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On the stability to perturbations of Stieltjes continued fractions with complex elements

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This paper investigates the conditions for stability to perturbations of Stieltjes continued fractions with complex elements. An analytical approach is proposed that allows for the evaluation of the influence of perturbations of the fraction's coefficients and its complex variable on the value of its approximant. Recurrence relations for the relative errors of the approximants are established and presented in the form of a continued fraction. Sufficient conditions for stability to perturbations are obtained, as well as explicit estimates for the relative errors of the approximants. Sets of stability to perturbations are constructed, in particular, semi-circular and circular sets in the complex plane.

Key words and phrases: Stieltjes continued fraction, stability to perturbations, set of stability to perturbations, relative error, approximant of continued fraction.

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Introduction

Continued fractions [3, 12, 39, 41] and their multidimensional generalizations, particularly branched continued fractions [7, 27, 28], are an important tool for approximating functions in various fields of science, including the theory of special functions [1, 2, 16, 18, 22], economics [38], quantum mechanics [10, 44], computational mathematics [17, 31, 32], and computer science [5, 6, 11, 43, 45], and education [14]. One of the key advantages of continued fractions and their generalizations are their approximative properties, convergence, and speed of convergence, which in many cases make them more effective than classical approximation by power series (see, for example, [4, 20, 21] and also [24, 26, 29, 36]).

However, their effectiveness also depends on the stability of the algorithms used to find the values of the approximants. Even minor rounding errors that occur during implementation on a computer can lead to a significant accumulation of errors and, consequently, a loss of accuracy in the results [27, 31]. The backward recurrence algorithm deserves special attention in this context, as it often demonstrates stability in contrast to the forward recurrence algorithm [9, 13, 42]. The stability of algorithms for computing continued fractions has been the subject of research by W. Jones, W.J. Thron, W. Gautschi, and others [30, 40]. The stability of expansions of some hypergeometric functions has been considered in works [19, 34, 35].

At the same time, the study of the stability to perturbations of continued fractions and their multidimensional generalizations remains a relevant and less-studied area. The works

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investigating this issue focus on numerical continued fractions [15, 33, 37]. However, the case of functional continued fractions with complex elements requires the development of new approaches to the investigation, particularly for Stieltjes continued fractions (S-fractions) and their generalizations [8, 23, 25, 27].

The purpose of this work is to establish the conditions for stability to perturbations of S-fractions and to construct sets of stability to perturbations, within which the boundedness of the approximant errors is guaranteed, and to establish explicit estimates for the approximant errors. For this, the analytical theory of continued fractions is used, in particular, the method of element sets and their corresponding value sets of the approximant tails.

The paper is organized as follows. Section 2 presents the main concepts, definitions, and formulas for the relative errors of the S-fraction approximants. In Section 3, the main theoretical result is proven – sufficient conditions for the stability of S-fractions are obtained, and estimates for the relative errors of the approximants are established. In Section 4, sets of stability to perturbations are constructed. Finally, Section 5 formulates the conclusions and outlines prospects for further research.

1 Basis notations and definitions

Let us consider the continued fraction

$$\cfrac{a_0}{1+\cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\ddots}}}, \tag{1}$$

which is called an S-fraction, where $a_k \in \mathbb{R}_+$, $k \geq 0$, $z \in \mathbb{C}$. The nth approximant $f_n(z)$ of (1) is denoted as

$$f_n(z) = \cfrac{a_0}{1+\cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\ddots+\cfrac{a_n z}{1}}}}.$$

The so-called tails of the nth approximant $f_n(z)$ are defined as follows

$$Q_k^{(n)}(z) = 1 + \cfrac{a_{k+1} z}{1+\cfrac{a_{k+2} z}{1+\ddots+\cfrac{a_n z}{1}}}, \quad 0 \leq k \leq n.$$

For the tails $Q_k^{(n)}(z)$, the following recurrence relations hold

$$Q_k^{(n)}(z) = 1 + \frac{a_{k+1} z}{Q_{k+1}^{(n)}(z)}, \quad 0 \leq k \leq n-1,$$

with initial condition $Q_n^{(n)}(z) = 1$.

Let $\hat{a}_k$, $\hat{z}$ be the perturbed values of the coefficients a_k and the variable z of the S-fraction (1), respectively. Then we obtain the perturbed S-fraction

$$\cfrac{\hat{a}_0}{1+\cfrac{\hat{a}_1 \hat{z}}{1+\cfrac{\hat{a}_2 \hat{z}}{1+\ddots}}} \tag{2}$$

to the S-fraction (1).

Definition 1. *The S-fraction* (1) *is called stable to perturbations at the point $z_0 \in \mathbb{C}$ if for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for every $\hat{a}_k \in \mathbb{C}$, $k \geq 0$, such that $|\hat{a}_k - a_k|/|a_k| < \delta_\varepsilon$, and every $\hat{z}_0 \in \mathbb{C}$, such that $|\hat{z}_0 - z_0|/|z_0| < \delta_\varepsilon$, the inequalities $|\hat{f}_n(\hat{z}_0) - f_n(z_0)|/|f_n(z_0)| < \varepsilon$ hold for all $n \geq 1$, where*

$$f_n(z_0) = \cfrac{a_0}{1 + \cfrac{a_1 z_0}{1 + \cfrac{a_2 z_0}{1 + \ddots + \cfrac{a_n z_0}{1}}}}, \qquad \hat{f}_n(\hat{z}_0) = \cfrac{\hat{a}_0}{1 + \cfrac{\hat{a}_1 \hat{z}_0}{1 + \cfrac{\hat{a}_2 \hat{z}_0}{1 + \ddots + \cfrac{\hat{a}_n \hat{z}_0}{1}}}}.$$

Let $\varepsilon_k^{(a)}$, $k \geq 0$, $\varepsilon^{(z)}$, $\varepsilon_n^{(f)}$, $n \geq 0$, $\varepsilon_{k,n}^{(G)}$, $0 \leq k \leq n$, be the relative errors of the coefficients a_k, the variable z, the approximant $f_n(z)$, and the quantities

$$G_0^{(n)}(z) = \frac{a_0}{Q_0^{(n)}(z)}, \quad G_k^{(n)}(z) = \frac{a_k z}{Q_k^{(n)}(z)}, \quad 1 \leq k \leq n,$$

respectively, i.e. the following relations hold

$$\hat{a}_k = a_k(1 + \varepsilon_k^{(a)}), \quad k \geq 0, \qquad \hat{z} = z(1 + \varepsilon^{(z)}),$$

$$\hat{f}_n(\hat{z}) = f_n(z)(1 + \varepsilon_n^{(f)}), \quad n \geq 0, \qquad \hat{G}_k^{(n)}(\hat{z}) = G_k^{(n)}(z)(1 + \varepsilon_{k,n}^{(G)}), \quad 0 \leq k \leq n,$$

where

$$\hat{G}_0^{(n)}(\hat{z}) = \frac{\hat{a}_0}{\hat{Q}_0^{(n)}(\hat{z})}, \quad \hat{G}_k^{(n)}(\hat{z}) = \frac{\hat{a}_k \hat{z}}{\hat{Q}_k^{(n)}(z)}, \quad 1 \leq k \leq n,$$

and

$$\hat{Q}_k^{(n)}(z) = 1 + \cfrac{\hat{a}_{k+1}\hat{z}}{1 + \cfrac{\hat{a}_{k+2}\hat{z}}{1 + \ddots + \cfrac{\hat{a}_n \hat{z}}{1}}}$$

are the tails of the perturbed approximant $\hat{f}_n(\hat{z})$.

We will prove that the following recurrence formulas hold for the relative errors $\varepsilon_{k,n}^{(G)}$

$$\varepsilon_{0,n}^{(G)} = -1 + \frac{1 + \varepsilon_0^{(a)}}{1 + g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}}, \tag{3}$$

$$\varepsilon_{k,n}^{(G)} = -1 + \frac{(1 + \varepsilon_k^{(a)})(1 + \varepsilon^{(z)})}{1 + g_{k+1}^{(n)}(z)\varepsilon_{k+1,n}^{(G)}}, \quad 1 \leq k \leq n-1, \tag{4}$$

$$\varepsilon_{n,n}^{(G)} = \varepsilon_n^{(a)} + \varepsilon^{(z)} + \varepsilon_n^{(a)}\varepsilon^{(z)}, \tag{5}$$

where the quantities $g_k^{(n)}$ are given by the formula

$$g_k^{(n)}(z) = \frac{G_k^{(n)}(z)}{Q_{k-1}^{(n)}(z)}, \quad 1 \leq k \leq n. \tag{6}$$

Since $Q_n^{(n)}(z)=\hat{Q}_n^{(n)}(\hat{z})=1$, then $G_n^{(n)}(z)=a_nz$, $\hat{G}_n^{(n)}(\hat{z})=\hat{a}_n\hat{z}$ and $\varepsilon_{n,n}^{(G)}=\varepsilon_n^{(a)}+\varepsilon^{(z)}+\varepsilon_n^{(a)}\varepsilon^{(z)}$. For $1\le k\le n-1$ we have

$$\begin{aligned}\varepsilon_{k,n}^{(G)}&=\frac{\hat{G}_k^{(n)}(\hat{z})-G_k^{(n)}(z)}{G_k^{(n)}(z)}=-1+\frac{\hat{a}_k\hat{z}}{a_kz}\frac{Q_k^{(n)}(z)}{\hat{Q}_k^{(n)}(\hat{z})}=-1+\frac{(1+\varepsilon_k^{(a)})(1+\varepsilon^{(z)})Q_k^{(n)}(z)}{1+\hat{G}_{k+1}^{(n)}(\hat{z})}\\&=-1+\frac{(1+\varepsilon_k^{(a)})(1+\varepsilon^{(z)})}{1/Q_k^{(n)}(z)+G_{k+1}^{(n)}(z)/Q_k^{(n)}(z)(1+\varepsilon_{k+1,n}^{(G)})}\\&=-1+\frac{(1+\varepsilon_k^{(a)})(1+\varepsilon^{(z)})}{1-g_{k+1}^{(n)}(z)+g_{k+1}^{(n)}(z)(1+\varepsilon_{k+1,n}^{(G)})}=-1+\frac{(1+\varepsilon_k^{(a)})(1+\varepsilon^{(z)})}{1+g_{k+1}^{(n)}(z)\varepsilon_{k+1,n}^{(G)}}.\end{aligned}$$

For $k=0$ we have

$$\varepsilon_{0,n}^{(G)}=\frac{\hat{G}_0^{(n)}(\hat{z})-G_0^{(n)}(z)}{G_0^{(n)}(z)}=-1+\frac{\hat{a}_0}{a_0}\frac{Q_0^{(n)}(z)}{\hat{Q}_0^{(n)}(\hat{z})}=-1+\frac{(1+\varepsilon_0^{(a)})}{1+g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}}.$$

From formulas (3)–(5), we obtain the formula for the relative error of the quantity $G_k^{(n)}(z)$ in the form of a continued fraction

$$\varepsilon_{k,n}^{(G)}=-1+\cfrac{(1+\varepsilon_k^{(a)})(1+\varepsilon^{(z)})}{1-g_{k+1}^{(n)}+\cfrac{g_{k+1}^{(n)}(1+\varepsilon_{k+1}^{(a)})(1+\varepsilon^{(z)})}{1-g_{k+2}^{(n)}+\cfrac{g_{k+2}^{(n)}(1+\varepsilon_{k+2}^{(a)})(1+\varepsilon^{(z)})}{1-g_{k+3}^{(n)}+\ddots+\cfrac{g_n^{(n)}(1+\varepsilon_n^{(a)})(1+\varepsilon^{(z)})}{1}}}},\tag{7}$$

where $1\le k\le n-1$.

Since $f_n(z)=G_0^{(n)}(z)$, $\hat{f}_n(\hat{z})=\hat{G}_0^{(n)}(\hat{z})$, then

$$\varepsilon_n^{(f)}=\varepsilon_{0,n}^{(G)}=-1+\frac{1+\varepsilon_0^{(a)}}{1+g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}},$$

and, by setting $k=1$ in formula (7), we obtain the formula for the relative error of the nth approximant of the S-fraction (1)

$$\varepsilon_n^{(f)}=-1+\cfrac{1+\varepsilon_0^{(a)}}{1-g_1^{(n)}+\cfrac{g_1^{(n)}(1+\varepsilon_1^{(a)})(1+\varepsilon^{(z)})}{1-g_2^{(n)}+\cfrac{g_2^{(n)}(1+\varepsilon_2^{(a)})(1+\varepsilon^{(z)})}{1-g_3^{(n)}+\ddots+\cfrac{g_n^{(n)}(1+\varepsilon_n^{(a)})(1+\varepsilon^{(z)})}{1}}}}.\tag{8}$$

2 Sufficient conditions for stability to perturbations of S-fractions

In the study of stability to perturbations of the S-fraction, we will use the following statements.

Lemma 1 ([34]). *Let p, q be positive real constants. The modified approximant*

$$h_n(\omega) = q - \left.\cfrac{p}{q - \cfrac{p}{q - \cfrac{p}{q - \ddots - \cfrac{p}{q+\omega}}}}\right\} n \tag{9}$$

takes a positive value if

$$4p \leq q^2 \tag{10}$$

and one of the following conditions is met

$$\omega > -\frac{q - \sqrt{q^2 - 4p}}{2} \tag{11}$$

or $\omega < -q$.

Lemma 2 ([34]). *The sequence of modified approximants $\{h'_n(\omega)\}_{n\in\mathbb{N}}$, where*

$$h'_n(\omega) = -\left.\cfrac{p}{q - \cfrac{p}{q - \cfrac{p}{q - \ddots - \cfrac{p}{q+\omega}}}}\right\} n \tag{12}$$

whose elements satisfy inequality (10)*, converges to one of the values:*
- $-(q - \sqrt{q^2 - 4p})/2$, *if* $\omega \neq -(q + \sqrt{q^2 - 4p})/2$ *and* $4p < q^2$,
- $-(q + \sqrt{q^2 - 4p})/2$, *if* $\omega = -(q + \sqrt{q^2 - 4p})/2$ *and* $4p < q^2$,
- $-q/2$, *if* $4p = q^2$.

The following theorem holds.

Theorem 1. *The S-fraction* (1) *is stable to perturbations if there exist constants α, β, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $\alpha + \beta \neq 0$, such that the relative errors of the coefficients a_k and the variable z of the S-fraction* (1) *satisfy the conditions*

$$|\varepsilon_k^{(a)}| \leq \alpha, \quad k \geq 0, \quad |\varepsilon^{(z)}| \leq \beta, \tag{13}$$

there exists a positive constant η, $0 < \eta < 1$, such that

$$|g_k^{(n)}| \leq \eta, \quad 1 \leq k \leq n, \quad n \geq 1, \tag{14}$$

where the quantities $g_k^{(n)}$ are determined according to (6)*, and*

$$4\eta\gamma \leq (1+\eta)^2, \tag{15}$$

where $\gamma = (1+\alpha)(1+\beta)$. In this case, for the relative errors of the approximants of the S-fraction (1)*, the following estimate holds*

$$|\varepsilon^{(n)}| \leq \frac{1 - \eta - 2\eta\beta - \sqrt{(1+\eta)^2 - 4\eta\gamma}}{2\eta(1+\beta)}, \quad n \geq 1, \tag{16}$$

if $4\eta\gamma < (1+\eta)^2$, and

$$|\varepsilon^{(n)}| \leq \frac{1 - (2\gamma - 1 - 2\sqrt{\gamma(\gamma-1)})(1+2\beta)}{2(2\gamma - 1 - 2\sqrt{\gamma(\gamma-1)})(1+\beta)}, \quad n \geq 1, \tag{17}$$

if $4\eta\gamma = (1+\eta)^2$.

Proof. Let n be a fixed natural number. We will estimate the relative errors $\varepsilon_{k,n}^{(G)}$ of the quantities $G_k^{(n)}$.

From formula (5), the estimate $|\varepsilon_{n,n}^{(G)}| \leq \alpha + \beta + \alpha\beta$ follows. Using the method of mathematical induction on k, for $k = n-1, k-2, \ldots, 1$, we will show that

$$|\varepsilon_{k,n}^{(G)}| \leq -1 + \frac{\gamma}{h_{n-k}(-\eta)}, \tag{18}$$

for $k = n-1, k-2, \ldots, 1$, where the modified approximant $h_{n-k}(-\eta)$ is determined according to (9), and its elements p, q are given by the formulas

$$p = \eta\gamma, \quad q = 1 + \eta. \tag{19}$$

From inequality (15), it follows that $\eta\gamma < (1+\eta)$. Then for $k = n-1$, considering (4), we have

$$\begin{aligned}|\varepsilon_{n-1,n}^{(G)}| &= \Big| -1 + \frac{(1+\varepsilon_{n-1}^{(a)})(1+\varepsilon^{(z)})}{1+g_n^{(n)}\varepsilon_{n,n}^{(G)}}\Big| = \Big|\frac{\varepsilon_{n-1}^{(a)} + \varepsilon^{(z)} + \varepsilon_{n-1}^{(a)}\varepsilon^{(z)} - g_n^{(n)}\varepsilon_{n,n}^{(G)}}{1+g_n^{(n)}\varepsilon_{n,n}^{(G)}}\Big| \\ &\leq -1 + \frac{(1+\alpha)(1+\beta)}{1-\eta(\alpha+\beta+\alpha\beta)} \leq -1 + \frac{\gamma}{1+\eta-\eta\gamma} = -1 + \frac{\gamma}{h_1(-\eta)}.\end{aligned}$$

Assuming that the estimates (18) hold for some $k = m+1$, $1 \leq m \leq n-2$,

$$|\varepsilon_{m+1,n}^{(G)}| \leq -1 + \frac{\gamma}{h_{n-m-1}(-\eta)},$$

we will prove them for $k = m$. From (15), it follows that the elements of the modified approximant $h_{n-m}(-\eta)$ satisfy conditions (1) and (10). Then, according to Lemma 1, $h_{n-m}(-\eta) > 0$ and

$$\begin{aligned}|\varepsilon_{m,n}^{(G)}| &= \Big| -1 + \frac{(1+\varepsilon_m^{(a)})(1+\varepsilon^{(z)})}{1+g_{m+1}^{(n)}\varepsilon_{m+1,n}^{(G)}}\Big| = \Big|\frac{\varepsilon_m^{(a)} + \varepsilon^{(z)} + \varepsilon_m^{(a)}\varepsilon^{(z)} - g_{m+1}^{(n)}\varepsilon_{m+1,n}^{(G)}}{1+g_{m+1}^{(n)}\varepsilon_{m+1,n}^{(G)}}\Big| \\ &\leq -1 + \frac{(1+\alpha)(1+\beta)}{1-\eta(-1+\gamma/h_{n-m-1}(-\eta))} = -1 + \frac{\gamma}{1+\eta-\eta\gamma/h_{n-m-1}(-\eta)} \\ &= -1 + \frac{\gamma}{h_{n-m}(-\eta)},\end{aligned}$$

which proves the estimates (18).

From formula (3), the estimate follows

$$\begin{aligned}|\varepsilon_n^{(f)}| = |\varepsilon_{0,n}^{(G)}| &= -1 + \frac{1+\varepsilon_k^{(a)}}{1+g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}} = \frac{\varepsilon_k^{(a)} - g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}}{1+g_1^{(n)}(z)\varepsilon_{1,n}^{(G)}} \\ &\leq -1 + \frac{1+\alpha}{1+\eta-\gamma/h_{n-1}(-\eta)} = -1 + \frac{1+\alpha}{h_n(-\eta)}.\end{aligned}$$

Let us consider the continued fraction

$$-1 + \cfrac{1+\alpha}{1+\eta - \cfrac{\eta\gamma}{1+\eta - \cfrac{\eta\gamma}{1+\eta - \ddots}}} \tag{20}$$

and the sequence of its modified approximants $\{h''_n(-\eta)\}_{n\in\mathbb{N}}$, where

$$h''_n(-\eta) = -1 - \frac{h'_n(-\eta)}{\eta(1+\beta)},$$

and $h'_n(-\eta)$ are determined according to (12), with elements p, q given by (19).

For the difference between the modified approximants $h''_{n+1}(-\eta)$ and $h''_n(-\eta)$ of the continued fraction (20), the following formula holds

$$h''_{n+1}(-\eta) - h''_n(-\eta) = \frac{h'_n(-\eta) - h'_{n+1}(-\eta)}{\eta(1+\beta)} = \frac{1}{\eta(1+\beta)} \frac{\eta^{n+1}\gamma^n(\gamma-1)}{h_n(-\eta)\sum\limits_{m=0}^{n-1} h^2_m(-\eta)}.$$

Since $\eta > 0$, $\gamma > 1$ and, according to Lemma 1, for all $n \geq 1$, $h_n(-\eta) > 0$, then $\{h''_n(-\eta)\}_{n\in\mathbb{N}}$ is an increasing sequence and

$$|\varepsilon_n^{(f)}| \leq \lim_{n\to+\infty} h''_n(-\eta).$$

Assume that $4\eta\gamma < (1+\eta)^2$. Since $-\eta \neq -(1+\eta+\sqrt{(1+\eta)^2-4\eta\gamma})/2$, then according to Lemma 2, we get

$$\begin{aligned}\lim_{n\to+\infty} h''{}_n(-\eta) &= -1 - \frac{1}{\eta(1+\beta)} \lim_{n\to+\infty} h'{}_n(-\eta) \\ &= -1 + \frac{1+\eta-\sqrt{(1+\eta)^2-4\eta\gamma}}{2\eta(1+\beta)} = \frac{1-\eta-2\eta\beta-\sqrt{(1+\eta)^2-4\eta\gamma}}{2\eta(1+\beta)}.\end{aligned}$$

Thus, for the relative errors of the approximants of the S-fraction (1), the estimate (16) holds.

Let us consider the function

$$\varphi(\alpha,\beta) = \frac{1-\eta-2\eta\beta-\sqrt{(1+\eta)^2-4\eta(1+\alpha)(1+\beta)}}{2\eta(1+\beta)},$$

continuous at the point $(0,0)$. Since

$$\lim_{(\alpha,\beta)\to(+0,+0)} \varphi(\alpha,\beta) = 0,$$

for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\alpha > 0$, $\beta > 0$ and $\sqrt{\alpha^2+\beta^2} < \delta$, the inequality $\varphi(\alpha,\beta) < \varepsilon$ holds. If $\alpha < \delta/\sqrt{2}$, $\beta < \delta/\sqrt{2}$, then for every $\hat{a}_k \in \mathbb{R}_+$, $k \geq 0$, and every $\hat{z} \in \mathbb{C}$, such that $|(\hat{a}_k - a_k)/a_k| \leq \alpha < \delta/\sqrt{2}$, $|(\hat{z}-z)/z| \leq \beta < \delta/\sqrt{2}$, the inequalities $|\varepsilon_n^{(f)}| \leq \varphi(\alpha,\beta) < \varepsilon$ hold for the relative errors of all approximants of the S-fraction (1). Thus, according to Definition 1, the S-fraction (1) is stable to perturbations, and the estimate (16) holds for its relative errors.

Assume that $4\eta\gamma = (1+\eta)^2$. Then according to Lemma 2, we have

$$\lim_{n\to+\infty} h''_n(-\eta) = \frac{1-\eta-2\eta\beta}{2\eta(1+\beta)},$$

where $\eta = 2\gamma - 1 - 2\sqrt{\gamma(\gamma-1)}$. Thus, for the relative errors of the approximants of the S-fraction (1), the estimate (17) holds, from which, as in the previous case, its stability to perturbations follows. □

3 Sets of stability to perturbations of S-fractions

Let us consider the continued fraction

$$\cfrac{c_0}{d_0+\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots}}}. \tag{21}$$

Let $\{P_k\}_{k\in\mathbb{N}\cup\{0\}}$, $\varnothing\neq P_k\subset\mathbb{C}^2$, $k\geq 0$, be a sequence of element sets of (21), i.e. $(d_k,c_{k+1})\in P_k$, $k\geq 0$.

Definition 2. *A sequence $\{W_k\}_{k\in\mathbb{N}\cup\{0\}}$, $\varnothing\neq W_k\subset\mathbb{C}$, $k\geq 0$, is called a sequence of value sets for the tails $Q_k^{(n)}$ of the approximants of the continued fraction* (21)*, corresponding to the sequence of element sets $\{P_k\}_{k\in\mathbb{N}\cup\{0\}}$, if*

$$d_k\in W_k \tag{22}$$

for every $(d_k,c_{k+1})\in P_k$, $k\geq 0$, and

$$d_k+\frac{c_{k+1}}{w}\in W_k \tag{23}$$

for every $w\in W_{k+1}$ and every $(d_k,c_{k+1})\in P_k$, $k\geq 0$.

The following theorem holds.

Theorem 2. *The set*

$$\Omega_M=\{z\in\mathbb{C}:\ |\arg z|\leq\pi/2,\ |z|\leq M\},\quad M>0, \tag{24}$$

is a set of stability to perturbations of the S-fraction (1)*, if there exist constants α, β, $0\leq\alpha<1$, $0\leq\beta<1$, $\alpha+\beta\neq 0$, such that for the relative errors of the coefficients a_k and the variable z, the inequalities* (13) *hold; there exists a constant A, $A>0$, such that*

$$a_k\leq A,\quad k\geq 0, \tag{25}$$

and the inequality (15) *holds, where $\eta=MA/\sqrt{1+(MA)^2}$. In this case, for the relative errors of the approximants of the S-fraction* (1)*, the estimate* (16) *holds if $4\eta\gamma<(1+\eta)^2$, and the estimate* (17) *holds if $4\eta\gamma=(1+\eta)^2$.*

Proof. Let us consider the continued fraction

$$\cfrac{r_0r_1a_0}{r_1+\cfrac{r_1^2a_1z}{r_1+\cfrac{r_1^2a_2z}{r_1+\ddots}}}, \tag{26}$$

where $r_0\neq 0$, $r_1\neq 0$. Let us denote by

$$f_n'(z)=\cfrac{r_0r_1a_0}{r_1+\cfrac{r_1^2a_1z}{r_1+\cfrac{r_1^2a_2z}{r_1+\ddots+\cfrac{r_1^2a_nz}{r_1}}}}$$

the nth approximant of the continued fraction (26), and by

$$Q_k'^{(n)}(z) = r_1 + \cfrac{r_1^2 a_{k+1} z}{r_1 + \cfrac{r_1^2 a_{k+2} z}{r_1 + \ddots + \cfrac{r_1^2 a_n z}{r_1}}}, \quad 0 \le k \le n,$$

the tails of the approximant $f_n'(z)$.

In [12, Theorem 4.1], it was proven that $g_k'^{(n)}(z) = g_k^{(n)}(z)$, $1 \le k \le n$, where $g_k'^{(n)}(z) = G_k'^{(n)}(z)/Q_{k-1}'^{(n)}(z)$, $G_k'^{(n)}(z) = r_1^2 a_k z / Q_k'^{(n)}(z)$, $1 \le k \le n$, and the quantities $g_k^{(n)}$ are determined by formula (6).

Let $r_0 = \sqrt{z}$, $r_1 = 1/\sqrt{z}$. Then the continued fraction (26) will have the form

$$\cfrac{a_0}{1/\sqrt{z} + \cfrac{a_1}{1/\sqrt{z} + \cfrac{a_2}{1/\sqrt{z} + \ddots}}}. \tag{27}$$

Let us make a change of variable $t = 1/\sqrt{z}$ in the continued fraction (27). Then we obtain the following continued fraction

$$\cfrac{a_0}{t + \cfrac{a_1}{t + \cfrac{a_2}{t + \ddots}}}. \tag{28}$$

Let us denote by

$$f_n''(t) = \cfrac{a_0}{t + \cfrac{a_1}{t + \cfrac{a_2}{t + \ddots + \cfrac{a_n}{t}}}}$$

the nth approximant of the continued fraction (28), and by

$$Q_k''^{(n)}(t) = t + \cfrac{a_{k+1}}{t + \cfrac{a_{k+2}}{t + \ddots + \cfrac{a_n}{t}}}, \quad 0 \le k \le n,$$

the tails of the approximant $f_n''(t)$, $G_k''^{(n)}(t) = a_k / Q_k''^{(n)}(t)$, $1 \le k \le n$.

If $z \in \Omega_M$, where the set Ω_M is determined according to (24), then $t \in W_M$, where

$$W_M = \{z \in \mathbb{C} : \ |\arg z| \le \pi/4, \ |z| \ge 1/\sqrt{M}\}.$$

Let us prove that W_M is a set of values for the tails of the continued fraction (28), corresponding to the set of elements $P_{M,A}$, where

$$P_{M,A} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{R}_+ : \ |\arg z_1| \le \pi/4, \ |z_1| \ge 1/\sqrt{M}, \ z_2 \le A\}.$$

Since $t \in W_M$, condition (22) is satisfied. Let us prove that condition (23) is satisfied.

The function $1/z$ maps the set W_M to the set

$$W_M' = \{z \in \mathbb{C} : \ |\arg| \le \pi/4, \ |z| \le \sqrt{M}\}.$$

Then

$$V_M = a_{k+1}W'_M = \{z \in \mathbb{C}: \ |\arg z| \leq \pi/4, \ |z| \leq a_{k+1}\sqrt{M}\}$$

and condition (23) is satisfied if

$$|\arg(t+z)| \leq \pi/4, \tag{29}$$

$$|t+z| \geq 1/\sqrt{M}, \tag{30}$$

for all $z \in V_M$. To prove (29), let us denote $t = \operatorname{Re} t + i \operatorname{Im} t$, $z = \operatorname{Re} z + i \operatorname{Im} z$. From the fact that $|\arg t| \leq \pi/4$, $|\arg z| \leq \pi/4$, it follows that $|\operatorname{Im} t| \leq \operatorname{Re} t$, $|\operatorname{Im} z| \leq \operatorname{Re} z$. Since $\operatorname{Re} t > 0$, $\operatorname{Re} z \geq 0$, then

$$\frac{|\operatorname{Im}(t+z)|}{\operatorname{Re}(t+z)} = \frac{|\operatorname{Im} t + \operatorname{Im} z|}{\operatorname{Re} t + \operatorname{Re} z} \leq \frac{|\operatorname{Im} t| + |\operatorname{Im} z|}{\operatorname{Re} t + \operatorname{Re} z} \leq 1.$$

Thus, inequality (29) holds for all $z \in V_M$.

Let us denote $t = |t|e^{i\psi_1}$, $z = |z|e^{i\psi_2}$. Then $|t+z| = \sqrt{|t|^2 + |z|^2 + 2|t||z|\cos(\psi_1 - \psi_2)}$. Since $|\psi_1 - \psi_2| \leq \pi/2$, then $|t+z| \geq |t| \geq 1/\sqrt{M}$. This proves that condition (30) is satisfied.

Let us transform the quantities $g''^{(n)}_k(t) = G''^{(n)}_k(t)/Q''^{(n)}_{k-1}(t)$, $1 \leq k \leq n$, namely

$$g''^{(n)}_k(t) = \frac{a_k}{Q''^{(n)}_{k-1}(t)Q''^{(n)}_k(t)} = \frac{a_k}{(t + a_k/Q''^{(n)}_k(t))Q''^{(n)}_k(t)} = \frac{1}{1 + tQ''^{(n)}_k(t)/a_k}.$$

Let us estimate the quantities $|1 + tQ''^{(n)}_k(t)/a_k|$ from below, for $1 \leq k \leq n$. Let us denote $t = |t|e^{i\psi_1}$, $Q''^{(n)}_k(t) = |Q''^{(n)}_k(t)|e^{i\psi^{(n)}_k}$. From the Definition 2, it follows that $Q''^{(n)}_k(t) \in W_M$, $0 \leq k \leq n$. Then

$$|1 + tQ''^{(n)}_k(t)/a_k| = \sqrt{1 + |t|^2|Q''^{(n)}_k(t)|^2/a_k^2 + 2|t||Q''^{(n)}_k|\cos(\psi_1 + \psi^{(n)}_k)/a_k}.$$

Since $|\psi_1 + \psi^{(n)}_k| \leq \pi/2$, then

$$|1 + tQ''^{(n)}_k(t)/a_k| \geq \sqrt{1 + |t|^2|Q''^{(n)}_k(t)|^2/a_k^2} \geq \sqrt{1 + 1/(MA)^2}.$$

Thus, $|g''^{(n)}_k(t)| \leq MA/\sqrt{1 + (MA)^2}$, $1 \leq k \leq n$. Since $g''^{(n)}_k(t) = g^{(n)}_k(z)$, $1 \leq k \leq n$, we get $|g^{(n)}_k(z)| \leq MA/\sqrt{1 + (MA)^2}$, $1 \leq k \leq n$.

Let us denote $\eta = MA/\sqrt{1 + (MA)^2}$. If $M > 0$, $A > 0$, then $0 < \eta < 1$. Furthermore, if for the relative errors of the coefficients a_k and the variable z, the inequalities (13) hold, and the quantities α, β, η satisfy condition (15), then according to Theorem 1, the S-fraction (1) is stable to perturbations, and for the relative errors of its approximants, the estimate (16) holds if $4\eta\gamma < (1+\eta)^2$, and the estimate (17) holds if $4\eta\gamma = (1+\eta)^2$. □

Finally, we have the following result.

Theorem 3. *The set*

$$\Omega_{\rho,A} = \{z \in \mathbb{C}: \ |z| \leq \rho(1-\rho)/A\}, \quad 0 < \rho < 1/2, \quad A > 0, \tag{31}$$

is a set of stability to perturbations of the S-fraction (1), *if there exist constants α, β, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $\alpha + \beta \neq 0$, such that for the relative errors of the coefficients a_k and the variable z, the inequalities* (13) *hold, the coefficients a_k, $k \geq 0$, satisfy condition* (25), *and the inequality* (15) *holds, where $\eta = (1-\rho)/\rho$. In this case, for the relative errors of the approximants of the S-fraction* (1), *the estimate* (16) *holds if $4\eta\gamma < (1+\eta)^2$, and the estimate* (17) *holds if $4\eta\gamma = (1+\eta)^2$.*

Proof. If the coefficients a_k of the S-fraction (1) are bounded by a constant $A > 0$, and $z \in \Omega_{\rho,A}$, where the set $\Omega_{\rho,A}$ is determined according to (31), then $|a_k z| \leq \rho(1-\rho)$, $k \geq 1$. Let us denote $c_k = a_k z,\ k \geq 1$. Consider the continued fraction

$$\cfrac{c_0}{1+\cfrac{c_1}{1+\cfrac{c_2}{1+\ddots}}}, \tag{32}$$

and denote by $Q_k^{(n)}$, $0 \leq k \leq n$, the tails of the nth approximant of (32).

Let us prove that the set

$$W_\rho = \{z \in \mathbb{C}:\ |z-1| \leq \rho\} \tag{33}$$

is a set of values for the tails of the approximants of the continued fraction (32), corresponding to the set of elements $P_\rho = \{(z_1, z_2) \in \mathbb{R} \times \mathbb{C}:\ z_1 = 1,\ |z_2| \leq \rho(1-\rho)\}$.

Since $1 \in W_\rho$, condition (22) is satisfied. Let us prove that if the elements of the continued fraction (32) satisfy the inequality

$$|c_k| \leq \rho(1-\rho), \quad k \geq 1, \tag{34}$$

then condition (23) is satisfied.

Since $0 < \rho < 1/2$, then $0 \notin W_\rho$, and the function $1/z$ maps the set W_ρ to the set

$$W'_\rho = \{z \in \mathbb{C}:\ |z - 1/(1-\rho^2)| \leq \rho/(1-\rho^2)\}.$$

Then

$$V_\rho = c_{k+1} W'_\rho = \{z \in \mathbb{C}:\ |z - c_{k+1}/(1-\rho^2)| \leq |c_{k+1}|\rho/(1-\rho^2)\} \tag{35}$$

and condition (23) is satisfied if

$$|C_{V_\rho} - C_{W_\rho}| + R_{V_\rho} \leq R_{W_\rho}, \tag{36}$$

where C_{W_ρ}, R_{W_ρ} are the center and radius of the circle (33), respectively, and C_{V_ρ}, R_{V_ρ} are the center and radius of the circle (35), respectively. Inequality (36) is equivalent to inequality (34). Since $Q_k^{(n)} \in W_\rho$, then $|Q_k^{(n)}| \geq 1-\rho$, $0 \leq k \leq n$, and for the quantities $g_k^{(n)}$, the following estimates hold $|g_k^{(n)}| = c_k/(Q_{k-1}^{(n)} Q_k^{(n)}) \leq \rho/(1-\rho)$, $1 \leq k \leq n$.

Let $\eta = \rho/(1-\rho)$. If $0 < \rho < 1/2$, then $0 < \eta < 1$. Furthermore, if for the relative errors of the coefficients a_k and the variable z, the inequalities (13) hold, and the quantities α, β, η satisfy condition (15), then according to Theorem 1, the S-fraction (1) is stable to perturbations, and for the relative errors of its approximant, the estimates (16) or (17) hold. □

4 Conclusions

In this paper, the stability to perturbations of S-fractions with complex elements has been investigated. A method for analyzing the influence of perturbations of both the fraction's coefficients and its complex variable on the value of the approximant has been developed, which is based on recurrence relations for the relative errors of the approximant of the continued fraction. The relative error of the approximant is represented in the form of a continued fraction (8), which allowed the application of results from the analytical theory of continued fractions to establish conditions for stability to perturbations (Theorem 1). It has been established that the S-fraction is stable to perturbations if the relative errors of its coefficients and variable are bounded, and the quantities $g_k^{(n)}$, which depend on the elements of the fraction, are also bounded. The obtained explicit estimates for the relative error of the approximant allow determining the maximum perturbation value of the approximant depending on the maximum perturbation value of the fraction's elements. Using the methodology of sequences of element sets and sequences of value sets of tails, sets of stability to perturbations have been established (Theorems 2 and 3), which is important for practical applications. The obtained results can be used in the study of stability to perturbations of solutions to problems presented in the form of S-fractions, particularly in the approximation of functions by such continued fractions. The fulfillment of the stability conditions guarantees the accuracy and reliability of computations that use the approximation of functions by S-fractions under conditions of machine arithmetic with limited precision.

A promising direction for further research is the study of such generalizations of S-fractions as C-fractions, particularly regular C-fractions, multidimensional S- and C-fractions, as well as other classes of functional continued fractions – J-fractions, P-fractions, and T-fractions.

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Гладун В.Р., Дмитришин М.В. *Про стійкість до збурень неперервних дробів Стілтьєса з комплексними елементами* // Карпатські матем. публ. — 2025. — Т.17, №2. — С. 565–578.

У статті досліджуються умови стійкості до збурень неперервних дробів Стільтьєса з комплексними елементами. Запропоновано аналітичний підхід, що дозволяє оцінити вплив збурень як коефіцієнтів дробу, так і його комплексної змінної. Виведено рекурентні співвідношення для відносних похибок апроксимант, які представлено у вигляді неперервного дробу. На основі цього підходу отримано достатні умови стійкості, а також явні оцінки для похибок апроксимант. Побудовано множини стійкості до збурень, зокрема півкругову та кругову множини в комплексній площині.

Ключові слова і фрази: неперервний дріб Стілтьєса, стійкість до збурень, множина стійкості до збурень, відносна похибка, апроксимація неперервного дробу.