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Some fixed point results for rational contraction on perturbed metric space

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This work is devoted to the study of fixed point theory for F-perturbed mappings in complete perturbed metric spaces. Specifically, we introduce the concept of rational type F-contraction and prove the existence and uniqueness of fixed points for such mappings. Our findings generalize several known results in the literature, extending the scope of the classical Banach contraction principle. The validity of the main results is demonstrated by means of an illustrative example.

Key words and phrases: fixed point, F-contraction, perturbed metric space.

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1 Intoduction and preliminaries

Fixed point theory is a cornerstone of nonlinear analysis, with Banach's contraction principle as a prototypical result. Banach's theorem states that any self-map $\mathcal{T}$ on a complete metric space $(\mathcal{Y}, \mathrm{d})$ satisfying a contractive inequality $\mathrm{d}(\mathcal{T}\eta, \mathcal{T}\mu) \leq \lambda \mathrm{d}(\eta, \mu)$ with $0 < \lambda < 1$ has a unique fixed point [4]. This principle is widely used to establish existence and uniqueness of solutions for differential, integral, and matrix equations in science and engineering (see, e.g., [8, 9, 11]).

Over the decades, many authors have generalized Banach's result. For example, D.W. Boyd and J.S.W. Wong [5], L.B. Ciric [6], and others introduced more general contractive conditions, while S. Czerwik [7] and later researchers studied fixed points in various generalized metric-like spaces such as b-metric or modular spaces. In this vein, D. Wardowski [12] introduced a new type of contraction, called an F-contraction, which further extends Banach's principle by using a real-valued function F to measure contractivity. According to D. Wardowski, any F-contraction on a complete metric space still admits a unique fixed point, thus generalizing Banach's theorem.

An F-contractions is defined via a function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying certain conditions:

(F1) F is strictly increasing;

(F2) $F(\alpha_n) \to -\infty$ if and only if $\alpha_n \to 0^+$;

(F3) there exists $k \in (0, 1)$ such that $\lim\limits_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

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A self-map $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is called an F-contraction if there exists $\tau > 0$ such that

$$\tau + F(\mathrm{d}(\mathcal{T}\eta, \mathcal{T}\mu)) \leq F(\mathrm{d}(\eta, \mu)), \quad \forall\, \eta \neq \mu.$$

D. Wardowski demonstrated that every F-contraction on a complete metric space has a unique fixed point. Importantly, the function F allows for a more flexible comparison than the linear factor in Banach's classical contraction. For instance, choosing $F(\alpha) = \ln \alpha$ recovers the standard Banach contractive inequality, i.e. $\mathrm{d}(\mathcal{T}\eta, \mathcal{T}\mu) \leq e^{-\tau}\mathrm{d}(\eta, \mu)$. Thus, the F-contraction concept not only generalizes Banach's classical result but also unifies many known contractive conditions, providing a versatile tool for fixed point theory.

As a further generalization of Banach's classical contraction principle, perturbed metric spaces provide a natural extension of classical metric spaces by allowing distances to include systematic perturbations while retaining an underlying exact metric. This framework offers a flexible setting for fixed point theory and has applications in areas where measurements are subject to intrinsic errors, such as physics, engineering, and applied mathematics (see [9]). In the following, we present essential definitions, lemmas, and illustrative results that form the foundation for the fixed point analysis conducted in this study.

In recent years, various generalizations of contraction principles, including results on generalized metric-like spaces [3] and rational-type F-contractions [1, 2], have been successfully established in the literature. The purpose of this research is to establish new fixed point results for rational-type F-contractions in the context of perturbed metric spaces. By leveraging the flexibility of F-functions, we provide convergence results that unify and generalize several known fixed point theorems.

Throughout this paper, $\mathcal{Y}$ denotes an arbitrary nonempty set, while $\mathbb{N}$ stands for the set of all non-negative integers.

Definition 1 ([10])**.** *Suppose that S and P are mappings defined on $\mathcal{Y} \times \mathcal{Y}$ with values in $[0, \infty)$. We define S as a perturbed metric on $\mathcal{Y}$ concerning P if*

$$S - P : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}, \quad (\eta, \mu) \to S(\eta, \mu) - P(\eta, \mu)$$

is a metric on $\mathcal{Y}$, i.e. for all $\eta, \mu, z \in \mathcal{Y}$ we have

(i) $(S - P)(\eta, \mu) \geq 0$;

(ii) $(S - P)(\eta, \mu) = 0 \iff \eta = \mu$;

(iii) $(S - P)(\eta, \mu) = (S - P)(\mu, \eta)$;

(iv) $(S - P)(\eta, \mu) \leq (S - P)(\eta, z) + (S - P)(z, \mu)$.

Then P is called a perturbed mapping, $\mathrm{d} = S - P$ *is called an exact metric, and* $(\mathcal{Y}, S, P)$ *is called a perturbed metric space (PMS).*

It is worth noting that a perturbed metric (PM) on $\mathcal{Y}$ does not necessarily satisfy the conditions of a metric on $\mathcal{Y}$. To illustrate this fact, consider the following examples.

Example 1 ([10])**.** *Let $S : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the mapping defined by*

$$S(\eta, \mu) = |\eta - \mu| + \eta^2\mu^4, \quad \eta, \mu \in \mathbb{R}.$$

Thus, S denotes a PM on $\mathbb{R}$ induced by the perturbed mapping $P : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by $P(\eta, \mu) = \eta^2\mu^4$, $\eta, \mu \in \mathbb{R}$.

Example 2 ([10]). *Define the mapping $S : C([0,1]) \times C([0,1]) \to [0,\infty)$ by*

$$S(k,l) = \int_0^1 |k(t) - l(t)|\, dt + (k(0) - l(0))^2, \quad k,l \in C([0,1]),$$

where $C([0,1])$ denotes the set of all functions $k : [0,1] \to \mathbb{R}$ that are continuous on the interval $[0,1]$. Therefore, S represents a PM on $C([0,1])$ corresponding to the perturbed mapping $P : C([0,1]) \times C([0,1]) \to [0,\infty)$ given by $P(k,l) = (k(0) - l(0))^2$, $k,l \in C([0,1])$.

Example 3 ([10]). *Let $S : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ be the mapping defined by $S(a,b) = (a-b)^2$, $a,b \in \mathbb{N}$. Accordingly, S represents a PM on $\mathbb{N}$ associated with the perturbed mapping $P : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ given by $P(a,b) = (a-b)^2 - |a-b|$, $a,b \in \mathbb{N}$, and the exact metric $\mathrm{d} : \mathbb{N} \times \mathbb{N} \to [0,\infty)$ is given by $\mathrm{d}(a,b) = |a-b|$, $a,b \in \mathbb{N}$.*

Proposition 1 ([10]). *Let $S, P, Q : \mathcal{Y} \times \mathcal{Y} \to [0,\infty)$ be three given mappings and $\alpha > 0$.*

(i) If $(\mathcal{Y}, S, P)$ and $(\mathcal{Y}, S, Q)$ are PMS's, then $(\mathcal{Y}, S, (P+Q)/2)$ forms a PMS as well.

(ii) For a PMS $(\mathcal{Y}, S, P)$, the triple $(\mathcal{Y}, \alpha S, \alpha P)$ is likewise a PMS.

Definition 2 ([10]). *Let $(\mathcal{Y}, S, P)$ be a PMS, $\{z_n\}$ be a sequence in $\mathcal{Y}$, and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$. Then*

(i) we define a sequence $\{z_n\}$ to be pertubed convergent in the space $(\mathcal{Y}, S, P)$ if it converges under the exact metric d, where $\mathrm{d} = S - P$, in the metric space $(\mathcal{Y}, \mathrm{d})$;

(ii) a sequence $\{z_n\}$ is considered perturbed Cauchy in $(\mathcal{Y}, S, P)$ if it satisfies the Cauchy property with respect to the exact metric d in $(\mathcal{Y}, \mathrm{d})$;

(iii) the perturbed metric space $(\mathcal{Y}, S, P)$ is called complete if the metric space $(\mathcal{Y}, \mathrm{d})$ is comlete; in other words, every perturbed Cauchy sequence in $(\mathcal{Y}, S, P)$ also converges as a perturbed convergent sequence within $(\mathcal{Y}, S, P)$;

(iv) a map $\mathcal{T}$ is said to be perturbed continuous if it is continuous when measured using the exact metric d.

Theorem 1 ([10]). *Suppose $(\mathcal{Y}, S, P)$ is a comlete PMS and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ is mapping assuming the following conditions are met:*

(i) $\mathcal{T}$ is a perturbed continuous mapping;

(ii) there exists $\lambda \in (0,1)$ such that $S(\mathcal{T}\eta, \mathcal{T}\mu) \leq \lambda S(\eta, \mu)$ for all $\eta, \mu \in \mathcal{Y}$.

Consequently, $\mathcal{T}$ has a unique fixed point.

2 Main results

Within this section, we first introduce the definition of a rational type F-contraction and establish a corresponding fixed point theorem derived from this notion. Subsequently, illustrative examples and related consequences are presented to demonstrate the applicability and significance of the obtained result.

Definition 3. *Let $(\mathcal{Y}, S, P)$ be a PMS and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a mapping. Then $\mathcal{T}$ is defined as a rational type F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(S(\mathcal{T}\eta, \mathcal{T}\mu)) \leq F(M(\eta, \mu))$$

for all $\eta, \mu \in \mathcal{Y}$, where

$$M(\eta, \mu) = \max\left\{S(\eta, \mu), \frac{S(\eta, \mathcal{T}\eta)[1 + S(\mu, \mathcal{T}\mu)]}{1 + S(\mathcal{T}\eta, \mathcal{T}\mu)}\right\}.$$

Theorem 2. *Let $(\mathcal{Y}, S, P)$ be a complete PMS, and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a rational type F-contraction with respect to* d. *Suppose the following conditions are satisfied:*

(i) $\mathcal{T}$ is a perturbed continuous mapping;

(ii) the mapping $\mathcal{T}$ is a rational type F-contraction.

Consequently, $\mathcal{T}$ has a unique fixed point.

Proof. Let $\eta_0 \in \mathcal{Y}$. We can show the existence of a fixed point. Take an $\eta_0 \in \mathcal{Y}$ and create the sequence $\{\eta_n\}$ starting at η_0. If $\eta_{n^*} = \eta_{n^*+1}$, then η_{n^*} is a fixed point of $\mathcal{T}$ and therefore, the proof is finished. So we suppose $\eta_n \neq \eta_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We have $\mathcal{T}\eta_{n-1} = \eta_n$ for all $n \in \mathbb{N}$. Since $\mathcal{T}$ is a rational type F-contration, we get

$$\begin{aligned} F(S(\eta_n, \eta_{n+1})) &= F(S(\mathcal{T}\eta_{n-1}, \mathcal{T}\eta_n)) \leq F(M(\eta_{n-1}, \eta_n)) - \tau \\ &= F\left(\max\left\{S(\eta_{n-1}, \eta_n), \frac{S(\eta_{n-1}, \mathcal{T}\eta_{n-1})[1 + S(\eta_n, \mathcal{T}\eta_n)]}{1 + S(\mathcal{T}\eta_{n-1}, \mathcal{T}\eta_n)}\right\}\right) - \tau \\ &= F(S(\eta_{n-1}, \eta_n)) - \tau \end{aligned} \tag{1}$$

for all $n \in \mathbb{N}$. Taking $a_n := S(\eta_{n-1}, \eta_n)$ for all $n \in \mathbb{N}$ and using (1), we obtain

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{2}$$

Based on statement (2), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Consequently, using (F2), we obtain

$$\lim_{n\to\infty} a_n = 0. \tag{3}$$

By the property (F3), there exist $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{4}$$

By (2), the following inequalities

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0 \tag{5}$$

hold for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (5) and using (3) and (4), we obtain $\lim_{n\to\infty} a_n^k n = 0$. Therefore, there exists $n_1 \in \mathbb{N}$ such that $na_n^k \leq 1$ for all $n \geq n_1$. So, we have $a_n \leq 1/n^{1/k}$.

By applying a standard reasoning, the inequality above demonstrates that $\{\eta_n\}$ forms a Cauchy sequence in the metric space $(\mathcal{Y}, \mathrm{d})$; in other words, $\{\eta_n\}$ is a perturbed Cauchy sequence within the PMS $(\mathcal{Y}, S, P)$. Due to the completeness of the PMS $(\mathcal{Y}, S, P)$, this implies the existence of $\eta^* \in \mathcal{Y}$ such that $\lim_{n\to\infty} \mathrm{d}(\eta_n, \eta^*) = 0$. We now show that η^* is a fixed point of $\mathcal{T}$. Since $\mathcal{T}$ is a perturped continuous mapping, this yields $\lim_{n\to\infty} \mathrm{d}(\mathcal{T}\eta_n, \mathcal{T}\eta^*) = 0$, that is $\lim_{n\to\infty} \mathrm{d}(\eta_{n+1}, \mathcal{T}\eta^*) = 0$. Since $\mathrm{d} = S - P$ is a metric on $\mathcal{Y}$, by the uniqueness property of the limit, we have $\eta^* = \mathcal{T}\eta^*$, namely η^* is a fixed point of $\mathcal{T}$.

Let us prove that $\mathcal{T}$ has a unique fixed point. We show by contraction assuming that $\eta, \mu \in \mathcal{Y}$ are two distinct fixed point of $\mathcal{T}$. Thus,

$$\begin{aligned} F(S(\eta,\mu)) &= F(S(\mathcal{T}\eta, \mathcal{T}\mu)) \leq F(M(\eta,\mu)) - \tau \\ &= F\left(\max\left\{ S(\eta,\mu), \frac{S(\eta,\mathcal{T}\eta)[1+S(\mu,\mathcal{T}\mu)]}{1+S(\mathcal{T}\eta,\mathcal{T}\mu)} \right\} \right) - \tau = F(S(\eta,\mu)) - \tau, \end{aligned}$$

which yields

$$F(d(\eta,\mu) + P(\eta,\mu)) \leq F(d(\eta,\mu) + P(\eta,\mu)) - \tau.$$

Since $\eta \neq \mu$, we get $\mathrm{d}(\eta,\mu) + P(\eta,\mu) \neq 0$ and from the inequality above it follows that $\tau \leq 0$, which conflicts with the condition $\tau > 0$. It follows that η^* is the unique fixed point of $\mathcal{T}$, thereby completing the proof of the theorem. □

Example 4. *Let* $S\colon [0,1] \times [0,1] \to [0,\infty)$ *be the mapping defined by*

$$S(\eta,\mu) = |\eta - \mu| + (\eta - \mu)^4.$$

Consequently, S *is considered a PM on the interval* $[0,1]$ *with the perturbed mapping* $P\colon [0,1] \times [0,1] \to [0,\infty)$ *defined as* $P(\eta,\mu) = (\eta-\mu)^4$.

Here, the exact metric $\mathrm{d} : [0,1] \times [0,1] \to [0,\infty)$ *is defined by*

$$\mathrm{d}(\eta,\mu) = |\eta - \mu|, \quad \forall \eta,\mu \in [0,1].$$

Note that S *is not a metric on* $[0,1]$. *For* $\eta = 0, \mu = 2/5, \rho = 1/5$, *we have*

$$S(0,2/5) = 0,4256, \quad S(0,1/5) = 0,2016, \quad S(2/5,1/5) = 0,2016,$$

and, therefore,

$$S(0,1/5) + S(2/5,1/5) = 0,4032,$$

which shows that

$$S(0,2/5) > S(0,1/5) + S(2/5,1/5).$$

Let us define a self-mapping $\mathcal{T} : [0,1] \to [0,1]$ *by* $\mathcal{T}\eta = \eta/2$, $\eta \in [0,1]$. *Take* $F(\eta) = \log \eta$, *which satisfies the conditions* $(F1)$–$(F3)$ *and* $\tau = \log 2 > 0$. *Then*

$$\begin{aligned} M(\eta,\mu) &= \max\left\{ S(\eta,\mu), \frac{S(\eta,\mathcal{T}\eta)[1+S(\mu,\mathcal{T}\mu)]}{1+S(\mathcal{T}\eta,\mathcal{T}\mu)} \right\} \\ &= \max\left\{ |\eta-\mu| + (\eta-\mu)^4, \frac{(|\eta|/2 + \eta^4/16)(1+|\mu|/2+\mu^4/16)}{1+|\eta-\mu|/2+(\eta-\mu)^4/16} \right\} \\ &= |\eta-\mu| + (\eta-\mu)^4 \end{aligned}$$

and

$$S(\mathcal{T}\eta,\mathcal{T}\mu) = |\mathcal{T}\eta - \mathcal{T}\mu| + (\mathcal{T}\eta - \mathcal{T}\mu)^4 = \frac{|\eta-\mu|}{2} + \frac{(\eta-\mu)^4}{16}.$$

Clearly, we have

$$\begin{aligned}\tau + F(S(\mathcal{T}\eta, \mathcal{T}\mu)) &= \log 2 + \log(S(\mathcal{T}\eta, \mathcal{T}\mu)) = \log(2 \cdot S(\mathcal{T}\eta, \mathcal{T}\mu)) \\ &= \log[2(|\eta - \mu|/2 + (\eta - \mu)^4/16)] \\ &= \log(|\eta - \mu| + (\eta - \mu)^4/8) \\ &\leq \log(|\eta - \mu| + (\eta - \mu)^4) = F(M(\eta, \mu)).\end{aligned}$$

Hence $\mathcal{T}$ is a F-perturbed mapping. So by the above theorem $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$, which is $\eta = 0$.

Corollary 1. *Let $(\mathcal{Y}, S, P)$ be a complete PMS, and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a F-contraction with respect to* d. *Suppose the conditions listed below are satisfied:*

(i) $\mathcal{T}$ is a continuous mapping subject to a perturbation;

(ii) there exist $F \in \mathcal{F}$ and a real number $\tau > 0$, such that

$$\tau + F(S(\mathcal{T}\eta, \mathcal{T}\mu)) \leq F(S(\eta, \mu))$$

for all $\eta, \mu \in \mathcal{Y}$ with $S(\eta, \mu) > 0$.

Consequently, $\mathcal{T}$ has a unique fixed point.

Corollary 2. *Let $(\mathcal{Y}, S, P)$ be a complete PMS, and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a mapping. Suppose the conditions listed below are satisfied:*

(i) $\mathcal{T}$ is a continuous mapping subject to a perturbation;

(ii) there exists $\lambda \in (0, 1)$ such that $S(\mathcal{T}\eta, \mathcal{T}\mu) \leq \lambda S(\eta, \mu)$ for all $\eta, \mu \in \mathcal{Y}$.

Consequently, $\mathcal{T}$ has a unique fixed point.

3 Conclusion

In this perspective, we establish a fixed-point theorem for rational type F-perturbed mappings in perturbed metric spaces, and then provide an illustrative example together with the corresponding results. These findings are expected to stimulate further research and contribute to the development of new results in the theory of PMS's.

References

[1] Acar Ö. *Fixed point result for rational type φ-Geraghty contraction.* Math. Morav. 2021, **25** (2), 35–41. doi: 10.5937/MatMor2102035A

[2] Acar Ö., Özkapu A.S. *Some results for rational type F-contraction with an applications to differential equations.* Adv. Stud. Euro-Tbil. Math. J. 2023, **16** (supp3), 131–138. doi:10.32513/asetmj/1932200823309

[3] Acar Ö. *Some recent and new fixed point results on orthogonal metric-like space*. Constr. Math. Anal. 2023, **6** (3), 184–197. doi:10.33205/cma.1360402

[4] Banach S. *Sur les fonctions dérivées des fonctions mesurables.* Fund. Math. 1922, **3** (1), 128–132.

[5] Boyd D.W., Wong J.S.W. *On nonlinear contractions*. Proc. Amer. Math. Soc. 1969, **20** (2), 458–464. doi: 10.2307/2035677

[6] Ciric L.B. *A generalization of Banach's contraction principle.* Proc. Amer. Math. Soc. 1974, **45** (2), 267–273. doi:10.1090/S0002-9939-1974-0356011-2

[7] Czerwik S. *Contraction mappings in b-metric spaces.* Acta Math. Inform. Univ. Ostraviensis 1993, **1** (1), 5–11.

[8] Huang H., Fu X. *Asymptotic properties of solutions for impulsive neutral stochastic functional integro-differential equations*. J. Math. Phys. 2021, **62** (1), article 013301. doi:10.1063/1.5139964

[9] Jleli M., Kawakami T., Samet B. *Critical behavior for a semilinear parabolic equation with forcing term depending on time and space.* J. Math. Anal. Appl. 2020, **486** (2), 123931. doi:10.1016/j.jmaa.2020.123931

[10] Jleli M., Samet B. *On Banach's fixed point theorem in perturbed metric spaces.* J. Appl. Anal. Comput. 2025, **15** (2), 993–1001. doi:10.11948/20240242

[11] Rus I.A. *Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey.* Carpathian J. Math. 2010, **26** (2), 230–258.

[12] Wardowski D. *Fixed points of a new type of contractive mappings in complete metric spaces.* Fixed Point Theory App. 2012, **2012**, article 94. doi:10.1186/1687-1812-2012-94

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Ердоган Е., Назам М., Акар О. *Деякі результати про нерухомі точки для раціональних стискуючих відображень у збуреному метричному просторі* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 402–408.

Ця робота присвячена вивченню теорії нерухомих точок для F-збурених відображень у повних збурених метричних просторах. Зокрема, ми вводимо поняття F-стискуючого відображення раціонального типу та доводимо існування та єдиність нерухомих точок для таких відображень. Наші результати узагальнюють кілька відомих результатів з літератури, розширюючи область застосування класичного принципу Банаха про стискуючі відображенння. Справедливість основних результатів демонструється за допомогою ілюстративного прикладу.

Ключові слова і фрази: нерухома точка, F-стискуюче відображення, збурений метричний простір.