



Entire functions of several variables: behaviour of directional derivatives

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Let h be a positive increasing on $[0; +\infty)$ function such that $h\left(x + \frac{1}{h(x)}\right) = O(h(x))$ as $x \rightarrow +\infty$. For a Lebesgue measurable set $E \subset [0; +\infty)$ of finite Lebesgue measure $\text{meas } E = \int_E dx$, we define the asymptotic h -density of E on $+\infty$ by

$$D_h(E) = \overline{\lim}_{R \rightarrow +\infty} h(R) \cdot \text{meas}(E \cap [R; +\infty)).$$

Consider the class \mathcal{H}^p of an entire functions in \mathbb{C}^p , that are bounded in an arbitrary domain $\Pi_R = \{z = (z_1, \dots, z_p) \in \mathbb{C}^p: \text{Re } z_j < R_j\}$, $R = (R_1, \dots, R_p) \in \mathbb{R}_+^p$ as well as in $G(r, A) = G + rA$ for every fixed $A \in \mathbb{R}^p$ and for each $r > 0$, where G is a complete polylinear domain. For a function $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$, let $F'_A(w)$ denotes the derivative of F in the direction of A at the point $w \in \mathbb{C}$. Let $F_A^{(k)}(w) = (F_A^{(k-1)}(w))'_A$ denotes the k th derivative in the direction of A at the point $w \in \mathbb{C}$. We also denote

$$S_F(r, A) := \sup \{|F(z)|: z \in G(r, A)\} = \sup \{|F(z)|: z \in \partial G(r, A)\}, \quad L_F(r, A) = (\ln S_F(r, A))'_+.$$

We prove the following statement. Let $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$ be such that $L_F(r, A) \uparrow +\infty$ as $r \rightarrow +\infty$. Suppose that Φ is a positive increasing on $[0; +\infty)$ function satisfying $u(r) \geq \Phi(r)$ for all $r \geq r_0$, and $h(r) = o(\Phi(r))$ as $r \rightarrow +\infty$. Then there exists a set $E \subset \mathbb{R}_+$ of zero asymptotical h -density, i.e. $D_h(E) = 0$, such that for every $k \in \mathbb{N}$ we have

$$F_A^{(k)}(w) = (1 + o(1)) L_F^k(r, A) F(w) \text{ as } r \rightarrow +\infty, \quad r \in \mathbb{R}_+ \setminus E,$$

for all points $w \in \partial G(r, A)$ satisfying the inequality $|F(w)| \geq S_F(r, A)/(1 + \varepsilon(r))$, where $\varepsilon(r)$ is a given arbitrary function such that $\varepsilon(r) \rightarrow +0$ as $r \rightarrow +\infty$.

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1 Introduction

The present paper is devoted to the analogues of the main relation of the Wiman-Valiron theory for entire functions of several complex variables which are bounded in the half-spaces. Let us first recall some facts from one-dimensional case, related to this topic.

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Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function of the form $f(z) = f_0 + \sum_{k=1}^{+\infty} f_k z^k$. For $r > 0$ we denote $M_f(r) = \max\{|f(z)| : |z| = r\}$, $\mu_f(r) = \max\{|f_k| r^k : k \geq 0\}$, $\nu_f(r) = \max\{k : |f_k| r^k = \mu_f(r)\}$ the maximum modulus, maximal term and central index, respectively.

The Wiman-Valiron theorem [18, 19, 26, 38] states that for every entire function f , for small $|\eta|$ in some sense and for any point z , $|z| = r$, such that $|f(z)| = M_f(r)$, we have

$$f(ze^\eta) \sim e^{\eta \nu_f(r)} f(z), \quad \frac{f^{(k)}(z)}{f(z)} \sim \left(\frac{\nu_f(r)}{z} \right)^k, \quad (1)$$

as $|z| = r \rightarrow +\infty$, $r \notin E$, and E is some set of finite logarithmic measure (f.l.m. for short), that is $\int_{E \cap [1, +\infty)} d \ln r < +\infty$.

Denote $K_f(r) = r(\ln M_f(r))'_+$, where $(\ln M_f(r))'_+$ is the right-hand derivative. From the theorem obtained in [27] for the entire Dirichlet series, it follows that for each entire function f we have

$$K_f(r) \sim \nu_f(r), \quad r \rightarrow +\infty, \quad r \notin E,$$

where $E = E(f)$ is some set of f.l.m. Various theorems about analogs of the second relation in (1) for an entire Dirichlet series with a positive monotonically increasing to infinity sequence of exponents can be found in [13, 24, 26, 27, 32, 34, 35], for absolutely convergent in a half-plane Dirichlet series in [36]. Analogous of the first relation in (1) for an entire Dirichlet series we find in [13, 24, 25], for entire functions of several complex variables in [15, 33, 37] and for multiple Dirichlet series in [22]. Also there are known first attempts [21] to deduce some results of this theory for multiple Dirichlet series with arbitrary complex exponents.

This article is devoted to proving counterparts of the second relation in (1) for the directional derivative of an entire multiple Dirichlet series with arbitrary non-negative exponents. We will obtain the relation as a consequence of the theorem on the behavior of the directional derivative of an entire function F of several complex variables $z = (z_1, \dots, z_p)$ at the neighborhood of a point w . This point w is chosen such that the value $F(w)$ is close to the supremum of its modulus on the topological boundary of polylinear domains. Similar results for entire multiple Dirichlet series with positive monotonically increasing to infinity exponents were deduced in book [37].

2 Notations and lemmas

Let \mathbb{R}^p and \mathbb{C}^p , $p \in \mathbb{N}$, be real and complex p -dimensional vector spaces, respectively; $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$. For $a = (a_1, \dots, a_p) \in \mathbb{R}^p$, $b = (b_1, \dots, b_p) \in \mathbb{R}^p$ we write $a < b$ and $a \leq b$, if $a_j < b_j$ and $a_j \leq b_j$ for all $1 \leq j \leq p$, respectively. For $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, $w = (w_1, \dots, w_p) \in \mathbb{C}^p$, we denote $(z, w) = z_1 w_1 + \dots + z_p w_p$, $\|z\| = z_1 + \dots + z_p$, $\|z\|_0 = z_1 + 2z_2 + \dots + pz_p$, $\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_p)$, and for $R = (r_1, \dots, r_p) \in \mathbb{R}^p$ we denote $\Pi_R = \{z \in \mathbb{C}^p : \operatorname{Re} z < R\}$.

Let $A = (A_1, \dots, A_p) \in \mathbb{R}^p \setminus \{0\}$ be a fixed vector. We say that the system of domains $\{G(r, A)\}_{r \geq 0}$, $G(r, A) \subset \mathbb{C}^p$, is an exhaustion of the space \mathbb{C}^p from the class Σ , if

- a) $\bigcup_{r \geq 0} G(r, A) = \mathbb{C}^p$;
- b) $G(r_1, A) \subset G(r_2, A)$, $0 \leq r_1 < r_2 < +\infty$;

c) if $z = (z_1, \dots, z_p) \in G(r, A)$, then for every $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ we get

$$z + iy = (z_1 + iy_1, \dots, z_p + iy_p) \in G(r, A);$$

d) if $z \in G(r, A)$, then $(z - rA) \in G(0, A)$;

e) for given $r \geq 0$ there exist $R_*, R^* \in \mathbb{R}^p$ such that $\Pi_{R_*} \subset G(r, A) \subset \Pi_{R^*}$.

We will denote the class of such exhaustions $\{G(r, A)\}_{r \geq 0}$ by Σ .

This definition is worth comparing with the definition of the exhaustions by system of A -like polylinear domains given in [37, p. 301], in which a similar concept is considered for exhaustions defined by a vector $A = (A_1, \dots, A_p) > 0$.

The condition $A > 0$ in the definition from [37] ensures the equality $\mathbb{C}^p = \bigcup_{r \geq 0} G_+(r, A)$ and the inclusion $G_+(r_1, A) \subset G_+(r_2, A)$, which in our definition of exhaustion are conditions a) and b), that allows us to abandon the a priori restriction on the positivity of the components of the vector A .

Consider the class \mathcal{H}^p of an entire functions in \mathbb{C}^p , that are bounded in an arbitrary domain Π_R , $R = (R_1, \dots, R_p) \in \mathbb{R}_+^p$. For a function $F \in \mathcal{H}^p$ and $x \in \mathbb{R}^p$ it is obvious that

$$M(x, F) := \sup\{|F(x + iy)| : y \in \mathbb{R}^p\} < +\infty,$$

and also that if the analytic function F is bounded in a polylinear domain G , then for every x such that $\{z \in \mathbb{C}^p : \operatorname{Re} z_1 = x_1, \dots, \operatorname{Re} z_p = x_p\} \subset G$ one has $M(x, F) < +\infty$.

For a function $F \in \mathcal{H}^p$ and $r > 0$ we also denote

$$S_F(r, A) := \sup\{|F(z)| : z \in G(r, A)\} = \sup\{|F(z)| : z \in \partial G(r, A)\}.$$

We note that $S_F(r, A) < +\infty, \forall r \geq 0$, in the case, when $F \in \mathcal{H}^p$, and the exhaustion is such that $\Pi_{R_*} \subset G(r, A) \subset \Pi_{R^*}$ for some $R_* < R^*$, i.e. condition e) is satisfied.

Lemma 1 ([33]). *Let $F \in \mathcal{H}^p$ and $\{G(r, A)\} \in \Sigma$. Then $\ln S_F(r, A)$ is a convex function in variable $r > 0$.*

Since $\ln S_F(r, A)$ is a convex function, it has everywhere a right-hand derivative

$$L_F(r, A) := (\ln S_F(r, A))'_+,$$

which is a non-decreasing function. It is obvious that for every $r_0 > 0$ there exists

$$L_+(r_0) := \lim_{r \rightarrow r_0+0} L_F(r, A) \geq \lim_{r \rightarrow r_0-0} L_F(r, A).$$

Therefore, without loss of the generality, we can assume that the function $L_F(r, A)$ is right semi-continuous, that is $L_F(r_0, A) = L_+(r_0)$ at every point $r_0 > 0$.

Let \mathcal{L} be the class of positive continuous functions which are increasing to $+\infty$ on $[0; +\infty)$ and \mathcal{L}_1 be the class of continuous positive non-decreasing on $[0, +\infty)$ functions h such that

$$h(x + O(1)) = O(h(x)), \quad x \rightarrow +\infty.$$

Let \mathcal{L}_2 be the class of continuous positive non-decreasing on $[0; +\infty)$ functions h such that

$$h\left(x + \frac{1}{h(x)}\right) = O(h(x)), \quad x \rightarrow +\infty.$$

Let h be a positive continuous non-decreasing function and $E \subset [0; +\infty)$ be a locally Lebesgue measurable set of finite measure with $\text{meas } E = \int_E dx < +\infty$. Then we define the asymptotic h -density of E as

$$D_h(E) = \overline{\lim}_{R \rightarrow +\infty} h(R) \cdot \text{meas}(E \cap [R, +\infty)).$$

Remark, if a locally Lebesgue measurable set E such that $h\text{-meas}(E) := \int_E h(r)dr < +\infty$, then $D_h(E) = 0$.

We need the following statement about the relation, similar to the first one in (1).

Theorem 1 ([33]). *Let $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$ be such that $L_F(r, A) \uparrow +\infty, r \rightarrow +\infty$, and $\Phi \in \mathcal{L}$, $h \in \mathcal{L}_2$ be the functions such that*

$$u(r) \geq \Phi(r), \quad r \geq r_0, \quad h(r) = o(\Phi(r)), \quad r \rightarrow +\infty. \quad (2)$$

Then there exist a set $E \subset \mathbb{R}_+$ of zero asymptotical h -density (i.e. $D_h(E) = 0$) and a function $\delta(u) \uparrow +\infty, u \rightarrow +\infty$, such that for all $r \in \mathbb{R}_+ \setminus E$ and all $\eta \in \mathbb{C}$ with $|\eta| \leq \delta(r)/L_F(r, A)$, it holds

$$F(w + A\eta) = (1 + \omega(\eta))F(w)e^{\eta L_F(r, A)},$$

where

$$|\omega(\eta)| < |\eta|c(r)L_F(r, A)/\delta(r), \quad c(r) := 1 + e(1 + \varepsilon(r))$$

and the point $w \in \partial G(r, A)$ is chosen such that

$$|F(w)| \geq S_F(r, A)/(1 + \varepsilon(r)), \quad (3)$$

and $\varepsilon(r)$ is a given function with $\varepsilon(r) \rightarrow +0$ as $r \rightarrow +\infty$.

In paper [33] (see also [24]) we find the following version of the Borel-Nevanlinna lemma. This lemma was the main tool to prove Theorem 1. Its wording is given for the completeness of the presentation of the paper.

Lemma 2 ([24,33]). *Let $u(r)$ be the right semi-continuous function which increases on $[r_0, +\infty)$, and $\Phi \in \mathcal{L}, h \in \mathcal{L}_2$ be the functions such that*

$$u(r) \geq \Phi(r), \quad r \geq r_0, \quad h(r) = o(\Phi(r)), \quad r \rightarrow +\infty.$$

Then there exists a function $\delta(u) \uparrow +\infty, u \rightarrow +\infty$, such that the set

$$E = \left\{ r \geq r_0 : \left| u \left(r \pm \frac{\delta(u(r))}{u(r)} \right) - u(r) \right| \geq u(r)/\delta(u(r)) \right\}$$

has zero asymptotic h -density, i.e. $D_h(E) = 0$. In other words, the inequality

$$|u(r + \tau) - u(r)| < u(r)/\delta(u(r))$$

holds for all $r \in (r_0, +\infty) \setminus E$ and all $\tau \in \mathbb{R}, |\tau| \leq \psi(r) := \delta(u(r))/u(r)$ and the set E has zero asymptotic h -density.

3 Main result

For a function $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$, let $F'_A(w)$ denotes the derivative of F in the direction A at the point $w \in \mathbb{C}$; $F_A^{(k)}(w) = (F_A^{(k-1)}(w))'_A$ denotes the k th derivative in the direction A at the point $w \in \mathbb{C}$.

Theorem 2. *Let $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$ be such that $L_F(r, A) \uparrow +\infty$, $r \rightarrow +\infty$. If $\Phi \in \mathcal{L}$, $h \in \mathcal{L}_2$ satisfy conditions (2), then there exists a set $E \subset \mathbb{R}_+$ of zero asymptotic h -density (i.e. $D_h(E) = 0$) such that for each $k \in \mathbb{N}$ one has*

$$F_A^{(k)}(w) = (1 + o(1)) L_F^k(r, A) F(w) \quad \text{as } r \rightarrow +\infty, r \in \mathbb{R}_+ \setminus E, \quad (4)$$

for every point $w \in \partial G(r, A)$ provided inequality (3) with a given function $\varepsilon(r)$ which is decreasing to $+0$ as $r \rightarrow +\infty$.

Proof. Let a given point $w \in \partial G(r, A)$ be such that condition (3) is satisfied. Consider the function

$$\omega(\eta) = \frac{F(w + A\eta)}{F(w)} e^{-\eta L_F(r, A)} - 1$$

of the variable $\eta \in \mathbb{C}$, $|\eta| \leq \psi(r)$, for fixed $r \in [0; +\infty) \setminus E$, where E is the set from Theorem 1. Let us denote again $\psi(r) = \delta(r)/L_F(r, A)$. From Theorem 1 for all η , $|\eta| \leq \psi(r)/c(r)$, $r \in [0; +\infty) \setminus E$ one has $|\omega(\eta)| < 1$. Thus,

$$\left| \frac{F(w + A\eta)}{F(w)} e^{-\eta L_F(r, A)} \right| = |1 + \omega(\eta)| \geq 1 - |\omega(\eta)| > 0.$$

Hence, $|F(w + A\eta)| > 0$ for all η , $|\eta| < \psi(r)/c(r)$, $r \notin E$. Therefore,

$$f(\eta) = \int_0^\eta \frac{F'_A(w + A\tau)}{F(w + A\tau)} d\tau - \eta L_F(r, A), \quad f(0) = 0,$$

is an analytic function in the unit disc $\{\eta : |\eta| < \psi(r)/c(r)\}$; here $F'_A(w + A\tau)$ is the derivative of the function F in the direction A at the point $w + A\tau$.

In addition, $f'(0) = F'_A(w)/F(w) - L_F(r, A)$, and

$$\operatorname{Re} f(\eta) = \ln \left| \frac{F(w + A\eta)}{F(w)} e^{-\eta L_F(r, A)} \right| = \ln |1 + \omega(\eta)| \leq \ln(1 + |\omega(\eta)|) \leq \ln \left(1 + \frac{qc(r)}{\psi(r)} \right)$$

for all η , $|\eta| \leq q < \psi(r)/c(r)$.

The statement of the following lemma was formulated in the solution of Problem 236 (see, [23, Part III, Ch. 5, § 2, 236, p.355]).

Lemma 3. *Let $f(z) = \sum_{n=0}^{+\infty} f_n z^n$ be an analytic function in the disk $\mathbb{D}_R = \{z : |z| < R\}$, $R > 0$. If $\operatorname{Re} f(z) < M$ for all $z \in \mathbb{D}_R$, then $|f_n| R^n \leq 2(M - \operatorname{Re} f_0)$ for all $z \in \mathbb{D}_R$.*

Applying Lemma 3 to the function f in the disc $\mathbb{D}_q = \{\eta : |\eta| \leq q\}$, $q < \psi(r)/c(r)$, we obtain

$$\left| \frac{F'_A(w)}{F(w)} - L_F(r, A) \right| = |f'(0)| = |f_1| \leq 2 \max\{\operatorname{Re} f(\eta) : |\eta| = q\} \leq 2 \ln \left(1 + \frac{qc(r)}{\psi(r)} \right) \leq \frac{2c(r)}{\psi(r)}.$$

So, as $r \rightarrow +\infty$, $r \notin E$, and for all $w \in \partial G_F(r, A)$ satisfying $|F(w)| \geq S_F(r, A)(1 + \varepsilon(r))^{-1}$ one has

$$\left| \frac{F'_A(w)}{F(w)} \frac{1}{L_F(r, A)} - 1 \right| \leq \frac{2c(r)}{L_F(r, A)\psi(r)} = \frac{c(r)}{\delta(r)} = o(1). \quad (5)$$

As above, applying Lemma 3 to the function f in the disc $\mathbb{D}_q = \{\eta: |\eta| \leq q\}$, $q < \psi(r)/c(r)$, we obtain for fixed $k \in \mathbb{N}$

$$\begin{aligned} \frac{1}{k!} |f^{(k)}(0)| &= |f_k| \leq \frac{2}{q^k} \max\{\operatorname{Re} f(\eta): |\eta| = q\} \\ &\leq 2q^{-k} \ln(1 + qc(r)/\psi(r)) =: \psi_k(r) \leq 2q^{-k+1} \frac{c(r)}{\psi(r)} \end{aligned} \quad (6)$$

for $|\eta| \leq q < \psi(r)/c(r)$. Continuing further, for $|\eta| \leq q$ one has

$$\begin{aligned} F_A(w + \eta A) &= F(w) \exp\{f(\eta) + \eta L_F(r, A)\} \\ &= F(w) \exp\left\{ \frac{F'_A(w)}{F(w)} \eta + \sum_{k=2}^{+\infty} \frac{f^{(k)}(0)}{k!} \eta^k \right\} =: F(w) F_0(\eta), \end{aligned} \quad (7)$$

where $F_0(\eta)$ is an analytic function in the variable η , $F_0(0) = 1$. We put

$$F_0(\eta) = 1 + \sum_{k=1}^{+\infty} F_k \eta^k, \quad |\eta| \leq q. \quad (8)$$

Since for the function $f_1(\eta) := f(\eta) + \eta L_F(r, A)$ we have $f_1^{(k)}(\eta) = f^{(k)}(\eta)$, $k \geq 2$, so

$$F_0(\eta) = 1 + \sum_{s=1}^{+\infty} \frac{1}{s!} \left(\frac{F'_A(z)}{F(z)} \eta + \sum_{k=2}^{+\infty} \frac{f^{(k)}(0)}{k!} \eta^k \right)^s = 1 + \sum_{s=1}^{+\infty} \frac{1}{s!} \left(\sum_{k=1}^{+\infty} \frac{f_1^{(k)}(0)}{k!} \eta^k \right)^s.$$

Therefore, the Taylor coefficients in (8) are formed by sums of the following form

$$F_k = \sum_{\|\alpha\|_0=k} B_\alpha (f_1'(0))^{\alpha_1} \cdot \left(\frac{f_1''(0)}{2!} \right)^{\alpha_2} \cdot \dots \cdot \left(\frac{f_1^{(k)}(0)}{k!} \right)^{\alpha_k}$$

with finite quantities of the summands, where $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k$ is the multi-index, and $\|\alpha\|_0 := \sum_{j=1}^k j\alpha_j$ its weight, $B_\alpha \in \mathbb{C}$. From the Taylor expansion $F(w + A\eta) = \sum_{k=0}^{+\infty} \frac{F_A^{(k)}(w)}{k!} \eta^k$ and (7)–(8) it follows that

$$F_A^{(k)}(w) = F(w) k! F_k, \quad k \geq 1.$$

Hence,

$$\frac{F_A^{(k)}(w)}{F(w)} \frac{1}{L_F^k(r, A)} - 1 = \left(\frac{F'_A(w)}{F(w)} \frac{1}{L_F(r, A)} \right)^k - 1 + \sum_{\substack{\|\alpha\|_0=k \\ \alpha_1 < k}} B_\alpha \prod_{j=1}^k \left(\frac{f_1^{(j)}(0)}{j!} \right)^{\alpha_j}. \quad (9)$$

Since $c(r) = O(1)$ and $\delta(r) \nearrow +\infty$ as $r \rightarrow +\infty$, thus $c(r)/\delta(r) = o(1)$ as $r \rightarrow +\infty$. So, from inequality (5) we have

$$\begin{aligned} \left| \left(\frac{F'_A(w)}{F(w)} \frac{1}{L_F(r, A)} \right)^k - 1 \right| &= \left| \frac{F'_A(w)}{F(w)} \frac{1}{L_F(r, A)} - 1 \right| \sum_{j=0}^{k-1} \left| \frac{F'_A(w)}{F(w)} \frac{1}{L_F(r, A)} \right|^j \\ &\leq \frac{c(r)}{\delta(r)} \sum_{j=0}^{k-1} \left(1 + \frac{c(r)}{\delta(r)} \right)^j = \left(1 + \frac{c(r)}{\delta(r)} \right)^k - 1 = o(1) \end{aligned} \quad (10)$$

as $r \rightarrow +\infty, r \notin E$. We put

$$q = \frac{1}{2} \frac{\psi(r)}{c(r)} = \frac{1}{2} \frac{\delta(r)}{c(r)L_F(r, A)}.$$

Using inequality (6), we get

$$\begin{aligned} \sum_{\substack{\|\alpha\|_0=k \\ \alpha_1 < k}} B_\alpha \prod_{j=1}^k \left(\frac{f_1^{(j)}(0)}{j!} \right)^{\alpha_j} &\leq \sum_{\substack{\|\alpha\|_0=k \\ \alpha_1 < k}} |B_\alpha| (qL_F(r, A))^{-k} \left(\frac{2qc(r)}{\psi(r)} \right)^{\|\alpha\|} \\ &= \sum_{\substack{\|\alpha\|_0=k \\ \alpha_1 < k}} |B_\alpha| \left(\frac{2c(r)}{\delta(r)} \right)^k = o(1) \end{aligned} \quad (11)$$

as $r \rightarrow +\infty, r \notin E$. Applying relations (10) and (11), from equality (9) we obtain

$$\left| \frac{F_A^{(k)}(w)}{F(w)} \frac{1}{L_F^k(r, A)} - 1 \right| = o(1)$$

as $r \rightarrow +\infty, r \notin E$. Thus, asymptotic relation (4) is proved. \square

Corollary 1. *Let $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$ be such that $L_F(r, A) \rightarrow +\infty$ as $r \rightarrow +\infty$. Then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesgue measure such that for each $k \in \mathbb{N}$ asymptotic relation (4) holds as $r \rightarrow +\infty, r \in \mathbb{R}_+ \setminus E$, for every point $w \in \partial G(r, A)$ satisfying the inequality $|F(w)| \geq S_F(r, A)/(1 + \varepsilon(r))$; here $\varepsilon(r)$ is a given function such that $\varepsilon(r) \rightarrow +0$ as $r \rightarrow +\infty$.*

4 Corollaries: analytical solutions of directional differential equations

Here we present a scheme of application of Theorem 2 to partial differential equations which contain directional derivatives.

There were deeply studied the growth, pseudo-convexity and pseudo-starlikeness of entire Dirichlet series in [30, 31] as solutions of the equation

$$F^{(n)}(z) + \left(\sum_{k=1}^n \gamma_k e^{khz} \right) F(z) = ae^{hz},$$

where $n \geq 3, h > 0, a \neq 0$. M.M. Sheremeta [28, 29] early considered similar problems in the case $n = 2, a = 0$.

We consider the directional differential equation of the form (see also, for example, [7])

$$F_A^{(N)}(w) + \sum_{k=0}^{N-1} g_k(w) F_A^{(k)}(w) = g_N(w), \quad w \in \mathbb{C}^p, \quad N \in \mathbb{N}, \quad (12)$$

where g_k, g are some analytic functions from the class \mathcal{H}^p . Equation (12) was intensively studied by methods of theory of functions having bounded index in direction. For this equation there are papers on its entire solutions [5], slice entire solutions [7], analytic solutions in the unit ball [2, 6], slice holomorphic solutions in the unit ball [3]. These methods are based by logarithmic criterion and analog of Hayman's theorem for various holomorphy domains [4]. The big advantage of these methods is possibility to estimate growth in such multidimensional holomorphy domains as unit ball, unit polydisc and in arbitrary Reinhardt domain.

Analogues of Wiman-Valiron's theory for the specified domains are known only in the case of unit disc [14] with some additional assumptions. However, the theory has advances in difference equations [20], meromorphic functions [8,9], for a polynomial series based on the Wilson operator [10,11], for random analytic functions [1,17], for fractional differentiation operator [12] and non-power series approach to deduce main estimate [16]. In particular, they can be exponential polynomials of the form

$$g_k(w) = \sum_{\|n\|=0}^{m_k} g_n^{(k)} e^{(w, \lambda_n^{(k)})}, \quad w \in \mathbb{C}^p, \quad m_k \in \mathbb{Z}_+, \quad 0 \leq k \leq N,$$

and $g_n^{(k)} \neq 0$ at least for one $n \in \mathbb{N}^p$ with $\|n\| = m_k$, $\lambda_n^{(k)} \in \mathbb{R}^p$. In the case, the directional differential equation may have solutions from the class \mathcal{H}^p .

Example 1. Consider the following simple differential equation $F'_A(w) = P(w)F(w)$, where $P \in \mathcal{H}^p$. Then it easy to see, that the function F such that

$$F(w + A\eta) = \exp \left\{ \int_0^\eta P(w + A\tau) d\tau \right\}, \quad w \in \mathbb{C}^p,$$

is a solution from the class \mathcal{H}^p . This function is also the solution to the following equation

$$F''_A(z) = P(z)F'_A(z) + P'_A(z)F(z).$$

It is not difficult to verify that the function

$$F(w + A\eta) = \exp \left\{ \int_0^\eta P(w + A\tau) d\tau \right\} \cdot \int_0^\eta \left(Q(w + Az) \exp \left\{ - \int_0^z P(w + A\tau) d\tau \right\} \right) dz$$

is a solution of equation $F'_A(w) = P(w)F(w) + Q(w)$ with $P, Q \in \mathcal{H}^p$ and also $F \in \mathcal{H}^p$.

Therefore, the question of studying the properties of solutions from the class \mathcal{H}^p seems natural.

Example 2. To simplify the reasoning, let us assume that $G(0, A) = \Pi_a$, $A > 0$, $a \in \mathbb{R}^p$. Then $G(r, A) = \Pi_{a+rA}$. Let us assume that $L_F(R, A) \rightarrow +\infty$ as $R \rightarrow +\infty$ and also

$$g_N(w) = \sum_{\|n\|=0}^{m_N} g_n^{(N)} e^{(w, \lambda_n^{(N)})} = O\left(e^{(w, \hat{\lambda}^{(N)})}\right), \quad g_{N-1}(w) = (1 + o(1)) \hat{B}_N e^{(w, \hat{\lambda}^{(N-1)})},$$

$$g_k(w) = \sum_{\|n\|=0}^{m_k} g_n^{(k)} e^{(w, \lambda_n^{(k)})} = o(g_{N-1}(w)),$$

as $\operatorname{Re} w = a + rA \rightarrow +\infty$, $r \rightarrow +\infty$, for all $k \in \{0, 1, \dots, N-2\}$, where

$$(A, \hat{\lambda}^{(k)}) = \max\{(w, \lambda_n^{(k)}) : g_n^{(k)} \neq 0, 0 \leq \|n\| \leq m_k\}, \quad k \in \{N-1, N\}.$$

Let us now apply the Corollary 1 to equation (12). Since $w \in \partial G(r, A) \implies \operatorname{Re} w = a + rA$ for w such that $|F(w)| = (1 + o(1))S_F(r, A)$, we obtain

$$F(w) \left((1 + o(1))L_F^N(r, A) + (1 + o(1))g_{N-1}(w)L_F^{N-1}(r, A) \right) = (1 + o(1))\hat{B}_N e^{(w, \hat{\lambda}^{(N)})} \quad (13)$$

as $r \rightarrow +\infty$ outside some set E of finite Lebesgue measure. Remark that $R = o(\ln S_F(R, A))$ as $R \rightarrow +\infty$. Then $g_N(w) = o(S_F(r, A))$ as $R \rightarrow +\infty$, and (13) implies

$$(1 + o(1))L_F^N(r, A) + (1 + o(1))g_{N-1}(w)L_F^{N-1}(r, A) = o(1).$$

Hence, $L_F(r, A) = (1 + o(1))|g_{N-1}(w)|$ as $r \rightarrow +\infty, r \notin E$, thus,

$$\ln L_F(r, A) = \ln |\widehat{B}_N| + (a + rA, \widehat{\lambda}^{(N-1)}), \quad r \rightarrow +\infty, r \notin E.$$

Therefore,

$$\ln L_F(r, A) = r(A, \lambda) + O(1), \quad \lambda = \widehat{\lambda}^{(N-1)}, \quad (14)$$

as $r \rightarrow +\infty$ outside some set E of finite Lebesgue measure. Now, it is easy to see that for every point $r' \in E$ there exists two points $r_1, r_2 \notin E$ such that $r_1 < r < r_2$ and $r = r_k + o(1)$ as $r \rightarrow +\infty, k \in \{1, 2\}$. Hence, $r_k(A, \lambda) = r(A, \lambda) + o(1), r \rightarrow +\infty, k \in \{1, 2\}$, and by relation (14) we have

$$\begin{aligned} \ln L_F(r_1, A) &\leq \ln L_F(r, A) \leq \ln L_F(r_2, A) = r_2(A, \lambda) + O(1) = r(A, \lambda) + O(1), \\ \ln L_F(r, A) &\geq \ln L_F(r_1, A) = r_1(A, \lambda) + O(1) = r(A, \lambda) + O(1), \quad r \rightarrow +\infty. \end{aligned}$$

From the last relations it follows that $\ln L_F(r, A) = r(A, \lambda) + O(1), r \rightarrow +\infty$. If we now use the equality $\ln S_F(r, A) - \ln S_F(r_0, A) = \int_{r_0}^r L_F(t, A) dt$, we get

$$\ln \ln S_F(r, A) = r(A, \lambda) + O(1), \quad r \rightarrow +\infty.$$

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Нехай h – додатна зростаюча на $[0; +\infty)$ функція така, що $h\left(x + \frac{1}{h(x)}\right) = O(h(x))$, $x \rightarrow +\infty$. Для вимірної за Лебегом множини $E \subset [0; +\infty)$ скінченної міри Лебега $\text{meas } E = \int_E dx$ ми визначаємо асимптотичну h -щільність множини E на $+\infty$ формулою

$$D_h(E) = \overline{\lim}_{R \rightarrow +\infty} h(R) \cdot \text{meas}(E \cap [R; +\infty)).$$

Розглянемо клас \mathcal{H}^p цілих в \mathbb{C}^p функцій, які є обмеженими в довільній області вигляду $\Pi_R = \{z = (z_1, \dots, z_p) \in \mathbb{C}^p : \text{Re } z_j < R_j\}$, $R = (R_1, \dots, R_p) \in \mathbb{R}_+^p$ та для кожного фіксованого $A \in \mathbb{R}^p$ і кожного $r > 0$ в $G(r, A) = G + rA$, де G – повна полілінійна область. Для функції $F \in \mathcal{H}^p$ та $A \in \mathbb{R}^p$ позначимо $F'_A(w)$ похідну F у напрямку A у точці $w \in \mathbb{C}$. Тоді $F_A^{(k)}(w) = (F_A^{(k-1)}(w))'_A$ позначає k -ту похідну в напрямку A у точці $w \in \mathbb{C}$. Позначимо також $S_F(r, A) := \sup \{|F(z)| : z \in G(r, A)\} = \sup \{|F(z)| : z \in \partial G(r, A)\}$, $L_F(r, A) = (\ln S_F(r, A))'_+$.

Ми доводимо наступне твердження. Нехай $F \in \mathcal{H}^p$ та $A \in \mathbb{R}^p$ такі, що $L_F(r, A) \uparrow +\infty$, $r \rightarrow +\infty$. Якщо Φ – додатна зростаюча на $[0; +\infty)$ функція, для якої виконуються умови $u(r) \geq \Phi(r)$, $r \geq r_0$, та $h(r) = o(\Phi(r))$, $r \rightarrow +\infty$, то існує множина $E \subset \mathbb{R}_+$ з нульовою асимптотичною h -щільністю (тобто $D_h(E) = 0$) така, що для кожного $k \in \mathbb{N}$ маємо

$$F_A^{(k)}(w) = (1 + o(1)) L_F^k(r, A) F(w) \text{ при } r \rightarrow +\infty, r \in \mathbb{R}_+ \setminus E,$$

для кожної точки $w \in \partial G(r, A)$ такої, що нерівність $|F(w)| \geq S_F(r, A)/(1 + \varepsilon(r))$ виконується з заданою довільною функцією $\varepsilon(r)$ такою, що $\varepsilon(r) \rightarrow +0$, $r \rightarrow +\infty$.

Ключові слова і фрази: ціла функція декількох комплексних змінних, полілінійна область, похідна за напрямком.