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Algebras of symmetric functions on Banach spaces and their applications in quantum physics

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We investigate relations between the group of symmetries of a subspace of the symmetric tensor product of a separable Hilbert space and a representation of the partition function of a quantum entangled family of particles associated with this subspace. Consider a system of N noninteracting identical bosons. In general, the system is described by the N-fold symmetric tensor product $\mathcal{E}^{\odot N}$, where $\mathcal{E}$ is the Hilbert space that describes the one-particle system. In some cases, the system can be described by some subspace of $\mathcal{E}^{\odot N}$. We consider the case in which this subspace is the closure of the linear span of some subset of the eigenbasis of the Hamiltonian of the system. We describe the group of symmetries S such that the partition function of the system is S-symmetric.

Key words and phrases: symmetric polynomial on Banach space, symmetric tensor product, quantum state, entanglement, partition function.

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Introduction

Let X be a complex Banach space and $\mathfrak{S}$ be a group of operators on X. A function $f\colon X \to \mathbb{C}$ is $\mathfrak{S}$-*symmetric* (or just *symmetric* if $\mathfrak{S}$ is fixed) if $f(A(x)) = f(x)$ for all $x \in X$, and $A \in \mathfrak{S}$. Symmetric polynomials on Banach spaces with respect to various groups have been extensively studied in recent years by many authors (see [1,3–5,9] and references therein). Symmetric polynomials and analytic functions have applications in different branches of mathematics, informatics, and physics. In [13, 14] it was observed that the canonical partition function of a quantum ideal gas can be represented by a basis of symmetric polynomials, and relations between different bases as Newton's identity, have physical meanings. Using this observation, in [6, 10] so-called supersymmetric polynomials on ℓ_p were applied for the description of the grand canonical partition function of an ideal gas consisting of both bosons and fermions.

In this paper, we consider how to represent the canonical partition function of a finite collection of entangled particles (bosons) by $\mathfrak{S}$-symmetric polynomials for some group $\mathfrak{S}$. The main result of the paper asserts that for any kind of entanglement (which can be defined by a subset of basis vectors in the symmetric tensor product of a Hilbert space) there exists a group of symmetries $\mathfrak{S} = S$ such that the partition function of the system is S-symmetric. Also, we show that for some particular cases, the S-symmetric polynomials are block-symmetric (the formal definitions are in the main part of the paper).

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For details on polynomials on Banach spaces, we refer the reader to [7]. Block-symmetric polynomials were studied in [2,8,11,15].

1 Results

Consider a system of N noninteracting identical bosons that can be entangled. Note that the situation when entangled particles do not interact with each other quantum mechanically is typical in quantum information: entangled noninteracting particles are key to quantum teleportation, superdense coding, and quantum cryptography. By postulates of Quantum Mechanics, the single-particle system is described by some complex Hilbert space $\mathcal{E}$, and the state of the system is represented by a ray with norm 1 in $\mathcal{E}$, i.e. by a set of the form $\{e^{i\theta}x : \theta \in [0, 2\pi)\}$, where $x \in \mathcal{E}$ is such that $\|x\| = 1$. Every physical observable corresponds to a self-adjoint (Hermitian) operator on $\mathcal{E}$ who's eigenvectors form a complete orthonormal basis in $\mathcal{E}$. Let $H : \mathcal{E} \to \mathcal{E}$ be the Hamiltonian (the operator that corresponds to the energy) of the single-particle system. Let $\{e_j : j \in \Lambda\}$, where Λ is some ordered index set, be the orthonormal basis in $\mathcal{E}$ consisting of eigenvectors of H, namely,

$$H(e_j) = \lambda_j e_j \tag{1}$$

for every $j \in \Lambda$, where $\lambda_j > 0$.

Consider the N-fold tensor product $\mathcal{E}^{\otimes N}$ (see [12, II.4, p.49]). By [12, II.4, Proposition 2, p. 50]), the set $\{e_{j_1} \otimes \cdots \otimes e_{j_N} : (j_1, \ldots, j_N) \in \Lambda\}$ is an orthonormal basis in $\mathcal{E}^{\otimes N}$. Since the particles in the system are bosons, it follows that the system is described by the N-fold symmetric tensor product $\mathcal{E}^{\odot N}$ (see [12, II.4, p.53]), which is the closed subspace of $\mathcal{E}^{\otimes N}$. Let us describe an orthogonal basis in $\mathcal{E}^{\odot N}$. First, let us introduce some notation. Let $\mathcal{S}(A)$ be the group of all bijections on a set A. We will also denote by $\mathcal{S}_N$ the group $\mathcal{S}(\{1, \ldots, N\})$. For every $\sigma \in \mathcal{S}_N$, let $\bar{\sigma} : \Lambda^N \to \Lambda^N$ be defined by

$$\bar{\sigma}((j_1, \ldots, j_N)) = (j_{\sigma(1)}, \ldots, j_{\sigma(N)}), \tag{2}$$

where $(j_1, \ldots, j_N) \in \Lambda^N$. For $(j_1, \ldots, j_N), (k_1, \ldots, k_N) \in \Lambda^N$ let

$$(j_1, \ldots, j_N) \sim (k_1, \ldots, k_N) \Leftrightarrow \exists \sigma \in \mathcal{S}_N : \ (k_1, \ldots, k_N) = \bar{\sigma}((j_1, \ldots, j_N)). \tag{3}$$

Lemma 1. *The relation "$\sim$" defined by* (3) *is an equivalence relation on* Λ^N.

Proof. Since the identity mapping on $\{1, \ldots, N\}$ belongs to $\mathcal{S}_N$, it follows that the relation is reflexive. Since for every element of $\mathcal{S}_N$ its inverse belongs to $\mathcal{S}_N$, it follows that the relation is symmetric.

Let us show that the relation is transitive. Let $(j_1, \ldots, j_N), (k_1, \ldots, k_N), (l_1, \ldots, l_N) \in \Lambda^N$ be such that $(j_1, \ldots, j_N) \sim (k_1, \ldots, k_N)$ and $(k_1, \ldots, k_N) \sim (l_1, \ldots, l_N)$. By (3), there exist $\sigma, \tau \in \mathcal{S}_N$ such that $(k_1, \ldots, k_N) = (j_{\sigma(1)}, \ldots, j_{\sigma(N)})$ and $(l_1, \ldots, l_N) = (k_{\tau(1)}, \ldots, k_{\tau(N)})$, i.e. $k_m = j_{\sigma(m)}$ and $l_m = k_{\tau(m)}$ for every $m \in \{1, \ldots, N\}$. Therefore $l_m = j_{\sigma(\tau(m))}$ for every $m \in \{1, \ldots, N\}$. Consequently, since $\sigma \circ \tau \in \mathcal{S}_N$, by (3), $(j_1, \ldots, j_N) \sim (l_1, \ldots, l_N)$. This completes the proof. $\square$

For $(j_1, \ldots, j_N) \in \Lambda^N$ let

$$e_{j_1} \odot \cdots \odot e_{j_N} = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(N)}}. \tag{4}$$

Note that
$$e_{j_1} \odot \cdots \odot e_{j_N} = e_{k_1} \odot \cdots \odot e_{k_N} \tag{5}$$
for $(j_1, \ldots, j_N), (k_1, \ldots, k_N) \in \Lambda^N$ such that $(j_1, \ldots, j_N) \sim (k_1, \ldots, k_N)$. For $\xi \in \Lambda^N/_\sim$ let
$$\mathbf{e}_\xi = e_{j_1} \odot \cdots \odot e_{j_N}, \tag{6}$$
where $(j_1, \ldots, j_N)$ is an arbitrary element of ξ. By (5), $\mathbf{e}_\xi$ does not depend on the representative $(j_1, \ldots, j_N)$ of the equivalence class ξ. It is known that the set $\{\mathbf{e}_\xi : \ \xi \in \Lambda^N/_\sim\}$ is an orthogonal basis in $\mathcal{E}^{\odot N}$.

In some cases, the state of an N-particle system can be described by some proper subspace of $\mathcal{E}^{\odot N}$. Suppose such the subspace for the N-particle system of bosons has the form
$$\mathcal{E}_{N,M} = \overline{\operatorname{span}} \{\mathbf{e}_\xi : \ \xi \in M\},$$
where M is some nonempty subset of $\Lambda^N/_\sim$. Let us establish some symmetry properties of $\mathcal{E}_{N,M}$. For $s \in \mathcal{S}(\Lambda)$ let $\widetilde{s} : \Lambda^N \to \Lambda^N$ be defined by
$$\widetilde{s}((j_1, \ldots, j_N)) = (s(j_1), \ldots, s(j_N)), \tag{7}$$
where $(j_1, \ldots, j_N) \in \Lambda^N$.

Lemma 2. *For every $s \in \mathcal{S}(\Lambda)$ the mapping $\widetilde{s}$ defined by (7) is a bijection.*

Proof. Let us show that $\widetilde{s}$ is an injection. Let $(j_1, \ldots, j_N), (k_1, \ldots, k_N) \in \Lambda^N$ be such that $(j_1, \ldots, j_N) \neq (k_1, \ldots, k_N)$. Then there exists $m \in \{1, \ldots, N\}$ such that $j_m \neq k_m$. Therefore, taking into account that s is injective, we get $s(j_m) \neq s(k_m)$. Consequently, we obtain $\widetilde{s}((j_1, \ldots, j_N)) \neq \widetilde{s}((k_1, \ldots, k_N))$. So, $\widetilde{s}$ is injective.

Let us show that $\widetilde{s}$ is a surjection. Let $(l_1, \ldots, l_N) \in \Lambda^N$. Since s is a surjection, for every $m \in \{1, \ldots, N\}$ there exists $j_m \in \Lambda$ such that $s(j_m) = l_m$. Therefore $\widetilde{s}((j_1, \ldots, j_N)) = (l_1, \ldots, l_N)$. So, $\widetilde{s}$ is surjective. □

Note that
$$\widetilde{s} \circ \bar{\sigma} = \bar{\sigma} \circ \widetilde{s} \tag{8}$$
for every $s \in \mathcal{S}(\Lambda)$ and $\sigma \in \mathcal{S}_N$, where $\bar{\sigma}$ and $\widetilde{s}$ are defined by (2) and (7), respectively.

Lemma 3. *Let $(j_1, \ldots, j_N), (k_1, \ldots, k_N) \in \Lambda^N$ be such that $(j_1, \ldots, j_N) \sim (k_1, \ldots, k_N)$. Then $\widetilde{s}((j_1, \ldots, j_N)) = \widetilde{s}((k_1, \ldots, k_N))$, where $\widetilde{s}$ is defined by (7).*

Proof. Since $(j_1, \ldots, j_N) \sim (k_1, \ldots, k_N)$, there is $\sigma \in \mathcal{S}_N$ such that $(k_1, \ldots, k_N) = \bar{\sigma}((j_1, \ldots, j_N))$. Therefore
$$\widetilde{s}\left((k_1, \ldots, k_N)\right) = \widetilde{s}\left(\bar{\sigma}((j_1, \ldots, j_N))\right). \tag{9}$$
By (8), we get
$$\widetilde{s}\left(\bar{\sigma}((j_1, \ldots, j_N))\right) = \bar{\sigma}\left(\widetilde{s}((j_1, \ldots, j_N))\right). \tag{10}$$
By (9) and (10), we obtain
$$\widetilde{s}\left((k_1, \ldots, k_N)\right) = \bar{\sigma}\left(\widetilde{s}((j_1, \ldots, j_N))\right).$$
Therefore, by (3), we have $\widetilde{s}\left((k_1, \ldots, k_N)\right) \sim \widetilde{s}((j_1, \ldots, j_N))$. This completes the proof. □

For $s \in \mathcal{S}(\Lambda)$, by Lemma 3, taking into account that, by Lemma 2, $\widetilde{s}$ is a bijection, the set $\widetilde{s}(\xi)$ belongs to $\Lambda^N/_\sim$ for every $\xi \in \Lambda^N/_\sim$. So, the mapping

$$\widehat{s} : \xi \in \Lambda^N/_\sim \mapsto \widetilde{s}(\xi) \in \Lambda^N/_\sim \tag{11}$$

is well defined. Let

$$\mathcal{S}_M(\Lambda) = \{s \in \mathcal{S}(\Lambda) : \; \widehat{s}(\xi) \in M \text{ for every } \xi \in M\}. \tag{12}$$

Lemma 4. *The set $\mathcal{S}_M(\Lambda)$ defined by* (12) *is a subgroup of $\mathcal{S}(\Lambda)$.*

Proof. Let $s, t \in \mathcal{S}_M(\Lambda)$. Let us show that $s \circ t \in \mathcal{S}_M(\Lambda)$, i.e. $\widehat{s \circ t}(\xi) \in M$ for every $\xi \in M$. Let $\xi \in M$. By (11), we obtain

$$\widehat{s \circ t}(\xi) = \widetilde{s \circ t}(\xi). \tag{13}$$

Note that

$$\widetilde{s \circ t}(\xi) = \left\{\widetilde{s \circ t}((j_1, \ldots, j_N)) : \; (j_1, \ldots, j_N) \in \xi\right\}. \tag{14}$$

For every $(j_1, \ldots, j_N) \in \xi$, by (7), we get

$$\begin{aligned} \widetilde{s \circ t}((j_1, \ldots, j_N)) &= ((s \circ t)(j_1), \ldots, (s \circ t)(j_N)) = (s(t(j_1)), \ldots, s(t(j_N))) \\ &= \widetilde{s}\left((t(j_1), \ldots, t(j_N))\right) = \left(\widetilde{s} \circ \widetilde{t}\right)((j_1, \ldots, j_N)). \end{aligned} \tag{15}$$

By (14) and (15), we obtain

$$\widetilde{s \circ t}(\xi) = \left\{\left(\widetilde{s} \circ \widetilde{t}\right)((j_1, \ldots, j_N)) : \; (j_1, \ldots, j_N) \in \xi\right\}. \tag{16}$$

Note that

$$\left\{\left(\widetilde{s} \circ \widetilde{t}\right)((j_1, \ldots, j_N)) : \; (j_1, \ldots, j_N) \in \xi\right\} = \left(\widetilde{s} \circ \widetilde{t}\right)(\xi). \tag{17}$$

By (13), (16), and (17), we have

$$\widehat{s \circ t}(\xi) = \left(\widetilde{s} \circ \widetilde{t}\right)(\xi). \tag{18}$$

Since $t \in \mathcal{S}_M(\Lambda)$, it follows that $\widehat{t}(\xi) \in M$. Therefore, taking into account (11), $\widetilde{t}(\xi) \in M$. Consequently, since $s \in \mathcal{S}_M(\Lambda)$, it follows that $\widehat{s}(\widetilde{t}(\xi)) \in M$. Therefore, taking into account (11), $\widetilde{s}(\widetilde{t}(\xi)) \in M$, i.e. $\left(\widetilde{s} \circ \widetilde{t}\right)(\xi) \in M$. Consequently, by (18), $\widehat{s \circ t}(\xi) \in M$. So, $s \circ t \in \mathcal{S}_M(\Lambda)$. This completes the proof. □

Since we have the N-particle system of identical noninteracting bosons, the Hamiltonian of the system has the form

$$H^{\oplus N} = \overline{H}_1 + \ldots + \overline{H}_N, \tag{19}$$

where

$$\overline{H}_k = \underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes H \otimes \underbrace{1 \otimes \cdots \otimes 1}_{N-k} \tag{20}$$

for $k \in \{1, \ldots, N\}$. Let us obtain the canonical partition function of the system:

$$Z^{(b)}_{N,M}(\beta) = \operatorname{tr} \exp\left(-\beta H^{\oplus N}\right), \tag{21}$$

where $\beta > 0$. Let $\xi \in M$ and $(j_1, \ldots, j_N)$ be an arbitrary element of ξ. By (6), we have

$$H^{\oplus N}(\mathbf{e}_\xi) = H^{\oplus N}\left(e_{j_1} \odot \cdots \odot e_{j_N}\right). \tag{22}$$

By (4), we obtain

$$H^{\oplus N}\left(e_{j_1}\odot\cdots\odot e_{j_N}\right)=H^{\oplus N}\left(\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right). \tag{23}$$

By the linearity of $H^{\oplus N}$, we get

$$H^{\oplus N}\left(\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right)=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}H^{\oplus N}\left(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right). \tag{24}$$

Using (19), we obtain

$$\begin{aligned}\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}H^{\oplus N}\left(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right)&=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\left(\overline{H}_1+\ldots+\overline{H}_N\right)\left(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right)\\&=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}\overline{H}_k\left(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right).\end{aligned} \tag{25}$$

From (20) it follows

$$\begin{aligned}&\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}\overline{H}_k\left(e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\right)\\&\qquad=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(k-1)}}\otimes H\left(e_{j_{\sigma(k)}}\right)\otimes e_{j_{\sigma(k+1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}.\end{aligned} \tag{26}$$

By (1), we get

$$\begin{aligned}&\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(k-1)}}\otimes H\left(e_{j_{\sigma(k)}}\right)\otimes e_{j_{\sigma(k+1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\\&\qquad=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\sum_{k=1}^{N}\lambda_{j_{\sigma(k)}}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\left(\sum_{k=1}^{N}\lambda_{j_{\sigma(k)}}\right)e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}.\end{aligned} \tag{27}$$

For every $\sigma\in\mathcal{S}_N$, taking into account the definition of $\mathcal{S}_N$, we have

$$\sum_{k=1}^{N}\lambda_{j_{\sigma(k)}}=\sum_{k=1}^{N}\lambda_{j_k}.$$

Therefore, taking into account (4) and (6), we obtain

$$\begin{aligned}\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\left(\sum_{k=1}^{N}\lambda_{j_{\sigma(k)}}\right)e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}&=\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}\left(\sum_{k=1}^{N}\lambda_{j_k}\right)e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\\&=\left(\sum_{k=1}^{N}\lambda_{j_k}\right)\frac{1}{N!}\sum_{\sigma\in\mathcal{S}_N}e_{j_{\sigma(1)}}\otimes\cdots\otimes e_{j_{\sigma(N)}}\\&=\left(\sum_{k=1}^{N}\lambda_{j_k}\right)e_{j_1}\odot\cdots\odot e_{j_N}\\&=\left(\sum_{k=1}^{N}\lambda_{j_k}\right)\mathbf{e}_\xi.\end{aligned} \tag{28}$$

By (22) – (28), we get

$$H^{\oplus N}(\mathbf{e}_\xi) = \left(\sum_{k=1}^{N} \lambda_{j_k}\right)\mathbf{e}_\xi.$$

So, $\mathbf{e}_\xi$ is an eigenvector of $H^{\oplus N}$ and $\sum_{k=1}^{N} \lambda_{j_k}$ is the corresponding eigenvalue. Consequently, for every $\xi \in M$, the element $\mathbf{e}_\xi$ is an eigenvector of the operator $\exp\left(-\beta H^{\oplus N}\right)$ and $\exp\left(-\beta \sum_{k=1}^{N} \lambda_{j_k}\right)$ is the corresponding eigenvalue, where $(j_1, \ldots, j_N)$ is an arbitrary element of ξ. Note that $\sum_{k=1}^{N} \lambda_{j_k}$ does not depend on the choice of $(j_1, \ldots, j_N) \in \xi$. But to be specific, we will work with the representative $\left(j_1^{(\xi)}, \ldots, j_N^{(\xi)}\right) \in \xi$ such that $j_1^{(\xi)} \le j_2^{(\xi)} \le \ldots \le j_N^{(\xi)}$. Thus,

$$\operatorname{tr}\exp\left(-\beta H^{\oplus N}\right) = \sum_{\xi \in M} \exp\left(-\beta \sum_{k=1}^{N} \lambda_{j_k^{(\xi)}}\right). \tag{29}$$

Evidently,

$$\exp\left(-\beta \sum_{k=1}^{N} \lambda_{j_k^{(\xi)}}\right) = \prod_{k=1}^{N} \exp\left(-\beta \lambda_{j_k^{(\xi)}}\right). \tag{30}$$

Let $x_j = \exp\left(-\beta\lambda_j\right)$ for every $j \in \Lambda$. Then

$$\prod_{k=1}^{N} \exp\left(-\beta\lambda_{j_k^{(\xi)}}\right) = \prod_{k=1}^{N} x_{j_k^{(\xi)}}. \tag{31}$$

By (21), (29), (30), and (31), we get

$$Z_{N,M}^{(b)}(\beta) = \sum_{\xi \in M} \prod_{k=1}^{N} x_{j_k^{(\xi)}}.$$

Lemma 5. *Let $s \in \mathcal{S}_M(\Lambda)$. Then*

$$\sum_{\xi \in M} \prod_{k=1}^{N} x_{s\left(j_k^{(\xi)}\right)} = \sum_{\xi \in M} \prod_{k=1}^{N} x_{j_k^{(\xi)}}. \tag{32}$$

Proof. Since $s \in \mathcal{S}_M(\Lambda)$, it follows that $\widehat{s}$ defined by (11) is a bijection on M. Consequently,

$$\sum_{\xi \in M} \prod_{k=1}^{N} x_{j_k^{(\xi)}} = \sum_{\xi \in M} \prod_{k=1}^{N} x_{j_k^{(\widehat{s}(\xi))}}. \tag{33}$$

Since $\left(j_1^{(\widehat{s}(\xi))}, \ldots, j_N^{(\widehat{s}(\xi))}\right) \in \widehat{s}(\xi)$, by (11), $\left(j_1^{(\widehat{s}(\xi))}, \ldots, j_N^{(\widehat{s}(\xi))}\right) \in \widetilde{s}(\xi)$. Therefore there exists $(l_1, \ldots, l_N) \in \xi$ such that $\widetilde{s}((l_1, \ldots, l_N)) = \left(j_1^{(\widehat{s}(\xi))}, \ldots, j_N^{(\widehat{s}(\xi))}\right)$, i.e. taking into account (7), we get

$$(s(l_1), \ldots, s(l_N)) = \left(j_1^{(\widehat{s}(\xi))}, \ldots, j_N^{(\widehat{s}(\xi))}\right).$$

Consequently,

$$\prod_{k=1}^{N} x_{j_k^{(\widehat{s}(\xi))}} = \prod_{k=1}^{N} x_{s(l_k)}. \tag{34}$$

Since $(l_1, \ldots, l_N) \in \xi$, it follows that $(l_1, \ldots, l_N) \sim \left(j_1^{(\xi)}, \ldots, j_N^{(\xi)}\right)$, i.e. $\left(j_1^{(\xi)}, \ldots, j_N^{(\xi)}\right)$ is a permutation of $(l_1, \ldots, l_N)$. Therefore

$$\prod_{k=1}^{N} x_{s(l_k)} = \prod_{k=1}^{N} x_{s\left(j_k^{(\xi)}\right)}. \tag{35}$$

The equalities (33), (34), and (35) imply the equality (32). This completes the proof. $\square$

Lemma 5 implies the following theorem.

Theorem 1. *For every subgroup S of the group $\mathcal{S}_M(\Lambda)$, the partition function $Z^{(b)}_{N,M}(\beta)$ is an S-symmetric polynomial with respect to variables x_j.*

Let us consider some cases in which the partition function is a so-called block-symmetric polynomial. First, let us define the respective group of symmetry. Let $m \in \mathbb{N}$. For $\tau \in \mathcal{S}(\mathbb{N})$, let $b_\tau : \mathbb{N} \to \mathbb{N}$ be defined by

$$b_\tau(k) = m\,(\tau(j) - 1) + s, \tag{36}$$

where $k \in \mathbb{N}$ and $(j, s) \in \mathbb{N} \times \{1, \ldots, m\}$ is the unique pair such that $k = (j-1)m + s$. In other words, b_τ permutes "blocks" $(1, 2, \ldots, m), (m+1, m+2, \ldots, 2m), \ldots$.

Let $B_m(\mathbb{N}) = \{b_\tau : \ \tau \in \mathcal{S}(\mathbb{N})\}$, where b_τ is defined by (36).

A polynomial $P(x_1, x_2, \ldots)$ is called block-symmetric if there exists $m \in \mathbb{N}$ such that

$$P\big(x_{s(1)}, x_{s(2)}, \ldots\big) = P(x_1, x_2, \ldots)$$

for every $s \in B_m(\mathbb{N})$. By Theorem 1, in the case $\Lambda = \mathbb{N}$ the partition function $Z^{(b)}_{N,M}(\beta)$ is block-symmetric if there exists $m \in \mathbb{N}$ such that $B_m(\mathbb{N})$ is a subgroup of the group $\mathcal{S}_M(\Lambda)$. Let us consider some example.

Example 1. *Consider the system of two noninteracting bosons such that the Hilbert space $\mathcal{E}$, that describes the states of the one-particle system, is infinite-dimensional and separable. So, we have $N = 2$ and $\Lambda = \mathbb{N}$. Let $\{e_n : \ n \in \mathbb{N}\}$ be the orthonormal basis in $\mathcal{E}$ consisting of eigenvectors of the Hamiltonian of the one-particle system. Then the set*

$$\{e_{j_1} \odot e_{j_2} : \ j_1, j_2 \in \mathbb{N} \text{ such that } j_1 \le j_2\}$$

is an orthogonal basis in the symmetric tensor product $\mathcal{E}^{\odot 2}$ that describes the two-particle system. Suppose that the states of the system can take values only from the subspace

$$\mathcal{E}_{2,M} = \overline{\operatorname{span}}\{e_1 \odot e_1, e_1 \odot e_2, e_2 \odot e_2, e_3 \odot e_3, e_3 \odot e_4, e_4 \odot e_4, \ldots\}$$

In other words, if one of the bosons is in a state $j_1 \in \{2k-1, 2k\}$ for some $k \in \mathbb{N}$, then the other is in a state j_2 that also belongs to $\{2k-1, 2k\}$. Then the partition function has the form

$$Z^{(b)}_{N,M}(\beta) = \sum_{k=1}^{\infty} x_k^2 + \sum_{j=1}^{\infty} x_{2j-1} x_{2j}.$$

This function is $B_2(\mathbb{N})$-symmetric.

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Шарин С.В., Василишин Т.В. *Алгебри симетричних функцій на банахових просторах та їх застосування в квантовій фізиці* // Карпатські матем. публ. — 2026. — Т.18, №1. — С. 250–257.

Ми досліджуємо зв'язки між групою симетрій підпростору симетричного тензорного добутку сепарабельного гільбертового простору та представленням функції статистичної суми квантово заплутаної сім'ї частинок, пов'язаної з цим підпростором. Розглянемо систему, яка складається із N ідентичних бозонів, які не взаємодіють між собою. В загальному випадку, система описується N-тим симетричним тензорним степенем $\mathcal{E}^{\odot N}$, де $\mathcal{E}$ — гільбертів простір, який відповідає одночастинковій системі. В деяких випадках систему можна описати за допомогою деякого підпростору простору $\mathcal{E}^{\odot N}$. В роботі розглянуто випадок, у якому такий підпростір є замиканням лінійної оболонки деякої підмножини базису, складеного із власних векторів гамільтоніана системи. Описано групу симетрій S таку, що статистична сума системи є S-симетричною функцією.

Ключові слова і фрази: симетричний поліном на банаховому просторі, симетричний тензорний добуток, квантовий стан, заплутаність, статистична сума.