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Kolmogorov-type inequalities in semilinear metric spaces

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For functions that take values in an isotropic semilinear metric space we prove two sharp Kolmogorov-type inequalities. In the first one we obtain an estimate for the uniform norm of the derivative (in the Rådström sense) of a function using the uniform norm of the function and the H^ω-norm of the function's derivative; here ω is an arbitrary modulus of continuity. The second one gives an estimate of the uniform norm of a generalized fractional derivative of a function via its uniform norm and its H^ω-norm.

Key words and phrases: Kolmgorov-type inequality, inequality for derivatives, semilinear metric space, modulus of continuity, fractional derivative.

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1 Introduction

Inequalities for derivatives of real-valued functions of one and several variables (often called Kolmogorov-type inequalities) are important in many areas of mathematics and applications. A presentation of many sharp inequalities of this type, a description of their applications, and the corresponding references can be found in monographs [8, 11]. Inequalities for functions with more general ranges are also important. We are interested in inequalities for derivatives of functions taking values in semilinear metric spaces. Some known inequalities for Hukuhara-type derivatives of integer and fractional orders can be found in [2, 3].

In this article, we prove two sharp Kolmogorov-type inequalities for functions that take values in semilinear metric spaces, where the definition of the derivative is based on the well-known Rådström embedding theorem [13].

Let G be line $\mathbb{R}$, or semi-line $\mathbb{R}_+$, and (X, h_X) be a semilinear metric space (precise definitions are given in Section 2). Let $\omega(t)$ be a modulus of continuity, i.e. a non-decreasing on $\mathbb{R}_+$, continuous and semiadditive function such that $\omega(0) = 0$. Denote by $H^\omega(G, X)$ the set of continuous functions $f\colon G \to X$ such that

$$\|f\|_{H^\omega(G,X)} := \sup_{\substack{u,v\in G\\ u\neq v}} \frac{h_X(f(u), f(v))}{\omega(|u-v|)} < \infty.$$

We obtain a sharp inequality that estimates the uniform norm of the derivative (in the Rådström sense) of a function $f\colon G \to X$ using the uniform norm of the function and the

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H^ω-norm of the function's derivative. An analogue of this result for the derivatives in the sense of Hukuhara can be found in [2, Theorems 5.1 and 6.1]. Inequalities that estimate the uniform norm of a mixed derivative of a multivariate real-valued function using the uniform norm of the function and the $\|\cdot\|_{H^\omega}$-norm of the mixed derivative were obtained in [5, Theorem 6]; using the the uniform norm of the function and the $\|\cdot\|_{L_p}$-norm, $p \in [1, \infty]$, of the gradient of the mixed derivative were obtained in [4, Theorem 5] and [6, Theorem 4]. Articles [4–6] also contain related inequalities for charges.

We also prove a sharp inequality that estimates the uniform norm of a generalized fractional derivative of a function via its uniform norm and its H^ω-norm. Many results on inequalities for fractional derivatives are known. We refer to [7, Section 1.3], [11, Chapter 2], and [9] for a survey of results and further references.

The article is organized as follows. In Section 2 we give necessary definitions and adduce several theorems that will be needed in the article. Section 3 contains the main results of the article.

2 Semilinear metric spaces

2.1 Definitions

Definition 1. *A set X is called a semilinear space, if the operations of addition of elements and their multiplication on real numbers are defined in X, and the following conditions are satisfied for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$:*

1) $x + y = y + x$;

2) $x + (y + z) = (x + y) + z$;

3) $\exists\, \theta \in X \colon x + \theta = x$;

4) $\lambda(x + y) = \lambda x + \lambda y$;

5) $\lambda(\mu x) = (\lambda\mu)x$;

6) $1 \cdot x = x, 0 \cdot x = \theta$.

Definition 2. *We call an element $x \in X$ convex, if for all $\alpha, \beta \geq 0$ we have $(\alpha + \beta)x = \alpha x + \beta x$. Denote by X^{c} the subspace of the space X that consists of all convex elements.*

Definition 3. *We say that an element $x \in X$ is invertible if there exists an element $x' \in X$ such that $x + x' = \theta$. In this case the element x' is called the inverse to x. Denote by X^{inv} the set of all invertible elements of the space X.*

Definition 4. *A semilinear space X endowed with a metric h_X is called a semilinear metric space if the following conditions are satisfied for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$:*

1) $h_X(\lambda x, \lambda y) = |\lambda| \cdot h_X(x, y)$;

2) $h_X(x + z, y + z) \leq h_X(x, y)$.

Due to these properties and the triangle inequality one has

$$h_X(x+y,u+v) \le h_X(x+y,x+v) + h_X(x+v,u+v) \le h_X(y,v) + h_X(x,u)$$

for all $x,y,u,v \in X$.

Definition 5. *A semilinear metric space X is called isotropic if*

$$h_X(x+z,y+z) = h_X(x,y)$$

for all $x,y,z \in X$.

Everywhere below we assume that X is isotropic and $X = X^c$.

2.2 Differences in semilinear spaces. Rådström's theorem

We start from the definition of Hukuhara-type differences.

Definition 6. *Let X be a semilinear metric space. We say that $z \in X$ is a Hukuhara type difference of $x,y \in X$ if $x = y+z$. We denote this difference by $z = x -_H y$.*

Note that in an isotropic semilinear metric space, the Hukuhara difference $x -_H y$ is unique, provided it exists.

Another approach to the definition of a difference is based on Rådström's theorem [13]. Let X be an isotropic semilinear metric space that consists of convex elements. Note that if for some $a,b,c \in X$ one has $a+c = b+c$, then $0 = h_X(a+c,b+c) = h_X(a,b)$, hence $a+c = b+c \implies a = b$. It follows from Rådström's theorem [13] that the set X can be isometrically embedded into some normed space as a convex cone. The outline of the proof of this fact is as follows. Consider the set $X \times X$. Define an equivalence relation on this set by setting $(x,y) \sim (u,v) \iff x+v = y+u$. Let $\mathfrak{B}$ denote the quotient set $\mathfrak{B} := (X \times X)/\sim$ and $\langle x,y\rangle$ denote the equivalence class containing the pair (x,y). For $x,y \in X$ and $\alpha \in \mathbb{R}$ define addition in $\mathfrak{B}$ and scalar multiplication by

$$\langle x,y\rangle + \langle u,v\rangle := \langle x+u, y+v\rangle, \qquad \alpha\langle x,y\rangle := \begin{cases} \langle \alpha x, \alpha y\rangle, & \alpha \ge 0, \\ \langle |\alpha| y, |\alpha| x\rangle, & \alpha < 0. \end{cases}$$

Define a metric on $\mathfrak{B}$ by the formula $\delta(\langle x,y\rangle, \langle u,v\rangle) := h_X(x+v, y+u)$, where h_X is the given metric on X. Next, define the norm in $\mathfrak{B}$ as $\|\langle x,y\rangle\|_{\mathfrak{B}} := \delta(\langle x,y\rangle, \langle\theta,\theta\rangle) = h_X(x,y)$. Define the embedding operator $\pi : X \to \mathfrak{B}$ by $\pi(x) := \langle x,\theta\rangle$. We can then define the difference between elements $x,y \in X$ as $x -_\pi y := \langle x,\theta\rangle -_{\mathfrak{B}} \langle y,\theta\rangle = \langle x,y\rangle$. This difference is not necessarily an element of X, but if for elements $x,y \in X$ the Hukuhara-type difference $x -_H y$ exists, then $x -_\pi y = \langle x,y\rangle = \langle x -_H y, \theta\rangle$.

2.3 Derivatives of a function with values in semilinear metric spaces

Let $G = \mathbb{R}$ or $G = \mathbb{R}_+$. As it is known for a function $f\colon G \to \mathfrak{B}$ and a point $t \in G$, the derivative at the point $t \in G$ is defined by the equality

$$Df(t) = \lim_{G \ni \gamma \to t} \frac{f(\gamma) -_{\mathfrak{B}} f(t)}{\gamma - t},$$

where the limit is understood as the limit in the norm of the space $\mathfrak{B}$.

In particular, if we identify a function $f\colon G \to X$ with the function $t \mapsto \langle f(t), \theta\rangle$, then we arrive at the following definition.

Definition 7. *For a function $f\colon G \to X$ and a point $t \in G$, the derivative in the sense of Rådström (π-derivative) at the point $t \in G$ is defined by the equality*

$$\mathcal{D}_\pi f(t) = D\langle f(\cdot), \theta\rangle(t) = \lim_{G\ni\gamma\to t} \frac{1}{\gamma - t}\langle f(\gamma), f(t)\rangle,$$

where the limit is understood as the limit in the norm of the space $\mathfrak{B}$.

If $t \in G$, $t \neq 0$, and for all small enough $\gamma > 0$ there exist Hukuhara-type differences $f(t+\gamma) -_H f(t)$ and $f(t) -_H f(t-\gamma)$, and both limits

$$\lim_{\gamma\to+0} \gamma^{-1}(f(t+\gamma) -_H f(t)) \quad \text{and} \quad \lim_{\gamma\to+0} \gamma^{-1}(f(t) -_H f(t-\gamma))$$

exist and are equal to each other, then the function f is said to be differentiable in the sense of Hukuhara at the point t; in the case $t = 0$ one only requires existence of one limit. The Hukuhara-type derivative is defined as

$$\mathcal{D}_H f(t) := \lim_{\gamma\to+0} \gamma^{-1}(f(t+\gamma) -_H f(t)).$$

Observe that if a function f is differentiable in the sense of Hukuhara at a point t, then it is π-differentiable at t, and

$$\mathcal{D}_\pi f(t) = \langle \mathcal{D}_H f(t), \theta\rangle. \tag{1}$$

A discussion of various ways to define differences and derivatives of multi-valued mappings can be found in [12, Section 2.1] (see also references therein). In [1], elements of calculus for functions with values in semilinear metric spaces were developed based on the Hukuhara-type differences.

2.4 Integral of functions with values in the space $\mathfrak{B}$

Although $\mathfrak{B}$ is a normed space and one might try to use standard definitions of integrals for such functions, the following problem arise: completeness of X does not, generally speaking, imply completeness of $\mathfrak{B}$, see e.g. [10, p. 363].

We will use the following definition of the integral (we give this definition only for continuous functions).

Let $f_1, f_2 : \mathbb{R} \to X$ be continuous functions, and let $a, b \in \mathbb{R}$, $a < b$. Then the functions $f_1, f_2\colon [a,b] \to X$ are Riemann integrable. Define

$$\int_a^b \langle f_1(t), f_2(t)\rangle\, dt := \Big\langle \int_a^b f_1(t)\, dt, \int_a^b f_2(t)\, dt \Big\rangle.$$

Note that this definition is consistent with the definition of the Riemann integral as the limit of Riemann sums. This definition is correct, and the integral so defined will have all the properties we require. These properties are as follows:

1) if $0 < a < b < c < \infty$, then

$$\int_a^c \langle f_1(t), f_2(t)\rangle\, dt = \int_a^b \langle f_1(t), f_2(t)\rangle\, dt + \int_b^c \langle f_1(t), f_2(t)\rangle\, dt;$$

2)

$$\Big\| \int_a^b \langle f_1(t), f_2(t)\rangle\, dt \Big\|_{\mathfrak{B}} \le \int_a^b \|\langle f_1(t), f_2(t)\rangle\|_{\mathfrak{B}}\, dt.$$

We will also need the definition of the improper integral. Let $f_1, f_2\colon (0,\infty) \to X$ be continuous functions, and let $a, b \in \mathbb{R}, a < b$. Then the improper integral is defined as follows

$$\int_0^\infty \langle f_1(t), f_2(t)\rangle\, dt := \lim_{\substack{a\to 0\\ b\to\infty}} \int_a^b \langle f_1(t), f_2(t)\rangle dt$$

(if the limit on the right-hand side exists in the space $\mathfrak{B}$). It is not difficult to verify that analogues of the above properties of proper integrals are preserved for improper integrals.

2.5 Elements of calculus

We need the Finite-increment Theorem (see e.g. [15, Section 10.4, Theorem 1]), which we formulate in a convenient for our purposes form.

Theorem 1. *Assume that $f\colon [a,b] \to \mathfrak{B}$ is a continuously differentiable on $[a,b]$ function. Then*

$$\frac{1}{b-a}\|f(b) -_{\mathfrak{B}} f(a)\|_{\mathfrak{B}} \le \sup_{t\in[a,b]} \|Df(t)\|_{\mathfrak{B}}.$$

As a corollary, one obtains the following result.

Theorem 2. *Let us assume that $f, g\colon [a,b] \to \mathfrak{B}$ are two differentiable functions such that $Df(t) = Dg(t)$ for all $t \in [a,b]$. Then the function $f -_{\mathfrak{B}} g$ is constant on $[a,b]$.*

We also need an analogue of the Newton-Leibniz formula.

Theorem 3. *Let a function $f\colon [a,b] \to X$ be π-differentiable at each point $t \in [a,b]$, and $\mathcal{D}_\pi f(t) = \langle A(t), B(t)\rangle$, where the functions A and B are continuous. Then*

$$\langle f(b), f(a)\rangle = \int_a^b \mathcal{D}_\pi f(t)dt = \Big\langle \int_a^b A(t)dt, \int_a^b B(t)dt \Big\rangle.$$

Proof. Consider the function $\Phi\colon [a,b] \to \mathfrak{B}$, defined as follows

$$\Phi(t) = \int_a^t \mathcal{D}_\pi f(s)ds = \Big\langle \int_a^t A(t)dt, \int_a^t B(t)dt \Big\rangle.$$

We compute the derivative $D\Phi(t)$ for a fixed $t \in [a,b]$. Let $\gamma \neq 0$ be such that $t + \gamma \in [a,b]$ and consider the quantity $\Delta(f; t, \gamma) = (\Phi(t+\gamma) -_{\mathfrak{B}} \Phi(t))/\gamma$. Using additivity of the integral and the definitions of the elements of the space $\mathfrak{B}$ and of the operations on the elements, we prove that for $\gamma > 0$ we have

$$\Delta(f; t, \gamma) = \frac{1}{\gamma}\Big\langle \int_t^{t+\gamma} A(s)ds, \int_t^{t+\gamma} B(s)ds \Big\rangle,$$

and for $\gamma < 0$ we have

$$\Delta(f; t, \gamma) = \frac{1}{|\gamma|}\Big\langle \int_{t+\gamma}^t A(s)ds, \int_{t+\gamma}^t B(s)ds \Big\rangle.$$

We start with the proof of the first equality.

$$\begin{aligned}
\Delta(f;t,\gamma) &= \frac{1}{\gamma}\Big\langle \int_a^{t+\gamma} A(s)ds, \int_a^{t+\gamma} B(s)ds\Big\rangle -_{\mathfrak{B}} \frac{1}{\gamma}\Big\langle \int_a^{t} A(s)ds, \int_a^{t} B(s)ds\Big\rangle \\
&= \frac{1}{\gamma}\Big\langle \int_a^{t+\gamma} A(s)ds + \int_a^{t} B(s)ds, \int_a^{t+\gamma} B(s)ds + \int_a^{t} A(s)ds\Big\rangle \\
&= \frac{1}{\gamma}\Big\langle \int_a^{t} A(s)ds + \int_t^{t+\gamma} A(s)ds + \int_a^{t} B(s)ds, \int_a^{t} B(s)ds + \int_t^{t+\gamma} B(s)ds + \int_a^{t} A(s)ds\Big\rangle \\
&= \frac{1}{\gamma}\Big\langle \int_t^{t+\gamma} A(s)ds, \int_t^{t+\gamma} B(s)ds\Big\rangle.
\end{aligned}$$

If $\gamma < 0$, then analogously we obtain

$$\begin{aligned}
\Delta(f;t,\gamma) &= \frac{1}{|\gamma|}\Big\langle \int_a^{t+\gamma} B(s)ds, \int_a^{t+\gamma} A(s)ds\Big\rangle -_{\mathfrak{B}} \frac{1}{|\gamma|}\Big\langle \int_a^{t} B(s)ds \int_a^{t} A(s)ds\Big\rangle \\
&= \frac{1}{|\gamma|}\Big\langle \int_a^{t+\gamma} B(s)ds + \int_a^{t+\gamma} A(s)ds + \int_{t+\gamma}^{t} A(s)ds, \\
&\qquad \int_a^{t+\gamma} A(s)ds + \int_a^{t+\gamma} B(s)ds + \int_{t+\gamma}^{t} B(s)ds\Big\rangle \\
&= \frac{1}{|\gamma|}\Big\langle \int_{t+\gamma}^{t} A(s)ds, \int_{t+\gamma}^{t} B(s)ds\Big\rangle.
\end{aligned}$$

Next, for $\gamma > 0$ one has

$$\begin{aligned}
\|\Delta(f;t,\gamma) -_{\mathfrak{B}} \mathcal{D}_\pi f(t)\|_{\mathfrak{B}} &= \Big\|\frac{1}{\gamma}\Big\langle \int_t^{t+\gamma} A(s)ds, \int_t^{t+\gamma} B(s)ds\Big\rangle -_{\mathfrak{B}} \langle A(t), B(t)\rangle\Big\|_{\mathfrak{B}} \\
&= h_X\Big(\frac{1}{\gamma}\int_t^{t+\gamma} A(s)ds + B(t), \frac{1}{\gamma}\int_t^{t+\gamma} B(s)ds + A(t)\Big) \\
&\le h_X\Big(\frac{1}{\gamma}\int_t^{t+\gamma} A(s)ds, A(t)\Big) + h_X\Big(\frac{1}{\gamma}\int_t^{t+\gamma} B(s)ds, B(t)\Big) \\
&= h_X\Big(\frac{1}{\gamma}\int_t^{t+\gamma} A(s)ds, \frac{1}{\gamma}\int_t^{t+\gamma} A(t)ds\Big) + h_X\Big(\frac{1}{\gamma}\int_t^{t+\gamma} B(s)ds, \frac{1}{\gamma}\int_t^{t+\gamma} B(t)ds\Big) \\
&\le \frac{1}{\gamma}\int_t^{t+\gamma} h_X(A(s), A(t))ds + \frac{1}{\gamma}\int_t^{t+\gamma} h_X(B(s), B(t))ds.
\end{aligned}$$

Thus, in this case we get

$$\|\Delta(f;t,\gamma) -_{\mathfrak{B}} \mathcal{D}_\pi f(t)\|_{\mathfrak{B}} \le \frac{1}{\gamma}\int_t^{t+\gamma} h_X(A(s), A(t))ds + \frac{1}{\gamma}\int_t^{t+\gamma} h_X(B(s), B(t))ds.$$

Analogously, in the case $\gamma < 0$ we obtain

$$\|\Delta(f;t,\gamma) -_{\mathfrak{B}} \mathcal{D}_\pi f(t)\|_{\mathfrak{B}} \le \frac{1}{|\gamma|}\int_{t+\gamma}^{t} h_X(A(s), A(t))ds + \frac{1}{|\gamma|}\int_{t+\gamma}^{t} h_X(B(s), B(t))ds.$$

Taking into account continuity of the functions A and B we obtain $\Delta(f;t,\gamma) \to \mathcal{D}_\pi f(t)$ as $\gamma \to 0$. Thus for all $t \in [a,b]$ we have

$$D\int_a^t \mathcal{D}_\pi f(s)ds = \mathcal{D}_\pi f(t) = D\langle f(t), \theta\rangle.$$

Using Theorem 2, we obtain

$$\langle f(t),\theta\rangle -_{\mathfrak{B}} \int_a^t \mathcal{D}_\pi f(s)ds = \langle f(a),\theta\rangle$$

for all $t\in[a,b]$, or, equivalently,

$$\int_a^t \mathcal{D}_\pi f(s)ds = \langle f(t),\theta\rangle -_{\mathfrak{B}} \langle f(a),\theta\rangle = \langle f(t), f(a)\rangle,$$

which implies the statement of the theorem. □

3 Kolmogorov-type inequalities

3.1 Auxiliary results

Recall that in this paper X is assumed to be an isotropic semilinear metric space that consists of convex elements. By G we denote a segment $[a,b]\subset\mathbb{R}$, line $\mathbb{R}$, or semi-line $\mathbb{R}_+$.

Definition 8. *For a function $f\colon G\to\mathbb{R}$ and $x\in X^{\rm inv}$ define a function*

$$f_x\colon G\to X, f_x(t) = f_+(t)\cdot x + f_-(t)\cdot x',$$

where for $t\in G$, $f_\pm(t) := (f(t))_\pm$, and for a real ξ, $\xi_\pm := \max\{\pm\xi, 0\}$.

In this section, we state several properties of the functions f_x, which will be used during the construction of extremal functions for inequalities for derivatives.

Denote by $C(G,X)$ the set of all continuous functions $f\colon G\to X$ and for a bounded function $f\in C(G,X)$ let

$$\|f\|_{C(G,X)} := \sup_{t\in G}\|f(t)\|_X.$$

Let $\omega(t)$ be some modulus of continuity.

Definition 9. *Denote by $H^\omega(G,X)$ the set of functions $f\in C(G,X)$ such that*

$$\|f\|_{H^\omega(G,X)} := \sup_{\substack{u,v\in G\\ u\neq v}} \frac{h_X(f(u),f(v))}{\omega(|u-v|)} < \infty.$$

Due to the definition of the metric in the space $\mathfrak{B}$, for a function $f\colon G\to\mathfrak{B}$, $f(\cdot) = \langle f_1(\cdot), f_2(\cdot)\rangle$, this definition can be rewritten as follows: we say that $f\in H^\omega(G,\mathfrak{B})$, if

$$\|f\|_{H^\omega(G,\mathfrak{B})} := \sup_{\substack{u,v\in G\\ u\neq v}} \frac{\|\langle f_1(u),f_2(u)\rangle -_{\mathfrak{B}} \langle f_1(v),f_2(v)\rangle\|_{\mathfrak{B}}}{\omega(|u-v|)} < \infty.$$

The following two lemmas are known, see [3, Lemmas 6,7].

Lemma 1. *Let $f\in H^\omega([a,b],\mathbb{R})$ and $x\in X^{\rm inv}$ be such that $h_X(x,\theta)=1$. Then $f_x\in H^\omega([a,b],X)$ and*

$$\int_a^b f_x(t)dt = \int_a^b f_+(t)dt\cdot x + \int_a^b f_-(t)dt\cdot x'.$$

Lemma 2. *Let $x\in X^{\rm inv}$, and $f\colon[a,b]\to\mathbb{R}$ be a continuously differentiable function. Then the derivative $\mathcal{D}_H f_x(t)$ exists at each point $t\in[a,b]$ and $\mathcal{D}_H f_x(t) = (f'(t))_x$.*

Lemma 3. *Let $x \in X$ and $A, B \geq 0$. Then $\langle Ax, Bx\rangle = (A-B)\langle x, \theta\rangle$.*

Proof. Assume that $A \geq B$. Then $(A-B)\langle x,\theta\rangle = \langle (A-B)x,\theta\rangle = \langle (A-B)x + Bx, Bx\rangle = \langle Ax, Bx\rangle$. If $A < B$, then $(A-B)\langle x,\theta\rangle = \langle \theta, (B-A)x\rangle = \langle Ax, (B-A)x + Ax\rangle = \langle Ax, Bx\rangle$. □

Lemma 4. *Let $\varphi, \psi\colon [a,b] \to \mathbb{R}$ be two continuous functions and $x \in X^{\mathrm{inv}}$. Then*

$$\int_a^b \langle \varphi_x(t), \psi_x(t)\rangle dt = \int_a^b (\varphi(t) - \psi(t))dt \cdot \langle x, \theta\rangle.$$

Proof. Using Lemmas 1 and 3 we obtain

$$\begin{aligned}
\int_a^b \langle \varphi_x(t), \psi_x(t)\rangle dt &= \Big\langle \int_a^b \varphi_x(t)dt, \int_a^b \psi_x(t)dt \Big\rangle \\
&= \Big\langle \int_a^b \varphi_+(t)dt \cdot x + \int_a^b \varphi_-(t)dt \cdot x', \int_a^b \psi_+(t)dt \cdot x + \int_a^b \psi_-(t)dt \cdot x' \Big\rangle \\
&= \Big\langle \int_a^b \varphi_+(t)dt \cdot x + \int_a^b \psi_-(t)dt \cdot x, \int_a^b \psi_+(t)dt \cdot x + \int_a^b \varphi_-(t)dt \cdot x \Big\rangle \\
&= \Big\langle \Big[\int_a^b \varphi_+(t)dt + \int_a^b \psi_-(t)dt \Big] x, \Big[\int_a^b \psi_+(t)dt + \int_a^b \varphi_-(t)dt \Big] x \Big\rangle \\
&= \Big[\int_a^b \varphi_+(t)dt + \int_a^b \psi_-(t)dt - \int_a^b \psi_+(t)dt - \int_a^b \varphi_-(t)dt \Big] \langle x, \theta\rangle \\
&= \int_a^b (\varphi(t) - \psi(t))dt \cdot \langle x, \theta\rangle.
\end{aligned}$$

□

3.2 Inequalities on the class of functions with given majorant of modulus of continuity

Theorem 4. *Let $G = \mathbb{R}$ or $G = \mathbb{R}_+$, $f\colon G \to X$ be a bounded function such that for every $t \in G$ the Rådström derivative $\mathcal{D}_\pi f(t)$ exists, and $\|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})} < \infty$. Then for any $\gamma > 0$ the following inequality*

$$\|\mathcal{D}_\pi f\|_{C(G,\mathfrak{B})} \leq \frac{1}{\gamma} \int_0^\gamma \omega(t)\, dt \cdot \|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})} + \frac{\alpha(G)}{\gamma} \|f\|_{C(G,X)} \tag{2}$$

holds, where $\alpha(G) = 1$ if $G = \mathbb{R}$, and $\alpha(G) = 2$ if $G = \mathbb{R}_+$. If $X^{\mathrm{inv}} \neq \{\theta\}$, then the inequality is sharp.

Proof. We start with the case $G = \mathbb{R}$. For arbitrary $t \in \mathbb{R}$ using the equality

$$\mathcal{D}_\pi f(t) -_{\mathfrak{B}} \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \mathcal{D}_\pi f(u)du = \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} (\mathcal{D}_\pi f(t) -_{\mathfrak{B}} \mathcal{D}_\pi f(u))du,$$

we obtain

$$\begin{aligned}
\Big\| \mathcal{D}_\pi f(t) -_{\mathfrak{B}} \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \mathcal{D}_\pi f(u)du \Big\|_{\mathfrak{B}} &\leq \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \|\mathcal{D}_\pi f(t) -_{\mathfrak{B}} \mathcal{D}_\pi f(u)\|_{\mathfrak{B}} du \\
&\leq \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})} \omega(|t-u|)du \\
&= \frac{1}{\gamma} \int_0^\gamma \omega(v)dv \cdot \|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})}.
\end{aligned}$$

Using Theorem 3, we obtain

$$\Big\|\frac{1}{2\gamma}\int_{t-\gamma}^{t+\gamma}\mathcal{D}_\pi f(u)du\Big\|_{\mathfrak{B}} = \frac{1}{2\gamma}\|\langle f(t+\gamma), f(t-\gamma)\rangle\|_{\mathfrak{B}} = \frac{1}{2\gamma}h_X(f(t+\gamma), f(t-\gamma)) \le \frac{1}{\gamma}\|f\|_{C(\mathbb{R},X)}.$$

Using the obtained estimates, we have

$$\Big\|\mathcal{D}_\pi f(t)\Big\|_{\mathfrak{B}} \le \Big\|\mathcal{D}_\pi f(t) -_{\mathfrak{B}} \frac{1}{2\gamma}\int_{t-\gamma}^{t+\gamma}\mathcal{D}_\pi f(u)du\Big\|_{\mathfrak{B}} + \Big\|\frac{1}{2\gamma}\int_{t-\gamma}^{t+\gamma}\mathcal{D}_\pi f(u)du\Big\|_{\mathfrak{B}} \le \frac{1}{\gamma}\int_0^\gamma \omega(v)dv\|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})} + \frac{1}{\gamma}\|f\|_{C(\mathbb{R},X)},$$

as required in the case $G = \mathbb{R}$.

In the case $G = \mathbb{R}_+$, we use averaging operator

$$f \mapsto \frac{1}{\gamma}\int_t^{t+\gamma}\mathcal{D}_\pi f(u)du \quad \text{instead of} \quad f \mapsto \frac{1}{2\gamma}\int_{t-\gamma}^{t+\gamma}\mathcal{D}_\pi f(u)du$$

and analogous arguments to obtain the inequality

$$\|\mathcal{D}_\pi f(t)\|_{\mathfrak{B}} \le \frac{1}{\gamma}\int_0^\gamma \omega(v)dv\|\mathcal{D}_\pi f\|_{H^\omega(G,\mathfrak{B})} + \frac{2}{\gamma}\|f\|_{C(G,X)}.$$

Next, we prove sharpness of the obtained inequalities. We start from the case $G = \mathbb{R}$ first. Choose $y \in X^{\mathrm{inv}}$ such that

$$\|\langle y, \theta\rangle\|_{\mathfrak{B}} = h_X(y, \theta) = 1. \tag{3}$$

Set

$$\phi_\gamma(t) = \begin{cases}\omega(\gamma) - \omega(|t|), & 0 \le |t| \le \gamma,\\ 0, & |t| \ge \gamma,\end{cases}$$

and

$$\psi_\gamma(t) = \Big(\int_0^t \phi_\gamma(s)ds\Big)_y = \begin{cases}\int_0^t \phi_\gamma(s)ds\cdot y, & t \ge 0,\\ \int_t^0 \phi_\gamma(s)ds\cdot y', & t < 0.\end{cases}$$

Using Lemma 2 and relation (1), we obtain

$$\mathcal{D}_\pi\psi_\gamma(t) = \langle(\phi_\gamma(t))_y, 0\rangle = \phi_\gamma(t)\cdot\langle y, \theta\rangle \tag{4}$$

for all $t \in \mathbb{R}$. From this equality and (3) it follows that

$$\|\mathcal{D}_\pi\psi_\gamma\|_{H^\omega(\mathbb{R},\mathfrak{B})} = \|\phi_\gamma\|_{H^\omega(\mathbb{R},\mathbb{R})} = 1 \quad \text{and} \quad \|\mathcal{D}_\pi\psi_\gamma\|_{C(\mathbb{R},\mathfrak{B})} = \|\phi_\gamma\|_{C(\mathbb{R},\mathbb{R})} = \omega(\gamma).$$

Moreover,

$$\|\psi_\gamma\|_{C(\mathbb{R},X)} = \|\phi_\gamma\|_{C(\mathbb{R},\mathbb{R})} = \gamma\omega(\gamma) - \int_0^\gamma \omega(s)ds.$$

Substituting these values into inequality (2) one can see that it becomes equality.

In order to prove sharpness of (2) in the case $G = \mathbb{R}_+$, set

$$\phi_\gamma(t) = \begin{cases}\omega(\gamma) - \omega(t), & 0 \le t \le \gamma,\\ 0, & t \ge \gamma.\end{cases}$$

Let $t_0 \in [0, \gamma]$ be such that

$$\int_0^{t_0} \phi(s)ds = \int_{t_0}^{\gamma} \phi(s)ds.$$

Define

$$\psi_\gamma(t) = \Big(\int_{t_0}^{t} \phi_\gamma(s)ds\Big)_y = \begin{cases} \int_t^{t_0} \phi_\gamma(s)ds \cdot y, & 0 \le t \le t_0, \\ \int_{t_0}^{t} \phi_\gamma(s)ds \cdot y', & t \ge t_0. \end{cases}$$

Again, by Lemma 2, we obtain that for all $t \ge 0$ equalities (4) hold, hence one has

$$\|\mathcal{D}_\pi \psi_\gamma\|_{H^\omega(\mathbb{R}_+, \mathfrak{B})} = 1$$

and $\|\mathcal{D}_\pi \psi_\gamma\|_{C(\mathbb{R}_+, \mathfrak{B})} = \omega(\gamma)$. Moreover,

$$\|\psi_\gamma(t)\|_{C(\mathbb{R}_+, X)} = \frac{1}{2}\gamma\omega(\gamma) - \frac{1}{2}\int_0^{\gamma} \omega(s)ds.$$

Substituting these values into inequality (2) one can see that it becomes equality. □

3.3 Inequalities for generalized derivatives of fractional order

Let $\Omega\colon (0, \infty) \to (0, \infty)$ be a non-negative continuous function. For a continuous function $f\colon [0, \infty) \to X$ define

$$D_{\pi,\Omega} f(t) := \int_0^{\infty} (f(t) -_\pi f(t+u))\Omega(u)du = \int_0^{\infty} \langle f(t), f(t+u)\rangle \Omega(u)du.$$

In the case when $\Omega(u) = 1/u^{1+\alpha}$, $0 < \alpha < 1$, and $X = \mathbb{R}$, this gives, up to a constant factor, the definition of the fractional derivative of order α in the sense of *Marchaud* (see e.g. [14, §5]).

Theorem 5. *Let a modulus of continuity ω and a continuous non-negative function*

$$\Omega\colon (0, \infty) \to (0, \infty)$$

be such that for some $\varepsilon > 0$ we have

$$\int_0^{\varepsilon} \omega(u)\Omega(u)du < \infty \quad \text{and} \quad \int_{\varepsilon}^{\infty} \Omega(u)du < \infty. \tag{5}$$

Then for any function $f \in H^\omega(\mathbb{R}_+, X)$ and arbitrary $\gamma > 0$ the following inequality

$$\|D_{\pi,\Omega} f\|_{C(\mathbb{R}_+, \mathfrak{B})} \le \int_0^{\gamma} \omega(u)\Omega(u)du \cdot \|f\|_{H^\omega(\mathbb{R}_+, X)} + 2\int_{\gamma}^{\infty} \Omega(u)du \cdot \|f\|_{C(\mathbb{R}_+, X)}$$

holds. If $X^{\rm inv} \ne \{\theta\}$, then the inequality is sharp and becomes an equality for the function $f_\gamma = (\phi_\gamma)_x$, where

$$\phi_\gamma(t) = \begin{cases} \frac{1}{2}\omega(\gamma) - \omega(t), & 0 \le t \le \gamma, \\ -\frac{1}{2}\omega(\gamma), & t \ge \gamma, \end{cases} \tag{6}$$

and $x \in X^{\rm inv}$ is such that $h_X(x, \theta) = 1$.

Proof. Observe that if condition (5) holds for some $\varepsilon > 0$, then it holds for all $\varepsilon > 0$. For any $t \in \mathbb{R}_+$ we get

$$D_{\pi,\Omega}f(t) = \int_0^\infty \langle f(t), f(t+u)\rangle \Omega(u)du$$
$$= \int_0^\gamma \langle f(t), f(t+u)\rangle \Omega(u)du + \int_\gamma^\infty \langle f(t), f(t+u)\rangle \Omega(u)du.$$

Therefore

$$\|D_{\pi,\Omega}f(t)\|_{\mathfrak{B}} \leq \int_0^\gamma \|\langle f(t), f(t+u)\rangle\|_{\mathfrak{B}} \Omega(u)du + \int_\gamma^\infty \|\langle f(t), f(t+u)\rangle\|_{\mathfrak{B}} \Omega(u)du$$
$$\leq \|f\|_{H^\omega(R_+,\mathfrak{B})} \int_0^\gamma \omega(u)\Omega(u)du + 2\|f\|_{C(R_+,X)} \int_\gamma^\infty \Omega(u)du.$$

Hence,

$$\|D_{\pi,\Omega}f\|_{C(\mathbb{R}_+,\mathfrak{B})} \leq \int_0^\gamma \omega(u)\Omega(u)du \cdot \|f\|_{H^\omega(\mathbb{R}_+,X)} + 2\int_\gamma^\infty \Omega(u)du \cdot \|f\|_{C(\mathbb{R}_+,X)}.$$

Next, we show that the inequality becomes an equality for the function f_γ. Note that due to definition (6), one has $\phi_\gamma \in H^\omega(\mathbb{R}_+,\mathbb{R})$, and hence by Lemma 1, $f_\gamma \in H^\omega(\mathbb{R}_+, X)$. Moreover, using Lemma 4, we obtain

$$\int_0^\gamma \langle f_\gamma(0), f_\gamma(u)\rangle \Omega(u)du = \int_0^\gamma \langle (\phi_\gamma(0)\Omega(u))_x, (\phi_\gamma(u)\Omega(u))_x\rangle du$$
$$= \int_0^\gamma (\phi_\gamma(0) - \phi_\gamma(u))\Omega(u)du \cdot \langle x, \theta\rangle = \int_0^\gamma \omega(u)\Omega(u)du \cdot x,$$

and analogously

$$\int_\gamma^\infty \langle f_\gamma(0), f_\gamma(u)\rangle \Omega(u)du = \omega(\gamma) \int_\gamma^\infty \Omega(u)du \cdot x.$$

Taking into account that

$$\|f_\gamma\|_{C(\mathbb{R}_+,X)} = \|\phi_\gamma\|_{C(\mathbb{R}_+,\mathbb{R})} = \frac{1}{2}\omega(\gamma) \quad \text{and} \quad \|f_\gamma\|_{H^\omega(\mathbb{R}_+,X)} = \|\phi_\gamma\|_{H^\omega(\mathbb{R}_+,\mathbb{R})} = 1,$$

we obtain that

$$\|D_{\pi,\Omega}f_\gamma\|_{C(\mathbb{R}_+,\mathfrak{B})} \geq \|D_{\pi,\Omega}f_\gamma(0)\|_{\mathfrak{B}}$$
$$= \|f_\gamma\|_{H^\omega(\mathbb{R}_+,X)} \int_0^\gamma \omega(u)\Omega(u)du + 2\|f_\gamma\|_{C(\mathbb{R}_+,X)} \int_\gamma^\infty \Omega(u)du,$$

which finishes the proof of the theorem. □

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Бабенко В., Колесник В., Коваленко О., Парфінович Н. *Нерівності типу Колмогорова у напівлінійних метричних просторах* // Карпатські матем. публ. — 2025. — Т.17, №2. — C. 579–590.

Для функцій, що набувають значень у напівлінійному метричному просторі, ми доводимо дві нерівності типу Колмогорова. У першій ми отримуємо оцінку для рівномірної норми похідної (у сенсі Радстрьома) функції, використовуючи рівномірну норму функції і H^ω-норму її похідної, де ω — це деякий модуль неперервності. Друга нерівність оцінює рівномірну норму узагальненої дробової похідної функції за допомогою рівномірної норми функції і її H^ω-норми.

Ключові слова і фрази: нерівність типу Колмогорова, нерівність для похідних, напівлінійний метричний простір, модуль неперервності, дробова похідна.