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Approximation with nonlinear Lototsky-Durrmeyer type operators

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The main objective of this study is to construct nonlinear Lototsky-Durrmeyer type operators and to investigate the approximation properties of these newly constructed operators. Research on Lototsky operators began with the work [Canad. J. Math. 1966, **18**, 89–91] by J.P. King and approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in 1983. In recent years, both theories have gained popularity through the work of many scientists. In this context, our primary motivation is to introduce the nonlinear Lototsky-Durrmeyer type operators arising from the combination of these two theories and subsequently examine the approximation properties of these operators, particularly for functions with bounded variation and derivatives with bounded variation.

Key words and phrases: Lototsky operator, nonlinear operator, bounded variation.

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Introduction

The basis of the theory of approximation of functions of a real variable is due to Weierstrass; it states that every continuous real-valued function h defined on $[a,b]$ can be approximated by algebraic polynomials. The original proof of this theorem was long and complicated. Therefore, the result of Weierstrass provoked many famous mathematicians to find simpler and more constructive proofs.

For a bounded real-valued function h defined on the interval $[0,1]$, the Bernstein operators $(B_n h)$ are defined by

$$(B_n h)(z) = \sum_{k=0}^{n} h\left(\frac{k}{n}\right) p_{n,k}(z), \quad n \geq 1,$$

where $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$ is the Bernstein basis, $0 \leq z \leq 1$. These polynomials were introduced by S.N. Bernstein [3] in 1912 to give the first constructive proof of the Weierstrass approximation theorem.

Even though these operators are probably the most studied linear positive operators, due to the importance of the Bernstein polynomials, a variety of their generalizations and related problems have been investigated. Let h be a Lebesgue integrable function defined on the interval $[0,1]$ and let $\mathbb{N} = \{1,2,3,\dots\}$. The classical Durrmeyer operators D_n applied to h are defined as

$$(D_n h)(z) = \int_0^1 h(t) H_n(z,t)\, dt, \quad 0 \leq z \leq 1,$$

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where
$$H_n(z,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(z)p_{n,k}(t)$$
and $\{p_{n,k}(z)\}$ is the Bernstein basis. These operators are the integral modification of the Bernstein polynomials and were introduced by J.L. Durrmeyer [4] to approximate Lebesgue integrable functions on the interval $[0,1]$.

Let $I \subset \mathbb{R}$. As usual, we denote by $B(I)$ the space of all bounded functions and by $C(I)$ the space of all continuous functions defined on I, endowed with the usual sup-norm. In 1966, J.P. King [9] introduced the Lototsky-Bernstein operators $L_n : B[0,1] \to C[0,1]$ for $n \in \mathbb{N}$ as follows
$$(L_n h)(z) = \sum_{k=0}^{n} h\Big(\frac{k}{n}\Big) a_{n,k}(z), \quad z \in [0,1],$$
using the basis functions $a_{n,k}(z)$ obtained from the following relation
$$\prod_{i=1}^{n} (g_i(z)y + 1 - g_i(z)) = \sum_{k=0}^{n} a_{n,k}(z)y^k, \quad y \in \mathbb{R},$$
where
$$a_{n,k}(z) = \sum_{\substack{J\cup\bar{J}=\mathbb{N}_n \\ \mathrm{Card}(J)=k}} \prod_{i\in\bar{J}} (1 - g_i(z)) \prod_{i\in J} g_i(z), \tag{1}$$
$(g_i)_{i\in\mathbb{N}}$ is a sequence of continuous functions $g_i : [0,1] \to [0,1]$, and $a_{0,0}(z) = 1$, $a_{0,k}(z) = 0$ for $k > 0$. J.P. King established necessary conditions on the sequence $(g_i)_{i\in\mathbb{N}}$ that guarantee that $(L_n)_{n\geq 1}$ forms an approximation process on $C[0,1]$. This result can be stated as follows: if
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g_i(z) = z \quad \text{uniformly for } z \in [0,1],$$
then
$$\lim_{n\to\infty} (L_n h)(z) = h(z) \quad \text{uniformly for } z \in [0,1],$$
for every $h \in C[0,1]$. It is clear that, in the special case $g_i(z) = z$ for all $i \in \mathbb{N}$, the functions $(a_{n,k})_{0\leq k\leq n}$, $n \in \mathbb{N}$, coincide with the Bernstein basis $(p_{n,k})_{0\leq k\leq n}$. Consequently, the operator L_n reduces to the Bernstein operator B_n. In recent years, many authors have studied these Lototsky-Bernstein operators (see, for example, [12–14]). Furthermore, U. Abel and O. Agratini [1] presented an integral Durrmeyer-type extension and investigated some approximation properties of these operators. They defined a particular Durrmeyer-type construction that involves both the Lototsky-Bernstein and the classical Bernstein bases.

Let $L_1([0,1])$ stand for the Banach space of all real-valued integrable functions on $[0,1]$ endowed with the norm $\|h\|_1 = \int_0^1 |h(z)|dz$. Then $LD_n : L_1([0,1]) \to C([0,1])$ can be defined by the formula
$$(LD_n h)(z) = (n+1)\sum_{k=0}^{n} a_{n,k}(z)\int_0^1 p_{n,k}(t)h(t)\,dt, \quad z \in [0,1], \tag{2}$$
where $a_{n,k}(z)$ is defined by (1).

In recent years, approximation by nonlinear operators has become as popular as its linear part. In fact, this popularity have started with the principal work of J. Musielak [11]. Thereafter, in [2], the approximation theory was extended to nonlinear operators through the significant contributions of C. Bardaro and G. Vinti. By using the techniques due to J. Musielak [11], the authors studied the rate of pointwise convergence of nonlinear Bernstein and nonlinear Durrmeyer type operators [5–8] for functions of bounded varation on $[0,1]$. Also, for further reading, we refer the reader to new paper [10] on Durrmeyer type Lototsky-Chlodowsky operators.

In this study, our primary motivation is to introduce the nonlinear Lototsky-Durrmeyer type operators arising from the combination of these theories and subsequently examine the approximation properties of these operators, particularly for functions with bounded variation and derivatives with bounded variation. Next section comprises essential notations and definitions. In Section 2, we present a result that is necessary for proving the main theorems. In the last section, we state the main theorems of this work and provide their proofs, along with some corollaries.

1 Preliminaries

Now we assemble the main definitions and notations which will be used in construction of the operator.

Let $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and non-decreasing function such that $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$. We call such a function a ψ-function, and denote the class of all ψ-functions by Ψ. Let X be the set of all bounded Lebesgue measurable functions $h : [0,1] \to \mathbb{R}$.

Let $\{H_n(z,t,u)\}_{n\in\mathbb{N}}$ be a sequence of functions $H_n(z,t,u)$ defined by

$$H_n(z,t,u) = (n+1)\sum_{k=0}^{n} a_{n,k}(z)p_{n,k}(t)K_n(u),$$

where $K_n : \mathbb{R} \to \mathbb{R}$ is a function such that $K_n(0) = 0$, $p_{n,k}(t)$ is the Bernstein basis, and $a_{n,k}(z)$ is defined by equation (1).

We formulate the conditions that will be used in the sequel.

(a) The function $K_n : \mathbb{R} \to \mathbb{R}$ is such that the inequality

$$|K_n(u) - K_n(v)| \le \psi(|u-v|)$$

holds for every $u, v \in \mathbb{R}$ and $n \in \mathbb{N}$, where $\psi \in \Psi$. That is, K_n satisfies a $(L\text{-}\psi)$-Lipschitz condition.

(b) Set

$$H_n(z,t) = (n+1)\sum_{k=0}^{n} a_{n,k}(z)\, p_{n,k}(t)$$

and

$$A_n(z) = \int_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{(1-z)}{n^{\gamma/\beta}}} H_n(z,t)\,dt$$

for $z \in (0,1)$, where $\beta > 0$, $\gamma \ge 1$. The use of the function $A_n(z)$ concerns the behaviour of the approximation near the point z. Suppose that there exist constants $\beta > 0$ and $\gamma \ge 1$ and a finite function $B_n(z)$ such that

$$n^{\gamma/\beta}\left(LD_n\,|t-z|^{\beta}\right)(z) \le B_n(z), \quad z \in (0,1)$$

uniformly with respect to $n \in \mathbb{N}$.

(c) Denote $r_n(u) = K_n(u) - u$ for $u \in \mathbb{R}$, $n \in \mathbb{N}$, and assume that

$$\lim_{n\to\infty} r_n(u) = 0$$

uniformly with respect to u. Equivalently, for all sufficiently large n, we have

$$\sup_u |r_n(u)| = \sup_u |K_n(u) - u| \le \frac{1}{\mu(n)},$$

where $\mu : \mathbb{N} \to \mathbb{R}^+$ is an increasing and continuous function satisfying $\lim_{n\to\infty} \mu(n) = \infty$.

Let $DBV(I)$ denote the class of differentiable functions on $I \subset \mathbb{R}$ whose derivatives are of bounded variation on I, that is,

$$DBV(I) = \{h : h' \in BV(I)\}.$$

2 Auxiliary Results

Now we present several results which are necessary to prove our theorems.

Lemma 1 ([1]). *Let LD_n be the operators defined by* (2). *The following identities*

$$LD_n e_0 = e_0,$$

$$LD_n e_1 = \frac{n}{n+2} L_n e_1 + \frac{1}{n+2},$$

$$LD_n e_2 = \frac{n^2}{(n+2)(n+3)} L_n e_2 + \frac{3n}{(n+2)(n+3)} L_n e_1 + \frac{2}{(n+2)(n+3)}$$

hold for each $n \in \mathbb{N}$.

Lemma 2 ([1]). *The second-order central moments of the operators LD_n satisfy the following relation*

$$\begin{aligned}(LD_n \varphi_z^2)(z) = {} & \frac{n^2}{(n+2)(n+3)} \left(\alpha_n^2(z) + \frac{1}{n^2} \sum_{i=1}^{n} g_i(z)(1 - g_i(z)) \right) \\ & + \frac{3n\alpha_n(z) + 2}{(n+2)(n+3)} - 2z\alpha_n(z) + z^2 \le (\alpha_n(z) - z)^2 + \frac{4}{n},\end{aligned}$$

where

$$\alpha_n(z) = \frac{1}{n} \sum_{i=1}^{n} g_i(z), \quad z \in [0,1], \qquad \text{and} \qquad \varphi_z(t) = t - z, \quad (t,z) \in [0,1] \times [0,1].$$

Lemma 3 ([5]). *For all $z \in (0,1)$ and each $n \in \mathbb{N}$, one has*

$$\lambda_n(z,w) = \int_0^w H_n(z,u)du \le \frac{B_n(z)}{(z-w)^\beta n^{\gamma/\beta}}, \quad 0 \le w < z, \tag{3}$$

and

$$1 - \lambda_n(z,w) = \int_w^1 H_n(z,u)du \le \frac{B_n(z)}{(w-z)^\beta n^{\gamma/\beta}}, \quad z < w < 1.$$

3 Convergence Results

Now we consider the following type of nonlinear Lototsky-Durrmeyer operators

$$(NLD_n h)(z) = \int_0^1 H_n(z,t,h(t))\, dt \tag{4}$$

with

$$H_n(z,t,h(t)) = (n+1)\sum_{k=0}^{n} a_{n,k}(z) p_{n,k}(t) K_n(h(t)) = H_n(z,t)K_n(h(t)).$$

We assume that these operators are defined for every $h \in \operatorname{Dom} NLD_n$, where $\operatorname{Dom} NLD_n$ is the subset of X on which NLD_n is well-defined. Some existence theorems for these type nonlinear operators can be found in [2]. For any function h with finite one-sided limits $h(z+)$ and $h(z-)$ at every point $z \in (0,1)$, let

$$h_z(t) = \begin{cases} h(t) - h(z+), & z < t \leq 1, \\ 0, & t = z, \\ h(t) - h(z-), & 0 \leq t < z. \end{cases} \tag{5}$$

Let $\bigvee\limits_0^1 \psi(|h_z|)$ denote the total variation of $\psi(|h_z|)$ on $[0,1]$, and let the symbol $[a]$ denote the greatest integer not exceeding a.

Theorem 1. *Let $\psi \in \Psi$ and $h \in X$ be such that $\psi \circ |h| \in BV([0,1])$. Suppose that $H_n(z,t,u)$ satisfies conditions (a), (b) and (c). Then, for each $z \in (0,1)$ such that the one-sided limits $h(z+)$ and $h(z-)$ exist, we have*

$$\begin{aligned}(NLD_n h)(z) - &\left[\psi\left(\left|\frac{h(z+)+h(z-)}{2}\right|\right) + \psi\left(\left|\frac{h(z+)-h(z-)}{2}\right|\right)\right] \\ &\leq B^*(z) n^{-\gamma/\beta}\left[\bigvee_0^1 \psi(|h_z|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|)\right] + A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|),\end{aligned}$$

where $B^*(z) = B_n(z)\max\left\{z^{-\beta}, (1-z)^{-\beta}\right\}$ and $\beta > 0$.

Proof. For any $h \in X$, using (5) we can write

$$\begin{aligned} h(t) = \frac{h(z+)+h(z-)}{2} &+ h_z(t) + \frac{h(z+)-h(z-)}{2}\operatorname{sgn}(t-z) \\ &+ \delta_z(t)\left[h(z) - \frac{h(z+)+h(z-)}{2}\right], \end{aligned} \tag{6}$$

where

$$\delta_z(t) = \begin{cases} 1, & z = t, \\ 0, & z \neq t. \end{cases}$$

Applying the operator (4) to (6), using condition (a), we have

$$\begin{aligned}(NLD_n h)(z) &= \int_0^1 H_n(z,t,h(t))\,dt = \int_0^1 H_n(z,t)K_n(h(t))\,dt \le \int_0^1 H_n(z,t)\psi(|h(t)|)\,dt \\ &\le \psi\left(\left|\frac{h(z+)+h(z-)}{2}\right|\right)\int_0^1 H_n(z,t)\,dt + \int_0^1 H_n(z,t)\psi(|h_z(t)|)\,dt \\ &+ \int_0^1 H_n(z,t)\psi\left(\left|\frac{h(z+)-h(z-)}{2}\right| |\operatorname{sgn}(t-z)|\right)dt \\ &+ \int_0^1 H_n(z,t)\psi\left(\left|h(z)-\frac{h(z+)+h(z-)}{2}\right|\delta_z(t)\right)dt.\end{aligned}$$

By the definition of $\delta_z(t)$, the last integral is zero. Then, using that $|\operatorname{sgn}(t-z)| \le 1$ and $\int\limits_0^1 H_n(z,t)dt = 1$, we get

$$(NLD_n h)(z) - \left[\psi\left(\left|\frac{h(z+)+h(z-)}{2}\right|\right) + \psi\left(\left|\frac{h(z+)-h(z-)}{2}\right|\right)\right] \le \int_0^1 H_n(z,t)\psi(|h_z(t)|)\,dt.$$

Using condition (b), we can separate the last integral as follows

$$\int_0^1 H_n(z,t)\psi(|h_z(t)|)\,dt = \left(\int_0^{z-\frac{z}{n^{\gamma/\beta}}} + \int_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} + \int_{z+\frac{1-z}{n^{\gamma/\beta}}}^1\right) H_n(z,t)\psi(|h_z(t)|)\,dt = I_1 + I_2 + I_3.$$

Since $h_z(z) = 0$ and $\psi(0) = 0$, using condition (b) we can estimate I_2 as

$$\begin{aligned}I_2 &= \int_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} [\psi(|h_z(t)|) - \psi(|h_z(z)|)]\, H_n(z,t)\,dt \\ &\le \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|) \int_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} H_n(z,t)dt \le A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|).\end{aligned} \tag{7}$$

Next we estimate I_1. By using Lebesgue-Stieltjes integration by parts with $y = z - \frac{z}{n^{\gamma/\beta}}$, we get

$$I_1 = \int_0^y \psi(|h_z(t)|)H_n(z,t)\,dt = \psi(|h_z(y)|)\lambda_n(z,y) - \int_0^y \lambda_n(z,t)\,d_t(\psi(|h_z(t)|)).$$

Using the fact that $\psi(|h_z(y)|) = |\psi(|h_z(y)|) - \psi(|h_z(z)|)| \le \bigvee\limits_y^z \psi(|h_z|)$ and (3), we obtain

$$\begin{aligned}|I_1| &\le \bigvee_y^z \psi(|h_z|)\,\lambda_n(z,y) + \int\limits_0^y \lambda_n(z,t)\,d_t\left(-\bigvee_t^z \psi(|h_z|)\right) \\ &\le \bigvee_y^z \psi(|h_z|)\, B_n(z)\, z^{-\beta} n^{\gamma(\beta-1)/\beta} + B_n(z)\, n^{-\gamma/\beta} \int\limits_0^y (z-t)^{-\beta}\, d_t\left(-\bigvee_t^z \psi(|h_z|)\right).\end{aligned}$$

In the last integral, first using integration by parts and then substituting $z-\frac{z}{u^{1/\beta}}$ for the variable t, we consequently get

$$\begin{aligned}
|I_1| &\leq \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z} \psi(|h_z|)B_n(z)z^{-\beta}n^{\gamma(\beta-1)/\beta} \\
&\leq B_n(z)n^{-\gamma/\beta}\Bigg[-z^{-\beta}n^{\gamma}\bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z}\psi(|h_z|)+z^{-\beta}\bigvee_{0}^{z}\psi(|h_z|) \\
&\qquad\qquad + \int_0^{z-\frac{z}{n^{\gamma/\beta}}}\bigvee_{t}^{z}\psi(|h_z|)\frac{\beta}{(z-t)^{\beta+1}}\,dt\Bigg] \\
&= B_n(z)n^{-\gamma/\beta}\Bigg[z^{-\beta}\bigvee_{0}^{z}\psi(|h_z|)+\int_0^{z-\frac{z}{n^{\gamma/\beta}}}\bigvee_{t}^{z}\psi(|h_z|)\frac{\beta}{(z-t)^{\beta+1}}\,dt\Bigg] \\
&= B_n(z)n^{-\gamma/\beta}\Bigg[z^{-\beta}\bigvee_{0}^{z}\psi(|h_z|)+\frac{1}{z^{\beta}}\int_1^{n^{\gamma}}\bigvee_{z-\frac{z}{u^{1/\beta}}}^{z}\psi(|h_z|)\,du\Bigg] \\
&\leq B_n(z)n^{-\gamma/\beta}z^{-\beta}\Bigg[\bigvee_{0}^{z}\psi(|h_z|)+\sum_{k=1}^{[n^{\gamma}]}\bigvee_{z-\frac{z}{k^{1/\beta}}}^{z}\psi(|h_z|)\Bigg].
\end{aligned} \tag{8}$$

Using the similar method, we find

$$|I_3| \leq B_n(z)\,n^{-\gamma/\beta}\,(1-z)^{-\beta}\left[\bigvee_{z}^{1}\psi(|h_z|)+\sum_{k=1}^{[n^{\gamma}]}\bigvee_{z}^{z+\frac{(1-z)}{n^{\gamma/\beta}}}\psi(|h_z|)\right]. \tag{9}$$

Finally, combining (7), (8) and (9), we obtain

$$\begin{aligned}
\int_0^1 H_n(z,t)\psi(|h_z(t)|)\,dt &\leq B_n(z)n^{-\gamma/\beta}z^{-\beta}\Bigg[\bigvee_{0}^{z}\psi(|h_z|)+\sum_{k=1}^{[n^{\gamma}]}\bigvee_{z-\frac{z}{k^{1/\beta}}}^{z}\psi(|h_z|)\Bigg] \\
&\quad + B_n(z)n^{-\gamma/\beta}(1-z)^{-\beta}\Bigg[\bigvee_{z}^{1}\psi(|h_z|)+\sum_{k=1}^{[n^{\gamma}]}\bigvee_{z}^{z+\frac{1-z}{k^{1/\beta}}}\psi(|h_z|)\Bigg] \\
&\quad + A_n(z)\bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}}\psi(|h_z|) \\
&\leq B^*(z)n^{-\gamma/\beta}\left[\bigvee_{0}^{1}\psi(|h_z|)+\sum_{k=1}^{[n^{\gamma}]}\bigvee_{z-\frac{z}{k^{1/\beta}}}^{z+\frac{1-z}{k^{1/\beta}}}\psi(|h_z|)\right] \\
&\quad + A_n(z)\bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}}\psi(|h_z|).
\end{aligned}$$

This completes the proof of the theorem. $\square$

Corollary 1. *If we choose $h \in C[0,1]$ in Theorem 1, then we get*

$$(NLD_n h)(z) - \psi(|h(z)|) \leq B^*(z) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|h_z|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{z-\frac{z}{k^{1/\beta}}}^{z+\frac{1-z}{k^{1/\beta}}} \psi(|h_z|) \right] + A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|).$$

Theorem 2. *Let $\psi \in \Psi$ and $h \in X$ be such that $\psi \circ |h| \in BV([0,1])$. Suppose that $H_n(z,t,u)$ satisfies conditions (a), (b) and (c). Then, for each $z \in (0,1)$ such that the one-sided limits $h(z+)$ and $h(z-)$ exist, we have*

$$\begin{aligned} \left| (NLD_n h)(z) - \frac{h(z+)+h(z-)}{2} \right| &\leq B^*(z) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|h_z|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{z-\frac{z}{k^{1/\beta}}}^{z+\frac{1-z}{k^{1/\beta}}} \psi(|h_z|) \right] \\ &+ A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} \psi(|h_z|) + \psi\left(\left| \frac{h(z+)-h(z-)}{2} \right| \right) + \frac{1}{\mu(n)}, \end{aligned}$$

where $B^*(z) = B_n(z) \max\left\{ z^{-\beta}, (1-z)^{-\beta} \right\}$, $\beta > 0$.

Proof. For any $h \in X$ we have

$$\begin{aligned} \left| (NLD_n h)(z) - \frac{h(z+)-h(z-)}{2} \right| &= \left| \int_0^1 H_n(z,t) K_n(h(t)) dt - \frac{h(z+)-h(z-)}{2} \right| \\ &\leq \int_0^1 H_n(z,t) \psi\left(\left| h(t) - \frac{h(z+)-h(z-)}{2} \right| \right) dt \\ &+ \int_0^1 H_n(z,t) \left| K_n\left(\frac{h(z+)-h(z-)}{2} \right) - \frac{h(z+)-h(z-)}{2} \right| dt \\ &\leq \int_0^1 H_n(z,t) \psi(|h_z(t)|) dt \\ &+ \int_0^1 H_n(z,t) \psi\left(\left| \frac{h(z+)-h(z-)}{2} \right| |\mathrm{sgn}(t-z)| \right) dt \\ &+ \int_0^1 H_n(z,t) \psi\left(\left| h(z) - \frac{h(z+)+h(z-)}{2} \right| \delta_z(t) \right) dt \\ &+ \int_0^1 H_n(z,t) \left| K_n\left(\frac{h(z+)-h(z-)}{2} \right) - \frac{h(z+)-h(z-)}{2} \right| dt. \end{aligned}$$

By using the required results as in the proof of the previous theorem and in view of (c) we get the desired result. □

Corollary 2. *If we choose $h \in C[0,1]$ in Theorem 2, then we get*

$$\begin{aligned} |(NLD_n h)(z) - h(z)| &\leq B^*(z) n^{-\gamma/\beta} \left[\bigvee_0^1 \psi(|h_z|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{(1-z)}{n^{\gamma/\beta}}} \psi(|h_z|) \right] \\ &+ A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{(1-z)}{n^{\gamma/\beta}}} \psi(|h_z|) + \frac{1}{\mu(n)}. \end{aligned}$$

Corollary 3. *If we choose $h \in C[0,1]$ and $\psi(t) = t$ (that is, strongly Lipschitz condition) in Theorem 2, then we obtain*

$$|(NLD_n h)(z) - h(z)| \leq B^*(z) n^{-\gamma/\beta} \left[\bigvee_0^1 (|h_z|) + \sum_{k=1}^{[n^\gamma]} \bigvee_{z-\frac{z}{k^{1/\beta}}}^{z+\frac{1-z}{k^{1/\beta}}} (|h_z|) \right] + A_n(z) \bigvee_{z-\frac{z}{n^{\gamma/\beta}}}^{z+\frac{1-z}{n^{\gamma/\beta}}} (|h_z|) + \frac{1}{\mu(n)}.$$

Theorem 3. *Let $\psi \in \Psi$ and h be a function with derivatives of bounded variation on $[0,1]$. Then, for every $z \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned} |(NLD_n h)(z) - h(z)| \leq & \left| \frac{h'(z+) + h'(z-)}{2} \right| \left(\frac{n\alpha_n(z) + 1 - (n+2)z}{n+2} \right) \\ & + \left| \frac{h'(z+) - h'(z-)}{2} \right| \sqrt{(\alpha_n(z) - z)^2 + \frac{4}{n}} \\ & + \left(\frac{2}{n} + \left((\alpha_n(z) - z)^2 + \frac{4}{n} \right) \frac{1}{z(1-z)} \right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z+\frac{1-z}{k}} (h_z') + \frac{1}{\mu(n)}. \end{aligned}$$

Proof. We can write

$$\begin{aligned} (NLD_n h)(z) - h(z) &= \int_0^1 K_n(h(t)) H_n(z,t)\, dt - h(z) \\ &= \int_0^1 [K_n(h(t)) - h(t)] H_n(z,t)\, dt + \int_0^1 [h(t) - h(z)] H_n(z,t)\, dt = I_1 + I_2. \end{aligned}$$

Since $h(t) \in DBV[0,1]$, we can write I_2 as

$$\begin{aligned} I_2 &= \int_0^z [h(t) - h(z)] H_n(z,t)\, dt + \int_z^1 [h(t) - h(z)] H_n(z,t)\, dt \\ &= -\int_0^z \left[\int_t^z h'(u) du \right] H_n(z,t)\, dt + \int_z^1 \left[\int_z^t h'(u) du \right] H_n(z,t)\, dt = -I_{21} + I_{22}. \end{aligned}$$

For any $h(t) \in DBV[0,1]$, we can decompose $h'(t)$ into four parts as in (6), so we can write

$$\begin{aligned} I_{21} = & \frac{h'(z+) + h'(z-)}{2} \int_0^z (z-t) H_n(z,t)\, dt + \int_0^z \left[\int_t^z h_z'(u)\, du \right] H_n(z,t)\, dt \\ & - \frac{h'(z+) - h'(z-)}{2} \int_0^z (z-t) H_n(z,t)\, dt \\ & + \left[h'(z) - \frac{h'(z+) + h'(z-)}{2} \right] \int_0^z \left[\int_t^z \delta_z(u)\, du \right] H_n(z,t)\, dt. \end{aligned}$$

From the fact $\int_t^z \delta_z(u)\, du = 0$ and using a similar method for I_{22}, we have

$$\begin{aligned} I_2 &= -I_{21} + I_{22} \\ &= \frac{h'(z+) + h'(z-)}{2} \int_0^1 (t-z) H_n(z,t)\, dt + \frac{h'(z+) - h'(z-)}{2} \int_0^z |t-z| H_n(z,t)\, dt \\ &\quad - \int_0^z \left[\int_t^z h_z'(u) du \right] H_n(z,t)\, dt + \int_z^1 \left[\int_z^t h_z'(u) du \right] H_n(z,t)\, dt. \end{aligned}$$

Since
$$\int_0^1 (t-z)H_n(z,t)\,dt = (LD_n(t-z))(z) = \frac{n\alpha_n(z)+1-(n+2)z}{n+2}$$
and
$$\int_0^1 |t-z|H_n(z,t)\,dt = (LD_n|t-z|)(z) \le \sqrt{(LD_n(t-z)^2)(z)} \le \sqrt{(\alpha_n(z)-z)^2+\frac{4}{n}},$$
we can write the absolute value of I_2 as
$$\begin{aligned}|I_2| \le & \left|\frac{h'(z+))+h'(z-)}{2}\right| |(LD_n(t-z))(z)| + \left|\frac{h'(z+))-h'(z-)}{2}\right| |(LD_n|t-z|)(z)| \\ & + \left| -\int_0^z \left[\int_t^z h'_z(u)\,du\right] H_n(z,t)\,dt\right| + \left|\int_z^1 \left[\int_z^t h'_z(u)\,du\right] H_n(z,t)\,dt\right|.\end{aligned}$$
Using (3) and partial integration, we get
$$\begin{aligned}\int_{z-\frac{z}{\sqrt{n}}}^{z} |h'_z(t)|\,\lambda_n(z,t)\,dt &= \int_{z-\frac{z}{\sqrt{n}}}^{z} |h'_z(t)-h'_z(z)|\,\lambda_n(z,t)\,dt \\ &\le \int_{z-\frac{z}{\sqrt{n}}}^{z} \bigvee_t^z (h'_z)\,dt \le \bigvee_{z-\frac{z}{u}}^{z} (h'_z)\int_{z-\frac{z}{\sqrt{n}}}^{z} dt.\end{aligned}$$
Thus,
$$\begin{aligned}\left| -\int_0^z \left[\int_t^z h'_z(u)\,du\right] H_n(z,t)\,dt\right| &\le \int_0^z |h'_z(t)|\,\lambda_n(z,t)\,dt \\ &= \int_0^{z-\frac{z}{\sqrt{n}}} |h'_z(t)|\,\lambda_n(z,t)\,dt + \int_{z-\frac{z}{\sqrt{n}}}^{z} |h'_z(t)|\,\lambda_n(z,t)\,dt.\end{aligned}$$
Using $h'_z(z)=0$, $\lambda_n(z,t)\le 1$ and the change of variables $t = z-\frac{z}{u}$, we have
$$\begin{aligned}\int_{z-\frac{z}{\sqrt{n}}}^{z} |h'_z(t)|\,\lambda_n(z,t)\,dt &= \int_{z-\frac{z}{\sqrt{n}}}^{z} |h'_z(t)-h'_z(z)|\,\lambda_n(z,t)\,dt \\ &\le \int_{z-\frac{z}{\sqrt{n}}}^{z} \bigvee_t^z (h'_z)\,dt \le \bigvee_{z-\frac{z}{u}}^{z} (h'_z)\int_{z-\frac{z}{\sqrt{n}}}^{z} dt.\end{aligned}$$
Besides from (3) and using the change of variable $t = z-\frac{z}{u}$ again, we have
$$\begin{aligned}\int_0^{z-\frac{z}{\sqrt{n}}} |h'_z(t)|\,\lambda_n(z,t)\,dt &\le \left((\alpha_n(z)-z)^2+\frac{4}{n}\right)\int_0^{z-\frac{z}{\sqrt{n}}} \bigvee_t^z (h'_z)\,\frac{dt}{(z-t)^2} \\ &= \left((\alpha_n(z)-z)^2+\frac{4}{n}\right)\int_1^{\sqrt{n}} \bigvee_{z-\frac{z}{u}}^{z} (h'_z)\,\frac{\left(\frac{z}{u^2}\right)du}{\left(-\frac{z}{u}\right)^2} \\ &= \left((\alpha_n(z)-z)^2+\frac{4}{n}\right)\frac{1}{z}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z} (h'_z).\end{aligned}$$

Hence, we have

$$\left| -\int_0^z \left[\int_t^z h'_z(u)\,du \right] H_n(z,t)\,dt \right| \leq \frac{z}{\sqrt{n}} \bigvee_{z-\frac{z}{k}}^{z} (h'_z) + \left((\alpha_n(z)-z)^2 + \frac{4}{n} \right) \frac{1}{z} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z} (h'_z).$$

Since

$$\frac{z}{\sqrt{n}} \bigvee_{z-\frac{z}{k}}^{z} (h'_z) \leq \frac{2z}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z} (h'_z),$$

we obtain

$$\left| -\int_0^z \left[\int_t^z h'_z(u)\,du \right] H_n(z,t)\,dt \right| \leq \left(\frac{2z}{n} + \left((\alpha_n(z)-z)^2 + \frac{4}{n} \right) \frac{1}{z} \right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z} (h'_z).$$

Using a similar method, we get

$$\left| \int_z^1 \left[\int_z^t h'_z(u)\,du \right] H_n(z,t)\,dt \right| \leq \left(\frac{2(1-z)}{n} + \left((\alpha_n(z)-z)^2 + \frac{4}{n} \right) \frac{1}{1-z} \right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z}^{z+\frac{1-z}{k}} (h'_z).$$

Collecting these estimates, we obtain

$$\begin{aligned} |I_2| &\leq \left| \frac{h'(z+)) + h'(z-)}{2} \right| \left(\frac{n\alpha_n(z) + 1 - (n+2)z}{n+2} \right) \\ &+ \left| \frac{h'(z+)) - h'(z-)}{2} \right| \sqrt{(\alpha_n(z)-z)^2 + \frac{4}{n}} \\ &+ \left(\frac{2}{n} + \left((\alpha_n(z)-z)^2 + \frac{4}{n} \right) \frac{1}{z(1-z)} \right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z+\frac{1-z}{k}} (h'_z), \end{aligned}$$

and, by condition (c), we get

$$|I_1| = \left| \int_0^1 [K_n(h(t)) - h(t)]\, H_n(z,t)\,dt \right| \leq \int_0^1 |K_n(h(t)) - h(t)|\, H_n(z,t)\,dt \leq \frac{1}{\mu(n)},$$

which completes the proof. $\square$

Theorem 4. *Let $\psi \in \Psi$ and $h \in X$ be such that $\psi \circ |h| \in DBV[0,1]$. Then, for every $z \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned} |(NLD_n h)(z) - h(z)| &\leq \left(\frac{(\psi\circ|h|)'(z+) + (\psi\circ|h|)'(z-)}{2} \right) \left(\frac{n\alpha_n(z) + 1 - (n+2)z}{n+2} \right) \\ &+ \left(\frac{(\psi\circ|h|)'(z+)) - (\psi\circ|h|)'(z-)}{2} \right) \sqrt{(\alpha_n(z)-z)^2 + \frac{4}{n}} \\ &+ \left(\frac{2}{n} + \left((\alpha_n(z)-z)^2 + \frac{4}{n} \right) \frac{1}{z(1-z)} \right) \sum_{k=1}^{[\sqrt{n}]} \bigvee_{z-\frac{z}{k}}^{z+\frac{1-z}{k}} (h'_z) + \frac{1}{\mu(n)}. \end{aligned}$$

Proof. We can write

$$\begin{aligned}
|(NLD_n h)(z) - h(z)| &= \left| \int_0^1 K_n(h(t)) H_n(z,t)\,dt - h(z) \right| \\
&\leq \int_0^1 |K_n(h(t)) - K_n(h(z))|\, H_n(z,t)\,dt + \int_0^1 |K_n(h(z)) - h(z)|\, H_n(z,t)\,dt \\
&\leq \int_0^1 \psi(|h(t) - h(z)|) H_n(z,t)\,dt + \int_0^1 |K_n(h(z)) - h(z)|\, H_n(z,t)\,dt \\
&= I_1 + I_2.
\end{aligned}$$

For a non-decreasing function ψ we have

$$-\psi\left(|h(t) - h(z)|\right) \leq \psi\left(|h(t)|\right) - \left(|h(z)|\right).$$

Using this and the fact that $\psi\left(|h(t)|\right) \in DBV\,[0,1]$, we can write

$$\begin{aligned}
I_1 &= \int_z^0 \left[\psi\left(|h(t)|\right) - \left(|h(z)|\right)\right] H_n(z,t)\,dt + \int_1^z \left[\psi\left(|h(t)|\right) - \left(|h(z)|\right)\right] H_n(z,t)\,dt \\
&= \int_0^z \left[\int_t^z \left(\psi \circ |h|\right)'(u)\,du\right] H_n(z,t)\,dt + \int_z^1 \left[\int_t^z \left(\psi \circ |h|\right)'(u)\,du\right] H_n(z,t)\,dt.
\end{aligned}$$

In the last two integrals, using the decomposition of $\left(\psi \circ |h|\right)'(t) \in DBV\,[0,1]$, namely

$$\begin{aligned}
\left(\psi \circ |h|\right)'(t) = {} & \frac{\left(\psi \circ |h|\right)'(z+)) + \left(\psi \circ |h|\right)'(z-)}{2} + \left(\psi \circ |h|\right)_z'(t) \\
& + \frac{\left(\psi \circ |h|\right)'(z+)) - \left(\psi \circ |h|\right)'(z-)}{2}\,\mathrm{sgn}(t-z) \\
& + \delta_z(t)\left[\left(\psi \circ |h|\right)'(z) - \frac{\left(\psi \circ |h|\right)'(z+)) + \left(\psi \circ |h|\right)'(z-)}{2}\right],
\end{aligned}$$

and following the same approach as in the proof of the previous theorem, we obtain the desired result. □

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Алтін Х.Е., Карслі Х. *Апроксимація нелінійними операторами типу Лотоцького-Дюррмеєра* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 345–357.

Основною метою цього дослідження є побудова нелінійних операторів типу Лотоцького–Дюррмеєра та вивчення апроксимаційних властивостей цих ново введених операторів. Дослідження операторів Лотоцького розпочалося з роботи [Canad. J. Math. 1966, **18**, 89–91] Й.П. Кінга, а апроксимація нелінійними інтегральними операторами згорткового типу була введена Й. Мусєляком у 1983 році. В останні роки обидві теорії набули значної популярності завдяки роботам багатьох науковців. У цьому контексті основною мотивацією цього дослідження є введення нелінійних операторів типу Лотоцького-Дюррмеєра, що виникають із поєднання цих двох теорій, та подальше вивчення їх апроксимаційних властивостей, зокрема для функцій зі скінченною варіацією та похідними зі скінченною варіацією.

Ключові слова і фрази: оператор Лотоцького, нелінійний оператор, обмежена варіація.