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Bi-Lipschitz embedding properties of lamplighter graphs on weighted and unweighted trees

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In 2021, F. Baudier, P. Motakis, Th. Schlumprecht and A. Zsák proved that if a sequence $(G_k)_{k\in\mathbb{N}}$ of graphs contains the sequence of complete graphs with uniformly bounded distortion, then the sequence of lamplighter graphs on G_k's contains Hamming cubes with uniformly bounded distortion and asked whether the converse holds.

They suggested that a sequence of trees with edges replaced by paths of "moderately growing" lengths may be a counterexample. We prove that indeed this is the case, and that a sequence of "moderately" weighted trees is another counterexample.

Further, we prove that diamond graphs do not embed with uniformly bounded distortion into lamplighter graphs on trees with edges replaced by paths with sufficiently fast growing lengths.

Key words and phrases: distortion of a bi-Lipschitz embedding, lamplighter group, Lipschitz map, metric embedding, Ribe program, superreflexivity.

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1 Introduction

Given metric spaces (M, d_M), (N, d_N), and a constant $C \geq 1$, a map $f : M \longrightarrow N$ is called a *bi-Lipschitz embedding with distortion at most* C if there exists a scaling factor $\lambda > 0$ so that for all $u, v \in M$ we have

$$\lambda d_M(u,v) \leq d_N(f(u), f(v)) \leq C\lambda d_M(u,v).$$

We denote

$$c_N(M) \stackrel{\text{def}}{=} \inf\left\{ dist(f) : f : M \to N \text{ is a bi-Lipschitz embedding}\right\}.$$

If $(M_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ are sequences of metric spaces, we say that $(M_k)_{k\in\mathbb{N}}$ *equi-bi-Lipschitzly embeds into* $(N_k)_{k\in\mathbb{N}}$ if there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $c_{N_n}(M_k) \leq C$.

The theory of bi-Lipschitz and metric embeddings has many interactions with geometry, analysis, probability, combinatorics, group theory, topology, and theoretical computer science. It has been studied intensively over the last four decades (see [19, 22]) and continues to be a very active and growing area of research (see, e.g., the recent workshop on Metric Embeddings

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at the American Institute of Mathematics and the Merkin Center for Pure and Applied Mathematics (https://aimath.org/pastworkshops/metricembeddings.html), organized by A. Eskenazis, A.Naor and S.-Y. Ryoo). In this theory, one aims to study a structure of an object by analyzing its embedding properties with respect to simpler metric or geometric objects, such as trees, which are one of the simplest, yet very rich and important objects, both in the geometric group theory and in computer science (cf. [15] and its references). Here, by a tree we mean a (weighted or unweighted) connected graph T that does not contain any cycles, i.e. such that for any $x, y \in T$ there exists exactly one shortest path from x to y. Trees play a fundamental role in hyperbolic geometry and also in computer science (see [15] and its references). The bi-Lipschitz structure of trees has been studied extensively. In particular, in a seminal paper [4], J. Bourgain proved that the sequence of finite complete unweighted binary trees B_k of depth k equi-bi-Lipschitzly embeds into a Banach space X if and only if the space X is not superreflexive (i.e. does not admit an equivalent uniformly convex norm), including $X = \ell_1$. This characterization of a linear property of Banach spaces in terms of their metric structure was one of the influential results that gave rise to the Ribe program and started a considerable amount of work in related directions (cf. [19, 22]). Notably, W.B. Johnson and G. Schechtman [14] proved that the sequence of diamond graphs $(\mathsf{D}_k)_{k\in\mathbb{N}}$ (see Definition 2.1 below) equi-bi-Lipschitzly embeds into a Banach space X if and only if X is not superreflexive.

In the present paper, we are interested in lamplighter graphs, whose systematic study was initiated by F. Baudier, P. Motakis, Th. Schlumprecht and A. Zsák in [2]. Given a graph $G = (V(G), E(G))$, the *lamplighter graph* $\mathsf{La}(G)$ is the graph whose vertices are of the form (A, x), where x is a vertex of G and A is a finite subset of $V(G)$. Two vertices $(A, x), (B, y)$ of $\mathsf{La}(G)$ are connected by an edge if and only if (i) $A = B$ and (x, y) is an edge in G, or (ii) $x = y$ and $A \triangle B = \{x\}$, where $A \triangle B$ denotes the symmetric difference of the sets A, B, i.e. $A \triangle B = (A \setminus B) \cup (B \setminus A)$ (see [2]). Intuitively, a vertex $(A, x) \in \mathsf{La}(G)$ "consists" of the set $A \subseteq V(G)$ that represents the set of lamps that are "on" and $x \in V(G)$ represents the current position of the lamplighter. The lamplighter can make one of two types of moves: either move to vertex connected by an edge to the current position, or switch the light "on"/"off" at the current position (cf. [27]). By [2, Proposition 2.1], the metric on $\mathsf{La}(G)$ is given by

$$d_{\mathsf{La}(G)}\big((A, x), (B, y)\big) = \mathrm{tsp}_G(x, A \triangle B, y) + |A \triangle B|, \tag{1}$$

where $|S|$ denotes the cardinality of the set S, and $\mathrm{tsp}_G(x, A \triangle B, y)$ denotes the solution of the traveling salesman problem, that is, the length of the shortest possible walk on G that starts at x, ends at y, and passes through all vertices $v \in A \triangle B$.

This definition of lamplighter graphs coincides with the standard definition of the wreath product $\mathbb{Z}_2 \wr G$ in the case when G is (a Cayley graph of) a group and it is its very natural generalization to the setting of arbitrary graphs (not necessarily Cayley graphs). Since their introduction, the lamplighter graphs have already been used to prove results about the lamplighter groups themselves (see [8, 9]).

Lamplighter groups are a very interesting class of groups that has been a rich source of important examples and counterexamples in geometric group theory (see [1, 8, 9] and their references). In particular, the study of embedding properties of wreath products into the Hilbert space L_2 and into L_1 has received a lot of attention in the last 20 years; we refer interested readers to an excellent overview of the literature in [21] (cf. also [2, 3, 8, 20] and their references). Here we mention only a few results most closely related to the topic of the present paper.

R. Lyons, R. Pemantle and Y. Peres [17] (see [2,23] for alternate short proofs) proved that the sequence of complete binary trees $(B_k)_k$ equi-bi-Lipschitzly embeds into the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ (and therefore, by the characterization of Bourgain mentioned above, $\mathbb{Z}_2 \wr \mathbb{Z}$ does not bi-Lipschitzly embed into any superreflexive Banach space). A. Naor and Y. Peres in [20] gave two proofs that the sequence of wreath products $(\mathbb{Z}_2 \wr \mathbb{Z}_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embeds into ℓ_1, and so does $\mathbb{Z}_2 \wr \mathbb{Z}$ (here $\mathbb{Z}_k$ denotes the cyclic group of order k). In [23], it was shown that the sequence $(\mathbb{Z}_2 \wr \mathbb{Z}_k)_{k\in\mathbb{N}}$ and $\mathbb{Z}_2 \wr \mathbb{Z}$ equi-bi-Lipschitzly embed into every nonsuperreflexive Banach space, thus identifying another metric characterization of superreflexivity. Y. Cornulier, Y. Stalder and A. Valette [6] showed that for every finitely generated free group F, the wreath product $\mathbb{Z}_2 \wr F$, and therefore the lamplighter group $\mathsf{La}(T)$ on any tree T, bi-Lipschitzly embeds into ℓ_1 (see also [2] for a purely metric, constructive proof).

The starting point for the present work is the paper [2]. One of the main results of [2] is the following theorem.

Theorem 1.1 ([2, Theorem 1.3]). *The sequences $(\mathsf{La}(B_k))_{k\in\mathbb{N}}$, $(H_k)_{k\in\mathbb{N}}$, and $(K_k)_{k\in\mathbb{N}}$ pairwise equi-bi-Lipschitzly embed into each other, where K_k is a complete graph on k vertices and H_k is the Hamming cube on the set $\{1,\dots,k\}$ for $k \in \mathbb{N}$ (for full definition see Section 2 below).*

As one of the tools for proving Theorem 1.1, the authors of [2] showed the next result.

Theorem 1.2 ([2, Lemmas 6.2 and 5.1]). *If a sequence of complete graphs $(K_k)_k$ equi-bi-Lipschitzly embeds into a sequence of graphs $(G_k)_k$, then the sequence $(H_k)_k$ of Hamming cubes equi-bi-Lipschitzly embeds into $(\mathsf{La}(G_k))_k$.*

The authors of [2] asked whether the converse of Theorem 1.2 is true.

Problem 1.3 ([2, Problem 7.1]). *Given a sequence $(G_k)_{k\in\mathbb{N}}$ of graphs such that the Hamming cubes $(H_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embed into $(\mathsf{La}(G_k))_{k\in\mathbb{N}}$, does it follow that $(K_k)_{k\in\mathbb{N}}$ equi-bi-Lipschitzly embed into $(G_k)_{k\in\mathbb{N}}$?*

They suggested that a counterexample might be a sequence of trees $W_{2,n}$, obtained by replacing each edge of level k of a binary tree B_n by a path of length 2^{n-k}, as described in [2, Example in Section 7], see Section 3 below for a precise definition. We prove that, indeed, this construction leads to a counterexample for Problem 1.3 and that another counterexample is a sequence of "moderately" weighted complete binary trees $(\mathsf{WB}_{2,n})_n$, see Section 3 below. Namely, we prove the following assertion.

Theorem 1.4. *For any $\theta \in \mathbb{N}$ with $\theta \geq 2$, the sequence $(K_k)_{k\in\mathbb{N}}$ of complete graphs does not equi-bi-Lipschitzly embed into the sequence of trees $(W_{\theta,n})_{n\in\mathbb{N}}$ or into the sequence of weighted trees $(\mathsf{WB}_{\theta,n})_{n\in\mathbb{N}}$.*

Theorem 1.5. *For every $\varepsilon > 0$, the sequence $(H_k)_{k\in\mathbb{N}}$ of Hamming cubes bi-Lipschitzly embeds into $(\mathsf{La}(W_{2,n}))_{n\in\mathbb{N}}$ and into $(\mathsf{La}(\mathsf{WB}_{2,n}))_{n\in\mathbb{N}}$ with distortion at most $(1+\varepsilon)$.*

As pointed out in [2], since there exist nonsuperreflexive Banach spaces that do not equi-bi-Lipschitzly contain the sequence of Hamming cubes (see [5,12,13,24]), the above counterexamples also provide a negative answer to [2, Problem 7.3].

Corollary 1.6. *The sequences $(W_{2,n})_n$ and $(\mathsf{WB}_{2,n})_n$ of trees do not contain the sequence $(K_k)_k$ equi-bi-Lipschitzly, but both sequences of lamplighter graphs on them $(\mathsf{La}(W_{2,k}))_{k\in\mathbb{N}}$ and $(\mathsf{La}(\mathsf{WB}_{2,n}))_n$ equi-bi-Lipschitzly contain the sequence of Hamming cubes, and thus do not equi-bi-Lipschitzly embed into every nonsuperreflexive Banach space.*

We note that both sequences $(W_{2,n})_n$ and $(\mathsf{WB}_{2,n})_n$ embed equi-bi-Lipschitzly into the Hilbert space L_2. This follows from [15, Theorem 1.1], which says that a metric tree admits a bi-Lipschitz embedding into Hilbert space if and only if it does not equi-bi-Lipschitzly contain the sequence $(B_n)_n$ of complete binary trees. Indeed, since the sequence $(K_k)_k$ of complete graphs equi-bi-Lipschitzly embeds into the sequence $(B_n)_n$ (see [18]), Theorem 1.4 implies that $(B_n)_n$ does not equi-bi-Lipschitzly embed into either $(W_{2,n})_n$ or $(\mathsf{WB}_{2,n})_n$ (alternatively, one can note that the trees $(W_{2,n})_n$ and $(\mathsf{WB}_{2,n})_n$ are doubling and use an earlier result [10, Theorem 2.4]).

Thus sequences $(W_{2,k})_k$ and $(\mathsf{WB}_{2,n})_n$ are new examples of graphs, other than $(K_n)_n$, that equi-bi-Lipschitzly embed into Hilbert space and such that the lamplighter graphs on them do not equi-bi-Lipschitzly embed into every nonsuperreflexive Banach space.

We note that, as is easy to see, the sequence $(W_{2,k})_k$ embeds equi-bi-Lipschitzly into the sequence of diamonds and also into the sequence of Laakso graphs. Thus, by [2, Lemma 5.1], Theorem 1.5 implies the following corollary.

Corollary 1.7. *The sequence of Hamming cubes embeds equi-bi-Lipschitzly into the sequence of lamplighter graphs on the diamond graphs and also into the sequence of lamplighter graphs on the Laakso graphs.*

Next we prove that Theorem 1.5 fails for $(\mathsf{La}(W_{\theta,n}))_n$, when $\theta \geq 3$, in the following much stronger sense (recall that the sequence of diamond graphs $(\mathsf{D}_k)_k$ equi-bi-Lipschitzly embeds into the sequence of Hamming cubes $(H_k)_k$, see [11, 14]).

Theorem 1.8. *For all $\theta \geq 3$, the sequence $(\mathsf{D}_k)_{k\in\mathbb{N}}$ of diamonds does not equi-bi-Lipschitzly embed into the sequence $(\mathsf{La}(W_{\theta,n}))_{n\in\mathbb{N}}$ of lamplighter graphs.*

Since $(W_{\theta,n})_n$ are trees, it follows from [6] (cf. [2, Theorem 1.1]), that the sequence $(\mathsf{La}(W_{\theta,n}))_n$ embeds equi-bi-Lipschitzly into ℓ_1. However, we do not know whether $(\mathsf{La}(W_{\theta,n}))_n$, when $\theta \geq 3$, embed equi-bi-Lipschitzly into every nonsuperreflexive Banach space. If yes, the sequences $(\mathsf{La}(W_{\theta,n}))_n$ for $\theta \geq 3$, would be a new metric characterization of superreflexivity. If no, this would be an even more interesting example.

We do not know whether Theorem 1.8 is true for the sequence of lamplighter graphs on the weigthed complete binary trees $(\mathsf{La}(\mathsf{WB}_{\theta,n}))_n$ for all or some $\theta \geq 3$.

2 Preliminaries

We use standard notation as may be found, e.g., in [16, 22].

Recall that a graph $G = (V, E)$ is a pair of the set of vertices $V = V(G)$ and the set of edges $E = E(G) \subseteq V \times V$. Frequently, with a slight abuse of notation, we will write $v \in G$ instead of writing $v \in V(G)$. A *walk* in G is a sequence $(x_0, x_1, \ldots, x_m)$ of vertices of G such that for every i with $1 \leq i \leq m$, (x_{i-1}, x_i) is an edge of G. We will only consider connected graphs, that is, graphs such that for every $x, y \in G$ there exists at least one walk in G from x to y.

The unweighted connected graph G is endowed with the shortest path metric d_G, where for all $x, y \in G$ we have

$$d_G(x,y) = \min\{m : \text{there exists a walk of length } m \text{ from } x \text{ to } y\}.$$

A walk from x to y is called a *(geodesic) path*, denoted $p(x,y)$, if its length is equal to the graph distance between x and y.

Our main interest here is in graphs known as trees, Hamming cubes, and diamonds.

Recall that a graph G is called a *tree*, if G is connected and does not contain any cycles, i.e. if for any $x, y \in G$ there exists exactly one path $p(x,y)$.

For an arbitrary set I, the *Hamming cube* H_I is a graph with the vertex set $\{0,1\}^{(I)}$ consisting of all functions $x : I \to \{0,1\}$ with finite support. Two vertices $x, y \in H_I$ are joined by an edge if and only if they differ in exactly one coordinate, i.e. there exists a unique $i \in I$ with $x_i \neq y_i$. The graph distance on H_I is the ℓ_1 metric given by

$$d_H(x,y) = \sum_{i \in I} |x_i - y_i|.$$

Definition 2.1 (cf. [22]). *The diamond graphs D_k, $k \in \mathbb{N}$, are constructed inductively as follows. D_0 is a single edge. If the graph D_{k-1} is already constructed, the diamond D_k is obtained by replacing each edge (u,v) of D_{k-1} by a a quadrilateral u, a, v, b, with edges (u,a), (a,v), (v,b), (b,u). Equivalently, for any $m < k$, one may think of a diamond D_k as obtained by replacing each edge of a diamond D_m by a diamond D_{k-m}. Each (isometric) copy of D_{k-m} inside D_k that replaced an edge of D_m, is called a subdiamond of D_k.*

We denote the graph metric on D_k by d_{D}.

The following lemma follows directly from the definition of the diamond graphs.

Lemma 2.2. *Let $k, j \in \mathbb{N}$ with $k > j + 3$, and $u, v \in \mathsf{D}_k$. If $d_{\mathsf{D}}(u,v) \geq 2^{k-j}$, then there exists a subdiamond $\bar{\mathsf{D}} = \mathsf{D}_{k-j-3} \subset \mathsf{D}_k$ that lies between u and v, that is, such that for all $w \in \bar{\mathsf{D}}$, we have $d_{\mathsf{D}}(u,w) + d_{\mathsf{D}}(w,v) = d_{\mathsf{D}}(u,v)$.*

3 Definitions of the trees $W_{\theta,n}$ and $\mathsf{WB}_{\theta,n}$

Given $n \in \mathbb{N}$ and $\theta \in \mathbb{N}$, $\theta \geq 2$, to define the tree $W_{\theta,n}$, we start with a rooted complete binary tree B_n. We say that a vertex of B_n is on the *level* k, $0 \leq k \leq n$, if its distance from the root is equal to k, and that an edge of B_n is on the level k, $1 \leq k \leq n$, if its lower (that is, further from the root) vertex is on the level k.

The tree $W_{\theta,n}$ is constructed by replacing each edge of B_n on the level k by a path of length θ^{n-k} (see Figure 1). Thus the tree $W_{\theta,n}$ may be seen, as described in [2], as a "binary tree with variable size legs". Alternatively, one may think of $W_{\theta,n}$ as a subgraph of the binary tree B_m, where $m = \sum_{k=1}^{n} \theta^{n-k} = (\theta^n - 1)/(\theta - 1)$ and multiple branches of B_m are removed. Note that $W_{\theta,n}$ is an unweighted tree and that $W_{2,n}$ is the same graph as the one described in the example in [2, Section 7].

Vertices of $W_{\theta,n}$ that came from the original tree B_n will be called *nodes* or *branching points* (note that all nodes, except for the root and the leaves of the tree, have degree equal to 3, and all other vertices of $W_{\theta,n}$ have degree at most 2). We say that a node of $W_{\theta,n}$ is on level k, $0 \leq k \leq n$, if it was a vertex of level k in B_n.

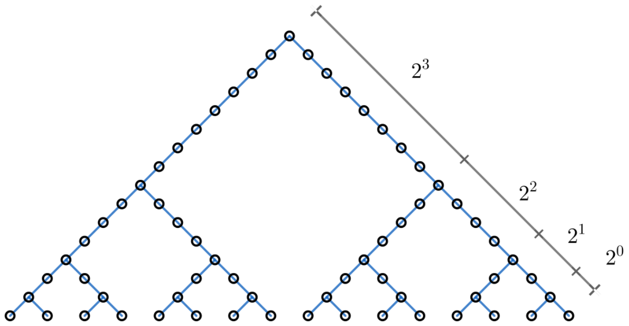

Figure 1. The tree $W_{2,4}$

For $n \in \mathbb{N}$ and $\theta \in \mathbb{N}$ with $\theta \geq 2$, the weighted tree $\mathsf{WB}_{\theta,n}$ is obtained by taking the complete binary tree B_n and assigning to each edge of B_n at level k a weight equal to θ^{n-k}. Note that the sets of edges and vertices of $\mathsf{WB}_{\theta,n}$ coincide with those of B_n, the only difference are the weights assigned to all nonterminal edges.

To simplify notation, when the values of n and θ are clear from the context, we will denote the metrics on $W_{\theta,n}$, $\mathsf{WB}_{\theta,n}$, $\mathsf{La}(W_{\theta,n})$, and on $\mathsf{La}(\mathsf{WB}_{\theta,n})$ by d_W, d_{WB}, d_L, and d_{LWB}, respectively.

4 Proof of Theorem 1.4

We start from a simple fact that is surely known. Since we could not find it in the literature, for the convenience of the reader, we inlude its short proof.

Lemma 4.1. *For any integer $m \geq 2$, distortion of any embedding of K_m into a path is at least $m-1$.*

Proof. If m points are mapped into a path, there are two which are furthest apart, and $m-2$ points between them. Denote by d the distance between the first and last points. The remaining points partition this distance into $m-1$ subpaths. Thus the minimum distance between two of these points is at most $\frac{d}{m-1}$. Hence, the distortion of an embedding of K_m is at least $d/(\frac{d}{m-1}) = m-1$. □

Theorem 4.2. *For any $n \in \mathbb{N}$ and $\theta \in \mathbb{N}$ with $\theta \geq 2$, the distortion of any embedding of K_4 into $W_{\theta,n}$ or into $\mathsf{WB}_{\theta,n}$ is at least $\frac{3}{2}$.*

Proof. Let $f : K_4 \to W_{\theta,n}$ be a map onto the set $\{v_i\}_{i=1}^4 \subset W_{\theta,n}$. By Lemma 4.1, if 3 or more points of this set are on a path, the distortion of f is at least 2. So, assume that there is never 3 points on a path. Thus there must exist a branching node s_1 between v_1 and v_2 such that s_1 is on the geodesic paths $p(v_3, v_1)$ and $p(v_3, v_2)$. Further, there must exist a branching node s_2 on one of $p(s_1, v_i)$, $i = 1, 2$, or 3, such that s_2 is on $p(v_4, v_i)$ for each $i = 1, 2, 3$. Without loss of generality we suppose that s_2 is on $p(s_1, v_2)$ and we denote, as marked on Figure 2, $a_1 = d_W(s_1, v_3)$, $a_2 = d_W(s_1, v_1)$, $c_1 = d_W(s_2, v_4)$, $c_2 = d_W(s_2, v_2)$, $a = \max\{a_1, a_2\}$, $c = \max\{c_1, c_2\}$, and $b = d_W(s_1, s_2)$.

Without loss of generality we suppose that the branching level k of s_1 is greater than or equal to the branching level of s_2. Then $b \geq \theta^{n-k}$ and both vertices v_1 and v_3 have to be below vertex s_1, and hence we get

$$a = \max\{a_1, a_2\} \leq \sum_{j=1}^{n-k} \theta^{n-(k+j)} = \frac{\theta^{n-k} - 1}{\theta - 1} < b. \tag{2}$$

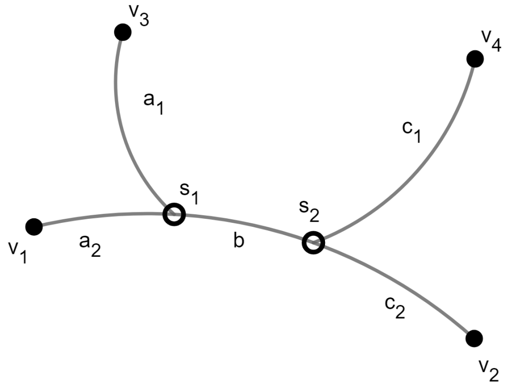

Figure 2. A possible configuration of an embedding of K_4 into $W_{2,n}$, where s_1 and s_2 are branching points

This implies that

$$a_1 + a_2 \leq \min\{a_1, a_2\} + \frac{\theta^{n-k} - 1}{\theta - 1} < \min\{a_1, a_2\} + b,$$

and therefore the shortest distance between vertices in the image of K_4 is at most

$$\min\{a_1 + a_2, c_1 + c_2\} \leq 2\min\{a, c\}.$$

Since the longest distance between vertices in the image of K_4 is at least $a + b + c$ and, by (2), $b \geq \min\{a, c\}$, we obtain

$$dist(f) \geq \frac{a + b + c}{2\min\{a, c\}} \geq \frac{3\min\{a, c\}}{2\min\{a, c\}} = \frac{3}{2},$$

which ends the proof for the case of $W_{\theta,n}$.

The case of the weighted trees $\mathsf{WB}_{\theta,n}$ is very similar. The only difference is that in $\mathsf{WB}_{\theta,n}$ every vertex is a branching node, but, due to the assigned weights to edges at level k, the distance between vertices s_1 and s_2 is still at least θ^{n-k}. Thus the rest of the proof is identical as in case of $W_{\theta,n}$. ☐

As a consequence of Theorem 4.2, by a standard Ramsey theory argument, we obtain a proof of Theorem 1.4.

Proof. Suppose for contradiction that there exists $C \in \mathbb{R}$ such that for all $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and a bi-Lipschitz embedding ϕ of K_k into $W_{\theta,n}$ (respectively, $\mathsf{WB}_{\theta,n}$) with distortion less than C.

Let $m \in \mathbb{N}$ be the smallest integer such that $(3/2)^m > C$. By the Ramsey's Theorem (see, e.g., [25, Theorem 3.6]) there exists the Ramsey number $R(m, 4) \in \mathbb{N}$ such that for every $k \geq R(m, 4)$, every m-coloring of edges of K_k contains a monochromatic copy of K_4.

Let $k > R(m, 4)$, and $n \in \mathbb{N}$ be such that there exists an embedding ϕ of K_k into $W_{2,n}$ with distortion at most C. Let $l, L > 0$ be optimal numbers such that for all $u, v \in K_k$ we have $l \leq d(\phi(u), \phi(v)) \leq L$. Then $L < Cl < (3/2)^m l$. We color an edge (u, v) of K_k with color $i \in \{1, \ldots, m\}$ iff

$$\left(\frac{3}{2}\right)^{i-1} l \leq d(\phi(u), \phi(v)) < \left(\frac{3}{2}\right)^{i} l.$$

Since $k > R(m, 4)$, there exist a monochromatic copy of $K_4 \subset K_k$ with respect to this m-coloring. Since ϕ restricted to any monochromatic subset has distortion less than $3/2$, this contradicts Theorem 4.2, and ends the proof. ☐

5 Proof of Theorem 1.5

Let $n \in \mathbb{N}$ with $n \geq 3$. For any $k \in \{0, \ldots, n-1\}$, there are 2^k nodes of level k in $W_{2,n}$, and there are 2 downward paths of length $2^{n-(k+1)}$ from each node. We index these paths from left to right by an $i \in \{1, \ldots, 2^{k+1}\}$. We define the set $F^n_{k,i} \subset W_{2,n}$ to be the set of all vertices that are in the ith path and below, see Figure 3.

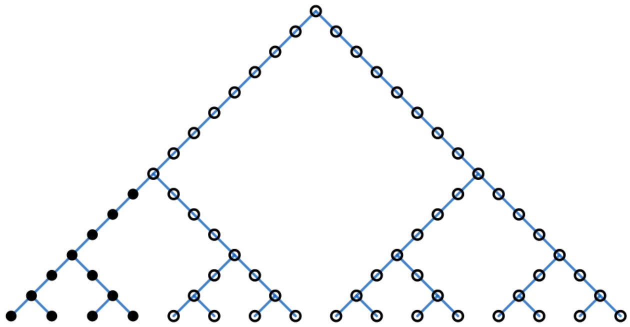

Figure 3. The set $F^4_{1,1} \subset W_{2,4}$

Lemma 5.1. *For any $n \in \mathbb{N}$, $0 \leq k < n$, and $1 \leq i \leq 2^{k+1}$, the cardinality of the set $F^n_{k,i} \subset W_{2,n}$ does not depend on i and equals*

$$|F^n_{k,i}| := w(n,k) = (n-k)2^{n-(k+1)}. \tag{3}$$

Proof. Indeed, the length of the path in $W_{2,n}$ from a node at level k to a node at level $k+1$ below is equal to $2^{n-(k+1)}$. The paths at the next level are of length $2^{n-(k+2)}$, but there are two of them. For each level, there are 2^{j-1} paths, each of length $2^{n-(k+j)}$. Thus

$$|F^n_{k,i}| = \sum_{j=1}^{n-k} 2^{j-1}2^{n-(k+j)} = (n-k)2^{n-(k+1)}.$$

□

Theorem 1.5 will follow immediately from our next result.

Theorem 5.2. *Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then H_{2^k} bi-Lipschitzly embeds into $\mathsf{La}(W_{2,n})$ with distortion at most $1+\varepsilon$, whenever $n > (2^{k+1}/3\varepsilon) + k - 1$.*

Proof of Theorem 5.2. Let $n > (2^{k+1}/3\varepsilon) + k - 2$. We define the map $f : H_{2^k} \to \mathsf{La}(W_{2,n})$ by setting for each $x \in H_{2^k}$

$$f(x) := \Big(\bigcup_{\{i : x_i = 1\}} F^n_{k-1,i}, v_0 \Big), \tag{4}$$

cf. Figure 4.

Given any $x, y \in H^k_2$, let $I = \{i \in [2^k] : x_i \neq y_i\}$. Then $|I| = d_H(x,y)$, and

$$d_L(f(x), f(y)) = d_L\Big(\big(\bigcup_{i \in I} F^n_{k-1,i}, v_0 \big), (\varnothing, v_0) \Big).$$

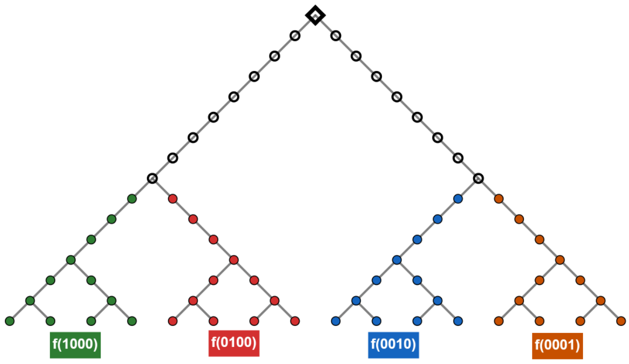

Figure 4. Images of the four basis elements of H_4 in $\mathsf{La}(W_{2,4})$; in each of them the lamplighter is at the root of $W_{2,4}$

Note that a move in $\mathsf{La}(W_{2,n})$ from the element $(\varnothing, v_0)$ to $(\bigcup_{i\in I} F^n_{k-1,i}, v_0)$ requires that we travel from the root to the top node of each of the sets $F^n_{k-1,i}$, $i \in I$, and back. The distance from the root v_0 to any node of level $k-1$ is equal to

$$\varrho_{k-1} = \sum_{j=1}^{k-1} 2^{n-j} = 2^n - 2^{n-k+1}.$$

The paths from the root to certain nodes may have significant overlaps depending on the relative positions of the nodes, but in any case, we have an upper estimate of the total travel distance to all nodes for $i \in I$ and back by $|I| \cdot 2\varrho_{k-1}$.

Also that for each $i \in I$, we need to travel down from the top node of $F^n_{k-1,i}$, $i \in I$, switch on all its lamps, and return back to the top node. As we computed in Lemma 5.1, such travel in each set $F^n_{k-1,i}$ requires $3w(n,k-1)$ steps, and gives the total of $|I| \cdot 3w(n,k-1)$ steps to cover all sets $F^n_{k-1,i}$, $i \in I$.

Since $|I| = d_H(x,y)$, we obtain

$$3w(n,k-1)d_H(x,y) \le d_L(f(x), f(y)) \le (3w(n,k-1) + 2\varrho_{k-1})d_H(x,y).$$

Therefore, by (3), we estimate the distortion of the map f as follows

$$\begin{aligned} dist(f) &\le \frac{3w(n,k-1)+2\varrho_{k-1}}{3w(n,k-1)} = 1 + \frac{2(2^n - 2^{n-k+1})}{3(n-k+1)2^{n-k}} \\ &< 1 + \frac{2^{k+1}}{3(n-k+1)} < 1 + \varepsilon, \end{aligned}$$

which ends the proof. □

Remark. *It is easy to check that Theorem 1.5 is valid also for the weighted trees* $\mathsf{WB}_{2,n}$*, with only minor changes in the proof.*

6 Proof of Theorem 1.8

We denote by F_n the top "fork of depth 2" contained in $W_{\theta,n}$, see Figure 5.

Lemma 6.1. *For all* $\theta, n \in \mathbb{N}$ *with* $\theta \ge 3$ *we have*

$$\mathrm{diam}(\mathsf{La}(W_{\theta,n}) \le 6\theta^n. \tag{5}$$

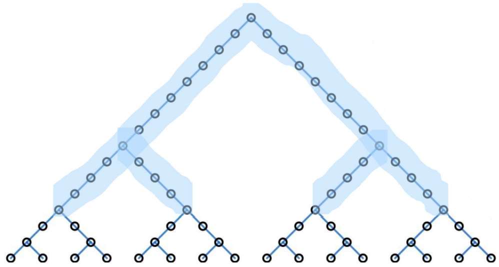

Figure 5. The set $F_n \subset W_{\theta,n}$ – the fork of depth 2

Proof. Since $W_{\theta,n}$ is an unweighted tree, by [2, Theorem 3.1], we have $\operatorname{diam}(\mathsf{La}(W_{\theta,n}) \le 3|W_{\theta,n}|$. We estimate the cardinality of $W_{\theta,n}$ similarly as in the proof of (3), namely

$$|W_{\theta,n}| \le \sum_{j=1}^{n} 2^j \theta^{n-j} = \theta^n \sum_{j=1}^{n} \left(\frac{2}{\theta}\right)^j \le \theta^n \cdot \frac{2}{\theta - 2} \le 2\theta^n,$$

since $\theta \ge 3$, which ends the proof of (5). □

M. Stein and J. Taback [26] showed that $\mathsf{La}(\mathbb{Z})$ embeds bi-Lipschitzly with distortion at most 4 into a direct product of two infinite binary trees B_∞ (see [2, Proposition 2.2] for a different proof).

Given two pointed graphs $G_i = (V_i, E_i, v_i)$ for $i = 1, 2$, we define the *vertex-coalescence* $G_1 * G_2$ of G_1 and G_2 by taking $V(G_1 * G_2)$ to be the disjoint union of V_1 and V_2 with vertices v_1 and v_2 identified with each other to form one vertex denoted by v_0, and $E(G_1 * G_2)$ is a disjoint union of E_1 and E_2. The graph metric on $G_1 * G_2$ is given by

$$d_{G_1 * G_2}(u, v) = \begin{cases} d_{G_i}(u, v), & \text{if } u, v \in G_i, \\ d_{G_1}(u, v_0) + d_{G_2}(v_0, v), & \text{if } u \in G_1 \text{ and } v \in G_2. \end{cases}$$

For any $n \in \mathbb{N}$, the fork graph $F_n \subset W_{\theta,n}$ can be obtained by coalescing three path graphs (at two different vertices), one of length $2 \cdot \theta^{n-1}$ and two of length $2 \cdot \theta^{n-2}$. It is well-known that for any finite $k \in \mathbb{N}$, the graph obtained by coalescing at a common vertex k copies of paths of arbitrary finite lengths embeds into the tree B_∞ with distortion at most 2 (cf. [18]). Hence [2, Theorem 4.2] shows that for any $n \in \mathbb{N}$, the lamplighter graph $\mathsf{La}(F_n)$ bi-Lipschitzly embeds with uniform distortion into a direct product of ten binary trees B_∞. A. Eskenazis [7, Chapter 2] showed that the sequence of diamonds $(\mathsf{D}_k)_{k \in \mathbb{N}}$ does not equi-bi-Lipschitzly embed into any finite direct product of infinite binary trees B_∞. The combination of these results gives the following assertion.

Corollary 6.2. *The sequence of diamonds* $(\mathsf{D}_k)_{k \in \mathbb{N}}$ *does not equi-bi-Lipschitzly embed into* $\mathsf{La}(\mathbb{Z})$ *or into the sequence* $(\mathsf{La}(F_n))_{n \in \mathbb{N}}$.

We are now ready for the proof of Theorem 1.8.

Proof. Fix $\theta \ge 3$. Suppose, for contradiction, that there exists $C \in \mathbb{N}$, such that for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with $c_{\mathsf{La}(W_{\theta,n})}(\mathsf{D}_k) \le 2^C$.

By Corollary 6.2, there exists $k_C \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, we have

$$c_{\mathsf{La}(\mathbb{Z})}(\mathsf{D}_{k_C}) > 2^C \quad \text{and} \quad c_{\mathsf{La}(F_n)}(\mathsf{D}_{k_C}) > 2^C. \tag{6}$$

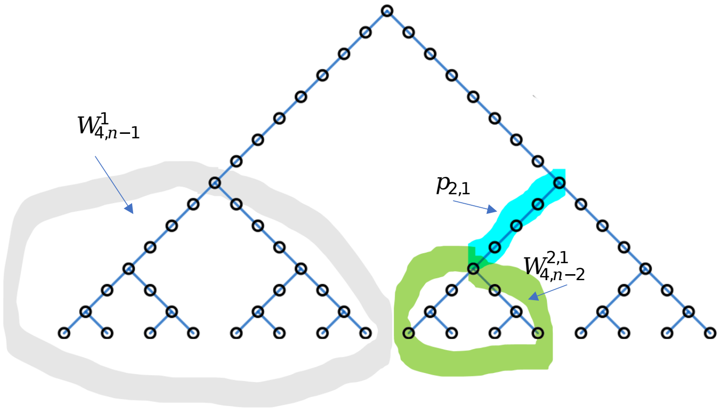

Figure 6. The subsets $W^1_{\theta,n-1}$, $W^{2,1}_{\theta,n-2}$, and $p_{2,1}$ of $W_{\theta,n}$

Fix $k > k_C + 2C + \theta + 9$, and let n be the smallest natural number such that there exist a map $\varphi : D_k \longrightarrow \mathsf{La}(W_{\theta,n})$ and $\lambda > 0$ so that for all $u, v \in \mathsf{D}_k$, we have

$$\lambda d_{\mathsf{D}}(u,v) \le d_L(\varphi(u),\varphi(v)) \le 2^C \lambda d_{\mathsf{D}}(u,v). \tag{7}$$

Thus, by (5) we get

$$\lambda 2^k \le d_L(\varphi(s),\varphi(t)) \le 6\cdot\theta^n = 6\theta^2\cdot\theta^{n-2}.$$

It is easy to check that if $\theta \ge 3$, then $6\theta^2 \le 2^{\theta+3}$. Therefore

$$\lambda \le 2^{\theta+3-k}\theta^{n-2}. \tag{8}$$

By (6) and the minimality of n, for any $\nu \in \{1,2\}$, we obtain

$$\varphi(\mathsf{D}_k) \not\subseteq \mathsf{La}(F_n) \quad \text{and} \quad \varphi(\mathsf{D}_k) \not\subseteq \mathsf{La}(W^\nu_{\theta,n-1}).$$

Thus there exist $u, v \in \mathsf{D}_k$, with $\varphi(u) = (a, A)$, $\varphi(v) = (b, B)$, such that there exist $x, y \in \{a,b\} \cup (A \triangle B)$ and $\nu, \mu \in \{1,2\}$ with $x \in W^{\nu,\mu}_{\theta,n-2}$ and $y \notin W^\nu_{\theta,n-1}$. Hence we obtain $d_W(x,y) > \theta^{n-2}$ and any walk in $W_{\theta,n}$ from x to y has to contain the path $p = p_{\nu,\mu}$ from $t \overset{\text{def}}{=} t_{\nu,\mu}$ to $\bar{t} \overset{\text{def}}{=} t_\nu$. Note that the length of p is equal to θ^{n-2}.

Define $z = z_{\nu,\mu}, \bar{z} = \bar{z}_{\nu,\mu}$ to be the points on p with

$$d_W(z,\bar{z}) = \frac{\theta^{n-2}}{2^{C+2}} \quad \text{and} \quad d_W(t,z) = d_W(\bar{z},\bar{t}) = \frac{1}{2}\Big(1 - \frac{1}{2^{C+2}}\Big)\theta^{n-2} \overset{\text{def}}{=} \bar{d}. \tag{9}$$

Note that, since $C \ge 1$, we have

$$d_W(z,\bar{z}) \le \frac{1}{2^C}\bar{d}. \tag{10}$$

Let $u = u_0, u_1, \ldots, u_m = v$ be a path in D_k from u to v, that is, $m = d_{\mathsf{D}}(u,v)$ and for each $i \le m$, (u_{i-1}, u_i) is an edge in D_k. Since $k - (C + \theta + 3) \ge C + 4$, by (8), for all i we have

$$d_L(\varphi(u_{i-1}),\varphi(u_i)) \le 2^C\lambda \le 2^{C+\theta+3-k}\theta^{n-2} \le \frac{1}{4} d_W(z,\bar{z}). \tag{11}$$

For each $i \le m$, let $\varphi(u_i) = (A_i, a_i)$ and let p_i denote a geodesic path from $\varphi(u_{i-1})$ to $\varphi(u_i)$ in $\mathsf{La}(W_{3,n})$. The length of each p_i is estimated from above in (11). Since $w = \bigcup_{i=1}^m p_i$ is a walk in

$\mathsf{La}(W_{\theta,n})$ from $\varphi(u) = (A,a)$ to $\varphi(v) = (B,b)$, w has to contain one or more elements of each of the forms (I,z) and $(J,\bar{z})$, where $I,J \subseteq W_{\theta,n}$. Let $\alpha \in \{1,\ldots,m\}$ be the largest index such that $(I,z) \in p_\alpha$ for some set I. Then $d_W(a_\alpha,z)$ is less than or equal to the length of the path p_α, so, by (9), (10) and (11), we obtain

$$\begin{aligned}\min\{d_W(a_\alpha,t),d_W(a_\alpha,\bar{t})\} &\geq \min\{d_W(t,z),d_W(\bar{t},z)\} - d_W(a_\alpha,z)\\ &\geq \bar{d} - \frac{1}{4}d_W(z,\bar{z}) \geq \bar{d} - \frac{1}{2^{\mathsf{C}+2}}\bar{d} > \frac{3}{4}\bar{d}.\end{aligned} \tag{12}$$

Let $\beta \in \{\alpha,\ldots,m\}$ be the smallest index larger than or equal to α so that $(J,\bar{z}) \in p_\beta$ for some set J. By (11), we obtain

$$\begin{aligned}d_L(\varphi(u_\alpha),\varphi(u_\beta)) &\geq d_L((I,z),(J,\bar{z})) - d_L((I,z),\varphi(u_\alpha)) - d_L((J,\bar{z}),\varphi(u_\beta))\\ &\geq d_W(z,\bar{z}) - \mathrm{length}(p_\alpha) - \mathrm{length}(p_\beta)\\ &\geq d_W(z,\bar{z}) - \frac{2}{4}d_W(z,\bar{z}) = \frac{1}{2}d_W(z,\bar{z}) = \frac{1}{2^{\mathsf{C}+3}}\theta^{n-2}.\end{aligned}$$

Let $\gamma \in \{\alpha,\ldots,\beta\}$ be the smallest index such that

$$d_L(\varphi(u_\alpha),\varphi(u_\gamma)) \geq \frac{1}{2^{\mathsf{C}+3}}\theta^{n-2} = \frac{1}{2}d_W(z,\bar{z}). \tag{13}$$

Then, by (10) and (11), we get

$$\begin{aligned}d_L(\varphi(u_\alpha),\varphi(u_\gamma)) &\leq d_L(\varphi(u_\alpha),\varphi(u_{\gamma-1})) + d_L(\varphi(u_{\gamma-1}),\varphi(u_\gamma))\\ &\leq \frac{1}{2}d_W(z,\bar{z}) + \frac{1}{4}d_W(z,\bar{z}) = \frac{3}{4}d_W(z,\bar{z}) \leq \frac{1}{2^{\mathsf{C}}}\cdot\frac{3}{4}\bar{d}.\end{aligned} \tag{14}$$

By (7), (8), and (13), we obtain

$$d_{\mathsf{D}}(u_\alpha,u_\gamma) \geq \frac{1}{2^{\mathsf{C}}\lambda}d_L(\varphi(u_\alpha),\varphi(u_\gamma)) \geq \frac{\theta^{n-2}}{2^{2\mathsf{C}+3}2^{\theta+3-k}\theta^{n-2}} = 2^{k-(2\mathsf{C}+\theta+6)}.$$

Thus, since $k \geq 2C+\theta+9$, by Lemma 2.2, there exists a subdiamond $\bar{\mathsf{D}} = \mathsf{D}_{k-(2\mathsf{C}+\theta+9)} \subset \mathsf{D}_k$ that lies between u_α and u_γ, that is, such that, for all $w \in \bar{\mathsf{D}}$, we have

$$d_{\mathsf{D}}(u_\alpha,w) + d_{\mathsf{D}}(w,u_\gamma) = d_{\mathsf{D}}(u_\alpha,u_\gamma). \tag{15}$$

Thus, by (7), (14), and (15), for all $w \in \bar{\mathsf{D}}$, we obtain

$$\begin{aligned}d_L(\varphi(u_\alpha),\varphi(w)) + d_L(\varphi(w),\varphi(u_\gamma)) &\leq 2^{\mathsf{C}}\lambda(d_{\mathsf{D}}(u_\alpha,w) + d_{\mathsf{D}}(w,u_\gamma))\\ &= 2^{\mathsf{C}}\lambda d_{\mathsf{D}}(u_\alpha,u_\gamma) \leq 2^{\mathsf{C}}d_L(\varphi(u_\alpha),\varphi(u_\gamma)) \leq \frac{3}{4}\bar{d}.\end{aligned} \tag{16}$$

Recall that we denoted $\varphi(u_\alpha) = (A_\alpha,a_\alpha)$ and $\varphi(u_\gamma) = (A_\gamma,a_\gamma)$. For each $w \in \bar{\mathsf{D}}$ we denote $\varphi(w) = (A_w,a_w)$. Then, by (1) and (16), for $j \in \{\gamma,w\}$ we have

$$\max\left\{d_W(a_\alpha,a_j), \max_{s\in A_\alpha\triangle A_j} d_W(a_\alpha,s)\right\} \leq d_L((A_\alpha,a_\alpha),(A_j,a_j)) \leq \frac{3}{4}\bar{d}.$$

Hence, by (12), the vertices a_γ, a_w belong to the path p connecting $t = t_{\nu,\mu}$ and $\bar{t} = t_\nu$, and $(A_\alpha \triangle A_\gamma) \cup (A_\alpha \triangle A_w) \subset p$. Since $A_\gamma \triangle A_w \subset (A_\alpha \triangle A_\gamma) \cup (A_\alpha \triangle A_w)$, we also have $A_\gamma \triangle A_w \subset p$. Thus $\varphi_p : \bar{\mathsf{D}} \longrightarrow \mathsf{La}(W_{\theta,n})$ defined by $\varphi_p(w) = (A_w \cap p, a_w)$ is a bi-Lipschitz embedding of $\bar{\mathsf{D}}$ into $\mathsf{La}(p)$ with distortion at most 2^{C}.

Since $\bar{\mathsf{D}} = \mathsf{D}_{k-(2\mathsf{C}+\theta+9)}$ and $k-(2C+\theta+9) \geq k_C$, this contradicts (6), and thus ends the proof of Theorem 1.8. $\square$

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References

[1] Bartholdi L. Growth of groups and wreath products. In: Ceccherini-Silberstein T., Salvatori M., Sava-Huss E. (Eds.) Groups, Graphs and Random Walks. London Math. Soc. Lecture Note Series, Cambridge University Press, 2017, 1–76. doi:10.1017/9781316576571.003

[2] Baudier F., Motakis P., Schlumprecht Th., Zsák A. *On the bi-Lipschitz geometry of lamplighter graphs.* Discrete Comput. Geom. 2021, **66** (1), 203–235. doi:10.1007/s00454-020-00184-1

[3] Baudier F., Motakis P., Schlumprecht Th., Zsák A. *Stochastic approximation of lamplighter metrics.* Bull. Lond. Math. Soc. 2022, **54** (5), 1804–1826. doi:10.1112/blms.12657

[4] Bourgain J. *The metrical interpretation of superreflexivity in Banach spaces.* Israel J. Math. 1986, **56** (2), 222–230. doi:10.1007/BF02766125

[5] Bourgain J., Milman V., Wolfson H. *On type of metric spaces.* Trans. Amer. Math. Soc. 1986, **294** (1), 295–317. doi:10.1090/S0002-9947-1986-0819949-8

[6] Cornulier Y., Stalder Y., Valette A. *Proper actions of wreath products and generalizations.* Trans. Amer. Math. Soc. 2012, **364** (6), 3159–3184. doi:10.1090/S0002-9947-2012-05475-4

[7] Eskenazis A. *Geometric inequalities and advances in the ribe program.* Ph.D. thesis, Princeton University, 2019.

[8] Genevois A. *Lamplighter groups, median spaces and Hilbertian geometry.* Proc. Edinb. Math. Soc. (2) 2022, **65** (2), 500–529. doi:10.1017/S0013091522000190

[9] Genevois A., Tessera R. *Asymptotic geometry of lamplighters over one-ended groups.* Invent. Math. 2024, **238** (1), 1–67. doi:10.1007/s00222-024-01278-w

[10] Gupta A., Krauthgamer R., Lee J.R. *Bounded geometries, fractals, and low-distortion embeddings*. In: Proc. of 44th Symposium on Foundations of Computer Science, 2003, 534–543. doi:10.1109/SFCS.2003.1238226

[11] Gupta A., Newman I., Rabinovich Y., Sinclair A. *Cuts, trees and ℓ_1-embeddings of graphs.* Combinatorica 2004, **24**, 233–269. doi:10.1007/s00493-004-0015-x

[12] James R.C. *A nonreflexive Banach space that is uniformly nonoctahedral.* Israel J. Math. 1974, **18** (2), 145–155. doi:10.1007/BF02756869

[13] James R.C. *Nonreflexive spaces of type 2.* Israel J. Math. 1978, **30** (1-2), 1–13. doi:10.1007/BF02760825

[14] Johnson W.B., Schechtman G. *Diamond graphs and super-reflexivity.* J. Topol. Anal. 2009, **1** (2), 177–189. doi:10.1142/S1793525309000114

[15] Lee J.R., Naor A., Peres Y. *Trees and Markov convexity.* Geom. Funct. Anal. 2009, **18** (5), 1609–1659. doi:10.1007/s00039-008-0689-0

[16] Lindenstrauss J., Tzafriri L. *Classical Banach spaces. I. Sequence spaces.* In: Ergebnisse der Mathematik und ihrer Grenzgebiete, **92**. Springer-Verlag, Berlin-New York, 1977.

[17] Lyons R., Pemantle R., Peres Y. *Random walks on the lamplighter group.* Ann. Probab. 1996, **24** (4), 1993–2006. doi:10.1214/aop/1041903214

[18] Matoušek J. *On embedding trees into uniformly convex Banach spaces.* Israel J. Math. 1999, **114** (1), 221–237. doi:10.1007/BF02785579

[19] Naor A. *An introduction to the Ribe program.* Jpn. J. Math. 2012, **7** (2), 167–233. doi:10.1007/s11537-012-1222-7

[20] Naor A., Peres Y. *Embeddings of discrete groups and the speed of random walks.* Int. Math. Res. Not. IMRN 2008, **2008**, rnn076. doi:10.1093/imrn/rnn076

[21] Naor A., Peres Y. *L_p compression, traveling salesmen, and stable walks.* Duke Math. J. 2011, **157** (1), 53–108. doi:10.1215/00127094-2011-002

[22] Ostrovskii M.I. Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces. In: de Gruyter Studies in Mathematics, **49**. Walter de Gruyter & Co., Berlin, 2013.

[23] Ostrovskii M.I., Randrianantoanina B. *A characterization of superreflexivity through embeddings of lamplighter groups.* Proc. Amer. Math. Soc. 2019, **147** (11), 4745–4755. doi:10.1090/proc/14526

[24] Pisier G. Martingales in Banach spaces. In: Cambridge Studies in Advanced Mathematics, **155**, Cambridge Univ. Press, Cambridge, 2016.

[25] Robertson A. Fundamentals of Ramsey theory. Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2021.

[26] Stein M., Taback J. *Metric properties of Diestel-Leader groups.* Michigan Math. J. 2013, **62** (2), 365–386. doi: 10.1307/mmj/1370870377

[27] Taback J. Lamplighter groups. In: Clay M., Margalit D. (Eds.) Office hours with a geometric group theorist, 310–330. Princeton Univ. Press, Princeton, NJ, 2017. doi:10.23943/princeton/9780691158662.003.0015

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Мелбі Ш., Рандріанантоаніна Б. *Властивості біліпшицевих вкладень графів ліхтарника на зважених і незважених деревах* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 222–235.

У 2021 році F. Baudier, P. Motakis, Th. Schlumprecht та A. Zsák довели, що якщо послідовність $(G_k)_{k\in\mathbb{N}}$ графів містить послідовність повних графів з рівномірно обмеженим спотворенням, то послідовність графів ліхтарника над графами G_k містить куби Геммінга з рівномірно обмеженим спотворенням. Вони також поставили питання, чи справджується обернене твердження.

Автори висловили припущення, що контрприкладом може бути послідовність дерев, у яких кожне ребро замінено шляхом довжини, що зростає "помірно". У цій роботі ми доводимо, що це справді так, а також показуємо, що іншим контрприкладом є послідовність "помірно" зважених дерев.

Крім того, ми доводимо, що діамантові графи не вкладаються з рівномірно обмеженим спотворенням у графи ліхтарника на деревах, у яких ребра замінено шляхами достатньо швидко зростаючих довжин.

Ключові слова і фрази: спотворення біліпшицевого вкладення, група ліхтарника, ліпшицеве відображення, метричне вкладення, програма Рібе, суперрефлексивність.