\ &< \tfrac{2}{3}\varepsilon\|x\| + \tfrac{1}{3}\varepsilon\|x\| = \varepsilon\|x\|,\end{aligned}$$

which contradicts the choice of ε. This shows that $\|t * (v * x) - rx\| = (r - tv)\|x\|$. The latter equality, Lemma 8 and the strict convexity of the metric group G imply

$$t * (v * x) \in B[0, tv\|x\|] \cap B[rx, (r - tv)\|x\|] = \{(tv) * x\}$$

and hence $t * (v * x) = (tv) * x$ if $t, v \ge 0$. This case will be called "positive".

If $t \ge 0$ and $v \le 0$, then $t * (v * x) = t * (|v| * (-x)) = (t|v|) * (-x) = (tv) * x$, by the "positive" case and the definition of multiplication by negative real numbers.

If $t \le 0$ and $v \ge 0$, then $t * (v * x) = |t| * (-(v * x)) = |t| * (v * (-x)) = (|t|v) * (-x)$ $= (tv) * x$, by the "positive" case, the definition of metric multiplication by a negative real number, and Lemma 10.

If $t, v \le 0$, then $t * (v * x) = t * (|v| * (-x)) = |t| * (-(|v| * (-x)) = |t| * (|v|(-(-x)))$ $= (|t| \cdot |v|) \cdot x = (tv) \cdot x$, by the "positive" case, the definition of metric multiplication by a negative real number, and Lemma 10.

Therefore, in all four cases, we obtain the equality $t * (v * x) = (tv) * x$, completing the proof of the property (2) of Theorem 1.

(3) Given any $x \in X$, we should prove that $(t + v) * x = t * x + v * x$ for all $t, v \in \mathbb{R}$. The continuity of the functions $*_x$ and $+ : G \times G \to G$ implies that the set

$$F = \{(t, v) \in \mathbb{R} : (t + v) * x = t * x + v * x\}$$

is closed in the real plane $\mathbb{R} \times \mathbb{R}$. Since the metric multiplication extends the dyadic multiplication and G is a $\mathbb{Z}[\frac{1}{2}]$-module, the closed set F contains the dense subset $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$ of $\mathbb{R} \times \mathbb{R}$ and hence $F = \mathbb{R} \times \mathbb{R}$, witnessing that $(t + v) * x = t * x + v * x$ for all $t, v \in \mathbb{R}$.

(4) Given any $x, y \in G$ with $x + y = y + x$, we have to prove that $t * (x + y) = t * x + t * y$ for all $t \in \mathbb{R}$. The continuity of the functions $*_x$, $*_y$, $*_{x+y}$, and $+ : G \times G \to G$ implies that the set

$$F := \{t \in \mathbb{R} : t * (x + y) = t * x + t * y\}$$

is closed in the real line. Since the metric multiplication extends the dyadic multiplication and G is a $\mathbb{Z}[\frac{1}{2}]$-comodule, the closed set F contains the dense subset $\mathbb{Z}[\frac{1}{2}]$ of dyadic rationals and hence it coincides with $\mathbb{R}$, witnessing that $t * (x + y) = t * x + t * y$ for all $t \in \mathbb{R}$.

8 Proof of Theorem 2

Let $(G, +, 0, d)$ be an $\mathbb{R}$-normable metric group with a scalar multiplication operation

$$\cdot : \mathbb{R} \times G \ni (t, x) \mapsto tx \in G.$$

Claim 9. *If $K \subseteq G$ is a compact subgroup of G, then $K = \{0\}$.*

Proof. For every $x \in K$ the function $\| \cdot \| : K \to \mathbb{R}$ is continous, hence the image

$$B = \{\|x\| : x \in K\}$$

is bounded in the real line. Assuming that $K \neq \{0\}$ we could find a positive integer n such that $\|nx\| = n \cdot \|x\| \notin B$ and conclude that $nx \notin K$, which is not possible as K is a subgroup of G. ☐

To finish the proof we will use the result [10] of K. Iwasawa stating that a connected locally compact topological group contains a compact invariant neighborhood of the identity if and only if the group is compact-by-abelian, i.e. contains a compact normal subgroup whose quotient group is abelian.

Claim 10. *Every $\mathbb{R}$-normable metric group G is path-connected.*

Proof. For every $x, y \in G$, the map $\gamma : [0, 1] \ni t \mapsto x + t(-x + y) \in G$ is a continuous path connecting the points x and y. ☐

Claim 10 ensures that G is connected. Since G is locally compact we can pick an open neighborhood U of identity $0 \in G$ such that $\overline{U}$ is compact in G. Let $B \subseteq U$ be a closed ball with center in $0 \in G$. Since $B \subseteq \overline{U}$ is closed, B is compact in G. Since the metric d is translation invariant, the ball B is a compact invariant neighborhood of the identity.

Then G contains a compact normal subgroup K whose quotient group is abelian by K. Iwasawa's result. Claim 9 ensures that $K = \{0\}$. Hence $G = G/K$ is abelian and being $\mathbb{R}$-normable, is a normed space. By Riesz's Theorem (see [6, Theorem 1.24]), the locally compact normed space G is finite-dimensional.

9 Proof of Theorem 3

By The Coincidence Theorem (see [5, Theorem 1.7.7]), the small inductive dimension and the covering dimension coincide for all metrizable separable spaces. In particular, those two dimensions coincide for all compact metrizable spaces. By the Subspace Theorem (see [5, Theorem 1.1.2]), the dimension is monotone in the sense that $\dim(X) \leq \dim(Y)$ for any closed subspace X of a metrizable separable space Y. By the Sum Theorem (see [5, Theorem 1.5.3]), the dimension is countably additive in the sense that $\dim(\bigcup_{n\in\omega} X_n) = \sup_{n\in\omega} \dim(X_n)$ for any sequence $(X_n)_{n\in\omega}$ of closed subspaces of a metrizable separable space.

Having in mind this information from Dimension Theory, we now are able to prove the equivalence of the conditions (1)–(3) of Theorem 3. In fact, the implication (1) ⇒ (2) has been proved in Theorem 2. The implication (2) ⇒ (3) follows from the Subspace Theorem (see [5, Theorem 1.1.2]). The implication (3) ⇒ (1) will be deduced from the following lemma.

Lemma 11. *For a compactly finite-dimensional metric group G the number*

$$\text{co-dim}(G) = \sup\big\{\dim(K) : K \textit{ is a compact subspace of } G\big\}$$

is finite.

Proof. Assume to the contrary that there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subspaces of G with $\dim(K_n) > n$. We aim to construct a compact subspace of G of infinite dimension. For every $n \in \mathbb{N}$, the compact K_n admits a finite cover $\mathcal{B}_n$ by closed balls of radius $\frac{1}{n}$ centered at points of the set K_n. By the countable additivity of dimension, $\dim(K_n) > n$ implies that for some ball $B_n \in \mathcal{B}_n$, the set $B_n \cap K_n$ has dimension $\dim(B_n) > n$. Let b_n be the center of the ball B_n. Observe that the compact set $C_n := -b_n + (B_n \cap K_n)$ is contained in the closed ball of radius $\frac{1}{n}$ around 0 and has dimension $\dim(C_n) = \dim(B_n \cap K_n) > n$.

We claim that the subspace $K := \bigcup_{n\in\mathbb{N}} C_n$ of G is compact. Given any open cover $\mathcal{U}$ of K, choose an open set $U_0 \in \mathcal{U}$ containing $0 \in K$ and find a number $m \in \mathbb{N}$ such that $\{x \in G : \|x\| \le \frac{1}{n}\} \subseteq U_0$. Then $\bigcup_{n\ge m} C_n \subseteq U_0$. By the compactness of the set $\bigcup_{n<m} C_n$, there exists a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup_{n=1}^{m-1} C_n \subseteq \bigcup \mathcal{V}$. Then $\{U_0\} \cup \mathcal{V} \subseteq \mathcal{U}$ is a finite subcover of K, witnessing that the subspace K of X is compact. Since $\dim(K) \ge \dim(C_n) > n$ for all $n \in \mathbb{N}$, the compact space K is infinite-dimensional, which contradicts the compact finite-dimensionality of the metric space G. □

Since G is path-connected by Claim 10 and compactly finite-dimensional, it is locally-compact by the Key Lemma in the paper [2] of T. Banakh and L. Zdomskyy.

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Строга опуклість є фундаментальною геометричною властивістю банахових просторів, що забезпечує єдиність найкращих апроксимацій, відіграє центральну роль у теорії дуальності та слугує підґрунтям для застосувань у оптимізації, теорії апроксимацій та теорії нерухомих точок. Незважаючи на те, що строга опуклість зазвичай означується для нормованих або банахових просторів, це чисто метрична властивість, яку можна визначити без залучення лінійної чи опуклої структури. У цій роботі доведено, що кожна строго опукла абелева метрична група має канонічну структуру нормованого простору над полем дійсних чисел. Щоб довести це твердження, ми спершу показуємо, що кожен строго опуклий метричний простір є геодезійним у тому сенсі, що будь-які дві точки з'єднуються єдиним прямолінійним сегментом, а також що кожна строго опукла абелева метрична група є однозначно 2-подільною. Це дозволяє нам наділити групу структурою $\mathbb{Z}[\frac{1}{2}]$-комодуля і продовжити операцію множення на двійково-раціональні числа до операції множення на дійсні числа, використовуючи строгу опуклість метрики.

Ключові слова і фрази: строго опуклий метричний простір, нормований простір, абелева метрична група, геодезійний метричний простір, скінченновимірний нормований простір, локально компактна метрична група, $\mathbb{R}$-нормована метрична група.