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LINEAR SUBSPACES IN ZEROS OF POLYNOMIALS ON BANACH SPACES

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Abstract. A survey of general results about linear subspaces in zeros of polynomials on real and complex Banach spaces.

Keywords: polynomials, linear subspaces, zeros of polynomials on Banach spaces.

1. Introduction

The paper is a survey of results related to linear subspaces in zero-sets (kernels) of real and complex polynomials on Banach spaces.

The study of the zeros of polynomials has a long history, which began with the results obtained in algebraic geometry and complex analysis. Zeros of polynomials on infinite demensional Banach spaces was studied in [2], [3], [4], [5], [7], [12], [13], [18] by R. Aron, B. Cole, T. Gamelin, D. Garsia, M. Maestre, A. Zagorodnyuk, A. Plichko, R. Gonzalo, J. Ferrer, P. Hajek and others.

Let X and Y be real or complex Banach vector spaces. For every positive integer numbers $n, m \in \mathbb{N}$ let X^nY^m will denote the Cartesian product of n copies of X and m copies of Y, and x^ny^m will denote the element $(x, \ldots, x, y, \ldots, y)$ from X^nY^m.

For $n \in \mathbb{N}$ we denote by $\mathcal{L}(^nX, Y)$ the vector space of all continuous n-linear mappings F from X to Y endowed with the norm of uniform convergence on the unit ball of X^n. An n-linear mapping F is called symmetric if

$$F(x_1, \ldots, x_n) = F\left(x_{s(1)}, \ldots, x_{s(n)}\right), \qquad s \in \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ means all permutations

$$s\colon \{1, \ldots, n\} \longmapsto \{s(1), \ldots, s(n)\}.$$

The subspace in $\mathcal{L}(^nX, Y)$ of all continuous symmetric n-linear maps will be denoted by $\mathcal{L}_s(^nX, Y)$. Clearly, $\mathcal{L}(^nX, Y)$ and $\mathcal{L}_s(^nX, Y)$ are Banach spaces. Further in the previous notations we will not write the index $n = 1$. In particular, $\mathcal{L}(X)$ denotes the algebra of all continuous linear operators and $\mathcal{L}(X, \mathbb{C}) := X'$ denotes the dual space of X.

Definition 1. Let us denote by Δ_n the natural embeddings called *diagonal mappings* from X to X^n defined as

$$\begin{aligned} \Delta_n \colon X &\longrightarrow X^n \\ x &\longmapsto (x, \ldots, x). \end{aligned}$$

A mapping P from X to Y is called a continuous n-homogeneous polynomial if

$$P(x) = (F \circ \Delta_n)(x) \quad \text{for some} \quad F \in \mathcal{L}(^nX, Y). \tag{1.1}$$

Let $\mathcal{P}(^nX, Y)$ denote the vector space of all continuous n-homogeneous polynomials endowed with the norm of uniform convergence on the unit ball B of X, i.e.,

$$\|P\| = \sup_{x \in B} \|P(x)\|$$

with $P \in \mathcal{P}(^nX, Y)$.

In the paper we consider cases $Y = \mathbb{R}$ or $Y = \mathbb{C}$ the fields of real or complex numbers. We use notation $\mathcal{P}(X)$ and $\mathcal{P}(^nX)$ for the space of scalar valued polynomials and n-homogeneous scalar valued polynomials respectively.

Let us denote by $\check{P}$ the unique symmetric n-linear map F which satisfies 1.1 for a given $P \in \mathcal{P}(^nX)$.

For detales on polynomials on Banach spaces we refer the reader to [9], [10], [17].

2. Linear Subspaces in Zeros of Complex Polynomials

If X is an arbitrary complex vector space (not necessarily normed), we define a n-homogeneous complex polynomial by the formula

$$P(x) = (F \circ \Delta_n)(x) \qquad x \in X,$$

where F is a complex n-linear (not necessarily continuous) functional on X.

It is clear that the kernel (i.e. the set of zeros) of an n-homogeneous complex polynomial P on X, where $n > 0$ and $\dim X > 1$, consists of one-dimensional subspaces. In [18] A. Plichko and A. Zagorodnyuk showed that one-dimensional subspaces consists of infinite-dimensional subspaces if $\dim X = \infty$.

Theorem 2.1. *Let X be an infinite-dimensional complex vector space and P is a complex n-homogeneous polynomial on X. Then there exists an infinite-dimensional subspace X_0 such that*

$$X_0 \subset \ker P.$$

Lemma 1. *Let Theorem 2.1 be proved for every homogeneous polynomial of degree $\leq n$. Then for arbitrary homogeneous polynomials $P_1, \cdots, P_m$ of degree $\leq n$ there exists a subspace*

$$X_0 \subset \ker P_1 \cap \ldots \cap \ker P_m$$

such that $\dim X_0 = \infty$.

Proof. Let $X_1 \subset \ker P_1$ with $\dim X_1 = \infty$. Then there exists a subspace $X_2 \subset X_1 \cap \ker P_2$ such that $\dim X_2 = \infty$. Continuing this process, we get the subspace

$$X_0 = X_m \subset X_{m-1} \subset \cdots \subset X_1$$

with $X_0 \subset \ker P_1 \cap \cdots \cap \ker P_m$ and $\dim X_0 = \infty$. □

Proof of Theorem 2.1. We will construct X_0 using the induction on n. Evidently that the theorem is true for linear functionals. Suppose that it is true for homogeneous polynomials of degree $< n$.

Let $x_1 \in X$ is chosen such that $P(x_1) \neq 0$ (if such x_1 does not exist then the assertion of theorems is true automatically). By the induction hypothesis and by Lemma 1 there exists a subspace $X_1 \subset X$ with $\dim X_1 = \infty$, on which each of the homogeneous polynomials

$$\begin{aligned} &P_{x_1}(x) := \check{P}\left(x_1, x^{n-1}\right), \\ &P_{x_1^2}(x) := \check{P}\left(x_1^2, x^{n-2}\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &P_{x_1^{n-1}}(x) := \check{P}\left(x_1^{n-1}, x\right) \end{aligned}$$

vanish for all $x \in X_1$, where $\check{P}$ is the symmetric n-linear functional associated with the n-homogeneous polynomial P.

On second step we choose an element $x_2 \in X_1$ such that $P(x_2) \neq 0$ (if x_2 does not exist then $X_1 \subset \ker P$ and the theorem is proved at once). By the induction hypothesis and by Lemma 1 there exists a subspace $X_2 \subset X_1$ with $\dim X_2 = \infty$ on which each homogeneous polynomials

$$P_{x_1^k, x_2^l}(x) := \check{P}\left(x_1^k, x_2^l, x^{n-k-l}\right), \qquad 0 < k + l < n$$

vanish for all $x \in X_2$.

We continue this process in the way written above. If it finishes on the i-th step (i.e. $P(X_i) \equiv 0$), then the theorem is proved. If it does not finish then we will get an infinite sequence (x_i) consisting of linearly independent terms such that $P(x_i) \neq 0$ for every $i \in \mathbb{N}$ and

$$\check{P}\left(x_1^{k_1}, x_2^{k_2}, \ldots, x_i^{k_i}\right) = 0$$

if $0 < k_i < n$ at least for one k_i.

Consequently, it follows that for any finite sequence of scalars (a_i),

$$P\left(\sum_i a_i x_i\right) = \sum_i a_i^n P(x_i).$$

Put $y_i = x_i / P(x_i)$ for all $i \in \mathbb{N}$. Then P vanishes on the linear span of elements

$$y_1 + \sqrt[n]{-1} y_2, \;\; y_3 + \sqrt[n]{-1} y_4, \;\; y_5 + \sqrt[n]{-1} y_6, \;\; \ldots .$$

The theorem is proved. □

In [18] it was proved the following:

Corollary 1. *For every polynomial functional P on a complex infinite dimensional vector space, for which $P(0) = 0$, there exists an infinite dimensional linear subspace X_0 such that $X_0 \subset \ker P$.*

Corollary 2. *If P is a polynomial functional on a complex infinite dimensional vector space and $P(x_0) = 0$, then there exists an infinite dimensional affine subspace $X_0 \subset \ker P$ with $x_0 \in X_0$.*

In [19] it was proved next corollary:

Corollary 3. *There is a function $\Phi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, $\Phi(m, d) = n$ with the following property.*

For every complex polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree d, there is a subspace $X \subset \mathbb{C}^n$ dimension m such that $P|_X \equiv P(0)$.

The real analogue of this result is obviously false, as can be seen by considering $P(x) = \sum x_j^2$. Despite this, a number of positive results hold. For example, one can show:

Theorem 2.2. *There is a function $\theta : \mathbb{N}_0 \to \mathbb{N}_0, \theta(m) = n$ with the following property:*

For every real polynomial $P : \mathbb{R}^n \to \mathbb{R}$ which is homogeneous of degree 3, there is a subspace $X \subset \mathbb{R}^n$ of dimension m such that $P|_X \equiv 0$.

Theorem 2.3. *If a real infinite dimensional Banach space E does not admit a 2-homogeneous positive definite polynomial, then every 2-homogeneous polynomial $P : E \to \mathbb{R}$ is identically 0 on an infinite dimensional subspace of E.*

3. Nonseparable Zero Subspaces

3.1. Nonseparable Subspaces in $\ker P \subset l_\infty$

All results of this subsection was proved in [11] by M. Fernandez-Unzueta. In particular in [11] was proved that every complex polynomial P defined on l_∞ such that $P(0) = 0$ necessarily vanishes on a non-separable subspace. In the real case, it was shown that if P vanishes on a copy of c_0, then it vanishes as well on a non-separable subspace.

Theorem 3.1. *Consider the Banach space l_∞ (real or complex) and a subspace $G \subset l_\infty$ isomorphic to c_0. Then, there exists a non-countable collection of vectors $(x_\alpha)_{\alpha\in A} \subset l_\infty$ satisfying the following condition: For every $\{P_i\}_{i=1}^\infty \subset \mathcal{P}(l_\infty)$ such that $G \subset \bigcap_{i=n}^\infty \ker P_i$, there exists a subset of indices $\Gamma \subset A$ with $A \setminus \Gamma$ at most countable such that the subspace*

$$F_\Gamma := \overline{Span\{x_\alpha; \alpha \in \Gamma\}}$$

is non-separable and contained in $\bigcap_{i=1}^\infty \ker P_i$.

Lemma 2. *It is enough to prove Theorem 3.1 for the case where the collection of polynomials $\{P_i\} \subset \mathcal{P}(^n l_\infty)$ reduces to a single homogeneous polynomial $P \in \mathcal{P}(^n l_\infty)$.*

Proof of Theorem 3.1. By Lemma 2 it is enough to consider the case of a single homogeneous polynomial P. The proof will be done by induction on n, the degree of the polynomial. At each inductive step n we will, however, assume that the result holds for a countable family of polynomials of degree strictly less than n. The case $n = 1$ asserts that for a fixed linear functional $x^* \in l_\infty$ such that $x^*|_{c_0} = 0$, there exists $\Gamma \subset A$, a subset of indices with countable complement in A, such that $x^*(x_\gamma) = 0$ if $\gamma \in \Gamma$.

We assume now that Theorem 3.1 holds for polynomials of degree $k < n$.

Let $P \in \mathcal{P}(^n l_\infty)$ be such that $c_0 \subset \ker P$ and let $(e_i)_i$ be the canonical basis of c_0. Consider the countable family of polynomials $P_{i_1,\ldots,i_k} \subset \mathcal{P}(^{n-k} l_\infty)$ defined as follows:

$$P_{i_1,\ldots,i_k}(x) := \check{P}(e_{i_k}, \ldots, e_{i_k}, x, \overset{n-k}{\ldots}, x) \text{ for } x \in l_\infty, \quad 1 \le k < n \text{ and } i_j \in \mathbb{N}. \tag{3.1}$$

Every polynomial in this countable collection has degree strictly less than n and satisfies $c_0 \subset \ker P_{i_1,\ldots i_k}$. The induction hypothesis allows us to choose some set of indices

$$\Gamma_1 \subset A \tag{3.2}$$

in such a way that $A \supset \Gamma_1$ is countable and $F_{\Gamma_1} = \overline{\text{Span}\{x_\gamma; \gamma \in \Gamma_1\}}$ is a non-separable subspace contained in $\bigcap \ker P_{i_1,\ldots i_k}$. A main step in the proof is the fact that the following set:

$$S = \{\gamma \in \Gamma_1;\ \text{there are } \alpha_2, \ldots, \alpha_n \in \Gamma_1 \text{ with } |\check{P}(x_\gamma \otimes x_{\alpha_2} \otimes \ldots \otimes x_{\alpha_n})| \neq 0\}$$

is countable, accoding to Lemma 4 (see below).

Assuming this, to finish the proof of Theorem 3.1, consider the non-countable set of indices obtained by removing from the set Γ_1 defined in (3.2), every index appearing in the countable set $S : \Gamma_2 = \Gamma_1 \backslash S$. Since S is countable, $A \backslash \Gamma_2$ is a countable set. Besides, whenever $\gamma_k \in \Gamma_2$ for $1 \leq k \leq n$, we have that $\check{P}(x_{\gamma_1} \otimes \ldots \otimes x_{\gamma_n})$. In particular if $\gamma \in \Gamma_2$ then $x_\gamma \in \ker P$. Let us finally check that not only these elements, but the subspace generated by them, F_{Γ_2}, satisfies $F_{\Gamma_2} \subset \ker P$. Let $\gamma_1, \ldots, \gamma_k \in \Gamma_2$ and $\lambda_1, \ldots \lambda_k \in \mathbb{C}$. We obtain the result for the elements in the linear span of $\{x_\gamma, \gamma \in \Gamma_2\}$ from the following computation:

$$P(\lambda_1 x_{\gamma_1} + \ldots + \lambda_k x_{\gamma_k}) = \sum_{\substack{i_1 = 1, \ldots, k \\ \ldots \\ i_n = 1, \ldots, k}} \lambda_{i_1} \ldots \lambda_{i_n} \check{P}(x_{\gamma_{i_1}} \otimes \ldots \otimes x_{\gamma_{i_n}}) = 0.$$

The result for the closure F_{Γ_2} of the linear span is obtained just from the continuity of P.

□

Lemma 3. *Let $k, n \in \mathbb{N}$, and let $\{e_i, i \in \mathbb{N}\}$ be the set of coordinate vectors in l_∞. For any indices $i_m^j \in \mathbb{N}, j = 1, \ldots, k, \quad m = 2, \ldots, n$, the set*

$$\{e_{i_j^1} \otimes e_{i_j^2} \otimes \ldots \otimes e_{i_j^n};\ 1 \leq j \leq k\} \tag{3.3}$$

defines a basis isometrically equivalent to the canonical basis of l_∞^k in $l_\infty \hat{\otimes}_\pi \overset{n}{\ldots} \hat{\otimes}_\pi l_\infty$.

Lemma 4. *Consider Γ_1 the subset of A defined in* (3.2). *Then, the set*

$$S = \{\Gamma_1;\ \textit{there are } \alpha_2, \ldots, \alpha_n \in \Gamma_1 \textit{ with } \check{P}(x_\gamma \otimes x_{\alpha_2} \otimes \ldots \otimes x_{\alpha_n}) \neq 0\}$$

is at most countable.

Corollary 4. *If $F \subset l_\infty$ is separable and $c_0 \subset F$, then F is not the intersection of any denumerable family of sets of zeroes of scalar-valued polynomials.*

The following theorem is an important consequence of Theorem 3.1. It asserts the existence of non-separable subspaces in the set of zeroes of every polynomial P on the complex l_∞ space, such that $P(0) = 0$.

Theorem 3.2. *Let E be a complex Banach space containing l_∞ and $P \in \mathcal{P}(E)$ be such that $P(0) = 0$. Then, there exists a non-separable subspace $F \subset \ker P$.*

For a fixed Banach space $E, \quad n \in \mathbb{N}$, and a polynomial $P \in \mathcal{P}(^n E)$ with $P(0) = 0$ we say that the subspace $F \subset E$ is *maximal among the subspaces contained* in $\ker P$ (or just maximal if the context is clear) if $F \subset \ker P$ and whenever $G \subset E$ is any subspace satisfying $F \subset G \subset \ker P$, then necessarily $F = G$.

The following proposition describes an arbitrary maximal subspace for a homogeneous polynomial P as the intersection of the sets of zeroes of a finite number of polynomials (generally non-scalar). This result is particularly interesting for our purposes if the maximal subspace is separable.

Proposition 1. *Let E be a Banach space (real or complex), $n \in \mathbb{N}$, $P \in \mathcal{P}(^nE)$ and $F \subset E$ a subspace such that $F \subset \ker P$. Then F is maximal among the subspaces contained in $\ker P$ if and only if*

$$F = \bigcap_{k=1}^{n} \ker Q_k, \tag{3.4}$$

where $Q_k \in \mathcal{P}(^kE, \mathcal{P}(^{n-k}F))$ is defined for every $x \in E$ and every $y \in F$ as

$$Q_k(x)(y) = \hat{P}(x, \overset{k}{\dots}, x, y, \overset{n-k}{\dots}, y),$$

where $1 \le k \le n$.

Proof. Assume first that F is a maximal subspace, and consider any $x \in F$. By hypothesis $Q_n(x) = P(x) = 0$. For $1 \le k \le n-1$ we have $Q_k(x) = 0$ if and only if for every $y \in F$, $Q_k(x)(y) = 0$. The condition $P|_F = 0$ is equivalent to $\check{P}|_{F\times \overset{n}{\dots} \times F} = 0$. Thus for every $y \in F$, $Q_k(x)(y) = \check{P}(x, \overset{k}{\dots}, x, y, \overset{n-k}{\dots}, y) = 0$. This implies that $x \in \ker Q_k$ and the contention $\subseteq$ in (3.4) is proved.

Observe that this proof does not make use of the maximality of F. However, to show the reverse inclusion this assumption is essential: Consider $x \in \bigcap_{k=1}^{n} \ker Q_k$ any scalar and $y \in F$. Then

$$\begin{aligned} P(\lambda x + y) &= \lambda^n P(x) + P(y) + \sum_{k=1}^{n-1} \lambda^k \binom{n}{k} \check{P}(x, \overset{k}{\dots}, x, y, \overset{n-k}{\dots}, y) \\ &= \lambda^n P(x) + P(y) + \sum_{k=1}^{n-1} \lambda^k \binom{n}{k} Q_k(x)(y) = 0. \end{aligned} \tag{3.5}$$

Since $F \subset [x] + F \subset \ker P$ and F is maximal, necessarily $x \in F$.

Assume now that $F \subset \ker P$ can be expressed as in (3.3). Let us prove that F is maximal. Consider $x \in \ker P$ such that $P(\lambda x + y) = 0$ for every scalar λ and every $y \in F$. Equation (3.5) still holds and says that for every fixed $y \in F$ we have a polynomial on $\lambda \in \mathbb{K}$ identically zero. Thus, every coefficient is zero. In this way it is proved that $x \in \ker Q_k$ for $k = 1, \dots, n$. From expression (3.3) we get that $x \in F$ and consequently that F is maximal. ☐

Corollary 5. *If $F \subset l_\infty$ is separable and $c_0 \subset F$, then F is not maximal for any $P \in \mathcal{P}(^nl_\infty)$.*

Observe that the description of a maximal separable subspace just given leads also to a proof of Theorem 3.2: As argued before, every complex polynomial with $P(0) = 0$ must be zero on a copy of c_0. This copy of c_0 is contained in a maximal subspace $F \subset \ker P$ which, by Corollary 5, is necessarily non-separable.

3.2. Zero Subspaces of Polynomials on $l_1(\Gamma)$

All results of this subsection was proved in [6].

The examples that we construct are defined on spaces $l_1(\Gamma)$. Over this space, all polynomials can be explicitly described. For instance, the general form of a quadratic functional $P : l_1(\Gamma) \to \mathbb{C}$ is

$$P(x) = \sum_{\alpha,\gamma\in\Gamma} \lambda_{\alpha\beta} x_\alpha x_\beta, \quad x = (x_\gamma)_{\gamma\in\Gamma} \in l_1(\Gamma),$$

where $(\lambda_{\alpha\beta})_{\alpha,\beta\in\Gamma}$ is a bounded family of complex scalars. Indeed in all our examples the coefficients $\lambda_{\alpha\beta}$ are either 0 or 1, they functionals of the form

$$P(x) = \sum_{\{\alpha,\beta\in G\}} x_\alpha x_\beta,$$

where G a certain set of couples of elements of Γ. On the other hand, we state the following elementary basic fact about polynomials that we shall explicitly use at some point.

Proposition 2. *Let $P : X \to Y$ be a homogeneous polynomial of degree n and norm K. Then*

$$||P(x) - P(y)|| \leq nKM^{n-1}||x||||y||$$

for every $x, y \in X$.

Let $\mathbf{\Omega}$ be a set and $\mathcal{A}$ be an almost disjoint family of subsets of $\mathbf{\Omega}$ (that is, $|A \cap A'| < |\mathbf{\Omega}|$ whenever $A, A' \in \mathcal{A}$ are different), and let $\mathcal{B} = \mathbf{\Omega} \cup \mathcal{A}$. We consider the following quadratic functional $P : l_1(\mathcal{B}) \to \mathbb{C}$ given by

$$P(x) = \sum\{x_n x_A : n \in \mathbf{\Omega}, \quad A \in \mathcal{A}, n \in A\}.$$

Theorem 3.3. *The space $X = l_1(\mathbf{\Omega}) \subset l_1(\mathcal{B})$ is maximal zero subspace for the polynomial P.*

Proof. The only point which requires explanation is that X is indeed maximal. So assume by contradiction that there is a vector y out of X such that $Y = \text{span}(X \cup \{y\})$ is a zero subspace for P. Without loss of generality, we suppose that y is supported in $\mathcal{A}$. Pick $A \in \mathcal{A}$ such that $|y_A| = \max\{|y_B| : B \in \mathcal{A}\}$ and $\mathcal{F} \subset \mathcal{A}$ a finite subset of $\mathcal{A}$ such that $\sum_{B\in\mathcal{A}\setminus\mathcal{F}} |y_B| < \frac{1}{9}|y_A|$. Now, because $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbf{\Omega}$, it is possible to find $n \in \mathbf{\Omega}$ such that $n \in A$ but $n \notin B$ whenever $B \in \mathcal{F} \setminus \{A\}$. Consider the element $y + 1_n \in Y$. We claim that $P(y + 1_n) \neq 0$ getting thus a contradiction

$$P(y + 1_n) = \sum_{n\in B} y_B = y_A + \sum_{n\in B, B\in\mathcal{A}\setminus\mathcal{F}} y_B.$$

The second term of the sum has modulus less than $\frac{1}{9}$ the modulus of the first term. So $P(y + 1_n) \neq 0$. □

We are interested in the case when $|\mathcal{A}| > |\mathbf{\Omega}|$. The subspace $l_1(\mathcal{A})$ is a zero subspace for P. It may not be maximal but this does not matter because, by a Zorn's lemma argument, it is contained in some maximal zero subspace. This fact together with Theorem 3.3 shows that P has maximal zero subspaces of both densities $|\mathbf{\Omega}|$ and $|\mathcal{A}|$.

There are two standard constructions of big almost disjoin families. One is by induction, and it shows that for every cardinal κ we can find an almost disjoint family of cardinality κ^+ on a set of cardinality κ. The other one is by considering the branches of the tree $\kappa^{<\omega}$, and this indicates that for every cardinal κ we can find an almost disjoint family of cardinality κ^ω(one construction or the other provides a better result depending on whether $\kappa^\omega = \kappa, \quad \kappa^\omega = \kappa^+$ or $\kappa^\omega > \kappa k^+$. Hence,

Corollary 6. *Let κ be an infinite cardinal and $\tau = \max(\kappa^+, \kappa^\omega)$. There exists a quadratic functional on $l_1(\tau)$ with a maximal zero subspace of density κ and another maximal zero subspace of density τ.*

Corollary 7. *There exists a quadratic functional on $l_1(\mathfrak{c})$ with a separable maximal zero subspace and a maximal zero subspace of density $\mathfrak{c}$.*

We denote by $[A]^2$ the set of all unordered pairs of elements of A

$$[A]^2 = \{t \subset A : |t| = 2\}.$$

We consider an ordinal α to be equal to the set of all ordinals less than α,so

$$\omega_1 = \{\alpha : \alpha < \omega_1\}$$

is the set of countable ordinals, and also for a nonnegative integer $n \in \mathbb{N}$, $n = \{0, 1, \ldots, n-1\}$.

We introduce some notations for subsets of a well ordered set Γ. If $a \subset \Gamma$ is a set of cardinality n, and $k < n$ we denote by $a(k)$ the $(k+1)$-th element of a according to the well order of Γ, so that

$$a = \{a(0), \ldots, a(n-1)\}.$$

Moreover, for $a, b \subset \Gamma$, we write $a < b$ if $\alpha < \beta$ for every $\alpha \in a$ and every $\beta \in b$.

We recall also that a Δ-system with root a is a family of sets such that the intersection of every two different elements of the family equals a. The well-known Δ-system lemma asserts that every uncountable family of finite sets has an uncountable subfamily which forms a Δ-system.

Definition 2. A function $f : [\Gamma]^2 \to 2$ is said to be a *partition of the first kind* if for every uncountable family A of disjoint subsets of Γ of some fixed finite cardinality n, and for every $k \in n$ there exist $a, b, a', b' \in A$ such that $f(a(k), b(k)) = 1$, $f(a'(k), b'(k)) = 0$ and $f(a(i), b(j)) = f(a'(i), b'(j))$ whenever $(i, j) \neq (k, k)$. Notice that, passing to a further uncountable subfamily A, we can choose such $a < b$ such that, in addition, $f(a(i), a(j)) = f(a'(i), a'(j)) = f(b(i), b(j)) = f(b'(i), b'(j))$ for all $\{i, j\} \in [n]^2$.

Theorem 3.4. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \to 2$ of the first kind.*

Theorem 3.5. *If $f : [\Gamma]^2 \to 2$ is a partition of the first king and if Y is a subspace of $l_1(\Gamma)$ with $Y \subset \ker P_f$, then Y is separable.*

We shall denote by $\Delta_n = \{(i, i) : i \in n\}$ the diagonal of the cartesian product $n \times n$, $n = \{0, 1, \ldots, n-1\}$. Also $B_X(x, r)$ or simply $B(x, r)$ will denote the ball of center x and radius r in a given Banach space X.

Definition 3. A function $f : [\Gamma]^2 \to \omega$ is said to be a *partition of the second kind* if for every uncountable family A of finite subsets of Γ all of some fixed cardinality n, we have the following two conclusions:

(a) there is an uncountable subfamily B of A and a function $h : n^2 \setminus \Delta_n \to \omega$ such that $f(a(i), b(j)) = h(i, j)$ for every $i \neq j, \quad i, j < n$ and every $a < b$ in B;

(b) for every function $h : n \to \omega$ there exists $a < b$ in A such that $f(a(i), b(i)) = h(i)$.

Theorem 3.6. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \to 2$ of the second kind as well.*

Theorem 3.7. *Suppose that $f : [\omega_1]^2 \to \omega$ is a partition of the second kind and let $P = P_f : l_1(\Gamma) \to V$ be the corresponding polynomial. Let Y be a nonseparable subspace of $l_1(\omega_1)$. Then $\overline{P(Y)}$ has nonempty interior in V.*

3.3. On the Zero-Set of Real Polynomials in Nonseparable Banach Spaces

All results of this subsection was proved in [12].

By $\mathcal{P}_f({}^nX)$ we denote the subspace of $\mathcal{P}({}^nX)$ formed by those polynomials which can be written as $P(x) = \sum_{j=1}^{m} \lambda_j \langle u_j^*, x\rangle^n$, with $\lambda_j \in \mathbb{R}, u_j^* \in X^*, 1 \le j \le m$, and they are called *finite type* polynomials. The space of approximable polynomials, $\mathcal{P}_A({}^nX)$, is given by the closure of $\mathcal{P}_f({}^nX)$ in $\mathcal{P}({}^nX)$. By $\mathcal{P}_\omega({}^nX)$ we represent the subspace of $\mathcal{P}({}^nX)$ formed by those polynomials that are weakly continuous on the bounded subsets of X. A polynomial $P \in \mathcal{P}({}^nX)$ is a *nuclear* polynomial whenever it has the form $P(x) = \sum_{j=1}^{\infty} a_j \langle u_j^*, x\rangle^n, \quad x \in X$, where $(a_j)_{j=1}^{\infty}$ and $(u_j^*)_{j=1}^{\infty}$ is a bounded sequence of X^*. Denoting by $\mathcal{P}_N({}^nX)$ the class of nuclear polynomials, it is quite clear that

$$\mathcal{P}_f({}^nX) \subset \mathcal{P}_N({}^nX) \subset \mathcal{P}_A({}^nX) \subset \mathcal{P}_\omega({}^nX) \subset \mathcal{P}({}^nX).$$

In what follows X will be an infinite-dimensional real Banach space and X^* its topological dual. We use the symbol $\langle \cdot, \cdot \rangle$ to denote the standard duality between X and X^*.

If $A \subset X$ and $B \subset X^*$, then we use the notation

$$A^\perp = \{x \in X : \langle x^*, \quad x\rangle = 0, x \in A\}, \quad B_\perp = \{x \in X : \langle x^*, x\rangle = 0, \quad x^* \in B\}.$$

For a polynomial $P \in \mathcal{P}({}^nX)$, the following conjugacy relationship between its first and $(n-1)$-th derivatives turned out to be relevant. The first derivative is the mapping $P' : X \to X^*$ such that

$$P'(x) = n\check{P}(x, \overset{(n-1)}{\dots}, x, \cdot),\ x \in X,$$

while the $(n-1)$-th derivative is given by the continuous linear map $P^{(n-1)} : X \to \mathcal{L}_s(X^{n-1})$ such that

$$P^{(n-1)}(x) = n!\check{P}(x, \cdot, \overset{(n-1)}{\dots}, \cdot), \quad x \in X,$$

where $\mathcal{L}_s(X^{n-1})$ denotes the space of symmetric continuous $(n-1)$-linear functionals on X. It is then straightforward to notice, using the Polarization formula, that

$$\ker P^{(n-1)} = P'(X)_\perp.$$

If Z is such a maximal subspace, then, for $x \in \ker P^{(n-1)}, \quad z \in Z$,

$$\begin{aligned} P(x+z) &= P(x) + P(z) + \sum_{j=1}^{n-1} \binom{n}{j} \check{P}(x, \overset{(j)}{\dots}, x, z, \overset{(n-j)}{\dots}, z) \\ &= \frac{1}{n!} P^{(n-1)}(x)(x, \overset{(n-1)}{\dots}, x) + \sum_{j=1}^{n-1} \frac{1}{j!(n-j)!} P^{(n-1)}(x)(x, \overset{(j-1)}{\dots}, x, z, \overset{(n-j)}{\dots}, z) = 0, \end{aligned}$$

i.e., $Z + \ker P^{(n-1)} \subset \ker P$, and the maximality of Z yields that $\ker P^{(n-1)}$ is contained in Z. Hence, if $\ker P^{(n-1)}$ were non-zero, we would easily obtain a non-zero linear subspace contained in $P^{-1}(0)$. Indeed, we will seek for conditions in order to guarantee that $\ker P^{(n-1)}$ is sufficiently big. For this purpose, recall that

$$(\ker P^{(n-1)})^{\perp} = (P'(X)_{\perp})^{\perp} = \overline{lin}^{\omega^*}(P'(X)),$$

and so, roughly speaking, the smaller $P'(X)$ is the bigger $\ker P^{(n-1)}$ will be. In particular, if $P'(X)$ were separable, then $(X/\ker P^{(n-1)})^* = (\ker P^{(n-1)})^{\perp}$ would have to be weak*-separable and this is mainly the reason why in the next section we shall be dealing with this type of space.

We say that a real Banach space X is in class $\mathcal{C}_H$ whenever there exists a one-to-one continuous linear map from X into a Hilbert space. When $X \in \mathcal{C}_H$ we shall say that X is injected into a Hilbert space. If X is injected into a separable Hilbert space, then we shall write $X \in \mathcal{W}^*$. Clearly, $\mathcal{W} \subset \mathcal{C}_H$. The following properties of the spaces in these two classes are quite straightforward.

Proposition 3. *The following conditions are equivalent for a space X :*

(i) $X \in \mathcal{W}^*$.

(ii) X^* is weak*-separable.

(iii) X^* has a countable total subset.

Proposition 4. *If X is in class $\mathcal{C}_H$ (respectively, in $\mathcal{W}^*$) and Y is a space that is injected linearly and continuously into X, then $Y \in \mathcal{C}_H$ (respectively, $Y \in \mathcal{W}^*$). Hence, every closed linear subspace of X is in the same class that X.*

Proposition 5. *If X is separable, then X and X^* are in $\mathcal{W}^*$.*

Proposition 6. *Let Y be a closed linear subspace of the Banach space X. If Y is in $\mathcal{W}^*$ and X/Y is in $\mathcal{C}_H$, then X is in $\mathcal{C}_H$.*

Proof. With no loss of generality, we may assume that we have two one- to-one bounded linear maps

$$S_1 : Y \to l_2, \quad S_2 : X/Y \to l_(\Gamma_0),$$

with Γ_0 being a set that is disjoint from the set of positive integers $\mathbb{N}$. Now, for each $j \in \mathbb{N}$, if e_j denotes the corresponding unit vector, we have that $S_1^* e_j \in Y^*$. Let $v_j^* \in X^*$ be the extension of $S_1^* e_j$ to X such that $||v_j^*|| = ||S_1^* e_j||$. Setting $\Gamma := \mathbb{N} \cup \Gamma_0$, we define the mapping $T : X \to l_2(\Gamma)$ such that, for $x \in X$, $\quad Tx := (\lambda_\gamma)_{\gamma \in \Gamma}$ where

$$\lambda_\gamma := \begin{cases} 2^{-\gamma}\langle v_\gamma^*, x\rangle, & \gamma \in \mathbb{N}, \\ \langle S_2(x+Y), e_\gamma\rangle, & \gamma \in \Gamma_0. \end{cases}$$

Then, T is a well defined linear map such that it is bounded. To see that it is one-to-one, let $x \in X$ be such that $Tx = 0$, then, $0 = \langle S_2(x+Y), e_\gamma\rangle, \quad \gamma \in \Gamma_0$, implies that $S_2(x+Y) = 0$, and so $x \in Y$; hence, from $0 = 2^{-j}\langle v_j^*, x\rangle, \quad j \in \mathbb{N}$, it follows that $0 = \langle S_1^* e_j, x\rangle = \langle e_j, S_1 x\rangle, \quad x \in \mathbb{N}$, therefore $S_1 x = 0$, and $x = 0$. ☐

Corollary 8. *Let Y be a closed linear subspace of X such that Y and X/Y are both in $\mathcal{W}^*$, then X is also in $\mathcal{W}^*$, i.e., being in $\mathcal{W}^*$ is a three-space property.*

We already know that, if X is separable then X and X^* are both in $\mathcal{W}^*$, let's take a look now at some other examples of spaces not belonging to $\mathcal{W}^*$, which will obviously be non-separable. Every non-separable weakly compactly generated space, and hence every non-separable reflexive one and $c_0(\Gamma)$, Γ an uncountable set, has a non-weak*-separable dual. This, plus the fact that $c_0(\Gamma)$ can be canonically injected into $l_\infty(\Gamma)$, yields that, for uncountable Γ, $c_0(\Gamma)$ and $l_\infty(\Gamma)$ are not in $\mathcal{W}^*$ and clearly $l_2(\Gamma) \in \mathcal{C}_H\mathcal{W}^*$. The easiest example of a space X such that $X \in \mathcal{W}^*$ and $X^* \notin \mathcal{W}^*$ is given by $X = l_\infty$: Being obvious that $l_\infty \in \mathcal{W}^*$, we show that $l_\infty^* \notin \mathcal{W}^*$: l_∞ contains a closed subspace F such that l_∞/F is isomorphic to a non-separable Hilbert space. Hence, $F^\perp = (l_\infty/F)^*$ is a subspace of l_∞^* which is also isomorphic to a non-separable Hilbert space. If l_∞^* were in class $\mathcal{W}^*$, then, from Proposition 4, there would be a non-separable Hilbert space in $\mathcal{W}^*$, which is clearly contradictory.

There are also examples satisfying the contrary, i.e., $X \notin \mathcal{W}^*$ and $X^* \in \mathcal{W}^*$. In particular, there is one which plays a somewhat outstanding role and we shall take a look at it right now. Let $X = c_0([0,1])$. Then $X^* = l_1([0,1])$, and to show that X^* is in $\mathcal{W}^*$, since the space of continuous functions $C[0,1]$, being separable, is a quotient of l_1, and therefore its topological dual $C[0,1]^*$ is isomorphic to a subspace of l_∞, it suffices to see that $l_1([0,1])$ can be continuously injected into $C[0,1]^*$. This is done by noticing that the mapping $T : l_1([0,1]) \to C[0,1]^*$ such that, if $x = (x_\gamma) \in l_1([0,1])$, $\quad Tx := \sum_{\gamma\in[0,1]} x_\gamma \delta_\gamma$ where δ_γ is the Dirac measure at the point $\gamma \in [0,1]$, is one-to-one bounded and linear.

Also, since $(l_\infty/c_0)^*$ admits no countable total subsets, it follows that l_∞/c_0 is not in $\mathcal{W}^*$.

Let us to show that, if $X \notin \mathcal{W}^*$, then every sequence of closed linear subspaces $(E_j)_{j=1}^\infty$ such that $X/E_j \in \mathcal{W}^*$, $j \geq 1$, satisfies that $\bigcap_{j=1}^\infty E_j \notin \mathcal{W}^*$.

Lemma 5. *Let E be a closed linear subspace of the Banach space X. Then $E^\perp$ is $\sigma(X^*, X)$-separable if and only if there is a sequence $(u_j^*)_{j=1}^\infty$ in X^* such that $E = \cap_{j=1}^\infty \ker u_j^*$.*

Proposition 7. *Let $(E_j)_{j=1}^\infty$ be a sequence of closed linear subspaces of X such that, for each j, $E_j^\perp$ is $\sigma(X^*, X)$-separable. Let $E := \cap_{j=1}^\infty E_j$, then:*

(i) $E^\perp$ is also $\sigma(X^*, X)$-separable.

(ii) If $X\sigma \notin \mathcal{W}^*$, then $E \notin \mathcal{W}^*$.

Proof. For each j, from the previous lemma, there is a sequence $(u_{jk}^*) \subset X^*$ such that $E_j = \bigcap_{k=1}^\infty \ker u_{jk}^*$. Hence $E^\perp = (\cap_{j=1}^\infty E_j)^\perp = (\cap_{j,k=1}^\infty \ker u_{jk}^*)^\perp = \overline{lin}^{w^*}\{u_{jk}^* : j,k \geq 1\}$ is clearly $\sigma(X, X^*)$-separable, thus obtaining (i). Besides, this yields $X/E \in \mathcal{W}^*$, and, if $X \notin \mathcal{W}^*$ the 3-space property shown in Corollary 8 guarantees (ii). ☐

Proposition 8. *If X is a Banach space which is not in class $\mathcal{W}^*$, then, if n is any positive integer, for each $P \in \mathcal{P}_w(^nX)$, $\ker P^{(n-1)}$ is not in $\mathcal{W}^*$.*

Proof. If $P \in \mathcal{P}_w(^nX)$, making use of the conjugacy relation mentioned in the first section, we have

$$\ker P^{(n-1)} = P'(X)_\perp.$$

From ([10], p. 88, Proposition 2.6), we know that P' is (weak-to-norm)-uniformly continuous on the bounded subsets and, since B_X is weakly precompact, it follows that $P'(X)$ is norm-separable in X^*. Clearly then, $\overline{\text{lin}}^{w^*}(P'(X))$ is weak*-separable and so, since

$$(X/P'(X)_\perp)^* = (P'(X)_\perp)^\perp = \overline{\text{lin}}^{w^*}(P'(X)),$$

we have that $X/P'(X)_\perp$ is in $\mathcal{W}^*$. From Corollary 8, since X is not in $\mathcal{W}^*$, it follows that $\ker P^{(n-1)} = P'(X)_\perp$ is not in $\mathcal{W}^*$. □

Recalling that $\ker P^{(n-1)}$ is contained in every maximal linear subspace contained in $\ker P$, the next result clearly follows.

Corollary 9. *If $X \notin \mathcal{W}^*$, then, for every integer n and every $P \in \mathcal{P}_w({}^nX)$, every maximal linear subspace Z contained in* $\ker P$ *is such that $Z \notin \mathcal{W}^*$.*

The next result gives us another characterization of the spaces in class $\mathcal{W}^*$.

Corollary 10. *For a Banach space X, the following conditions are equivalent:*

(i) $X \in \mathcal{W}^*$.

(ii) For any even integer n, X admits a positive definite polynomial $P \in \mathcal{P}_N({}^nX)$.

(iii) For any even integer n, X admits a positive definite polynomial $P \in \mathcal{P}_w({}^nX)$.

(iv) There is an even integer n such that X admits a positive definite polynomial $P \in \mathcal{P}_w({}^nX)$.

(v) There is an even integer n such that X admits a positive definite polynomial $P \in \mathcal{P}_N({}^nX)$.

As a by product of this last corollary, the author obtained a stronger version of part (i) in Theorem 16 of [1].

Corollary 11. *Let X be any infinite-dimensional real Banach space. Then, either X admits a positive definite nuclear polynomial of degree 2, or, for every positive integer n, the zero-set of every $P \in \mathcal{P}_w({}^nX)$ contains a closed linear subspace of X whose dual is not weak*-separable.*

The results previously obtained will be used in the following to show that, if $X \notin \mathcal{W}^*$, then every vector-valued polynomial, not necessarily homogeneous, which is weakly continuous on the bounded subsets of X admits a closed linear subspace not belonging to $\mathcal{W}^*$ where the polynomial is constant.

Lemma 6. *If $P \in \mathcal{P}_w({}^nX)$, then $(\ker P^{(n-1)})^\perp$ is $\sigma(X^*, X)$-separable.*

Proposition 9. *Let $(n_j)_{j=1}^\infty$ be a sequence of positive integers and let $(P_j)_{j=1}^\infty$ be a sequence of polynomials such that, for each j, $P_j \in \mathcal{P}_w({}^{n_j}X)$. If $X \in \mathcal{W}^*$, then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $Z \subset \cap_{j=1}^\infty P_j^{-1}(0)$.*

Proof. For each j, set $Z_j := \ker P_j^{(n_j-1)}$. Then, from the previous lemma we have that $Z_j^\perp$ is $\sigma(X^*, X)$-separable, $j \geq 1$. Setting $Z := \cap_{j=1}^\infty Z_j$, we know from Proposition 7 that $Z^\perp$ is (X^*, X)-separable, and so, since $X \notin \mathcal{W}^*$, we have that $Z \notin \mathcal{W}^*$. Now, since it is evident that $\ker P_j^{(n_j-1)} \subset \ker P_j$, $j \geq 1$, the result follows. □

For a Banach space Y and a positive integer n, the symbols $\mathcal{P}(^nX, Y)$ and $\mathcal{P}_w(^nX, Y)$ will denote the spaces of n-homogeneous continuous polynomials on X with values in Y and the subspace formed by those which are weakly continuous (to say it in a more explicit way, weak-to-norm continuous) on the bounded subsets of X, respectively. We see next that, when X is not in class $\mathcal{W}^*$, any countable family of polynomials in $\mathcal{P}_w(^nX, Y)$ vanishes simultaneously on quite a big linear subspace.

Corollary 12. *Let X, Y be Banach spaces with $X \notin \mathcal{W}^*$. Let $(n_j)_{j=1}^\infty$ be a sequence of positive integers and $(P_j)_{j=1}^\infty$ a sequence of polynomials such that, for each j, $P_j \in \mathcal{P}_w(^{n_j}X, Y)$. Then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $P_j|Z = 0, \quad j \geq 1$.*

Corollary 13. *Let $P : X \to Y$ be a polynomial, not necessarily homogeneous, which is weakly continuous on the bounded subsets of X. If $X \notin \mathcal{W}^*$, then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $P|_Z = P(0)$.*

In the results previously given we determine constructively the big linear subspace contained in the polynomial's zero-set. Nevertheless, noticing that what we really use is that weak zero-neighborhoods contain finite-codimensional linear subspaces, there is a natural extension of these existence results to a larger frame, namely that of the mappings which are weak-to-norm continuous on the bounded sets. More explicitelly, we have the following generalization.

Corollary 14. *Let $f : X \to Y$ be a weak-to-norm continuous mapping on the bounded subsets of X such that $f(0) = 0$. If $X \notin \mathcal{W}^*$, then there is a closed linear subspace Z in X, with $Z \notin \mathcal{W}^*$, such that $Z \subset f^{-1}(0)$.*

Corollary 15. *Let $(f_j)_{j=1}^\infty$ be a sequence of mappings from X into Y which are weak-to-norm continuous on the bounded subsets of X and such that $f_j(0) = 0, \quad j \geq 1$. Assume that, for all $x \in X$,*

$$f(x) := \lim_j f_j(x)$$

exists.

Conjecture. For a real Banach space X, either $X \in \mathcal{C}_H$, or, for every $P \in \mathcal{P}(^2X)$, $\ker P$ contains a non-separable linear subspace.

Proposition 10. *Let X be a space such that $X \notin \mathcal{C}_H$ and $X^* \in \mathcal{C}_H$. Then, if $P \in \mathcal{P}(^2X)$, $\ker P' \notin \mathcal{W}^*$.*

Proof. The first Fréchet derivative of P is the continuous linear map $P' : X \to X^*$ such that $\langle P'(x), y\rangle = 2\check{P}(x, y)$, $x, y \in X$. Assuming $\ker P'$ were in $\mathcal{W}^*$, then, from Proposition 6, since $X \notin \mathcal{C}_H$, we would have that $X/\ker P' \notin \mathcal{C}_H$. But the map $T : X/\ker P' \to X^*$ given by $T(x + \ker P') := P'(x)$ is well defined linear bounded and one-to-one, which would imply that $X/\ker P'$ is injected into X^*, but $X^* \in \mathcal{C}_H$, after Proposition 4, would then yield $X/\ker P' \in \mathcal{C}_H$, a contradiction. □

Corollary 16. *If $X \notin \mathcal{C}_H$ and $X^* \in \mathcal{C}_H$, then, for every $P \in \mathcal{P}(^2X)$, every maximal linear subspace Z contained in* $\ker P$ *is such that $Z \notin \mathcal{W}^*$.*

We show next that, for uncountable Γ, the spaces $c_0(\Gamma)$, $l_p(\Gamma)$, $\quad 2 < p < \infty$, are of the type just considered, i.e., $X \notin \mathcal{C}_H, \quad X^* \in \mathcal{C}_H$.

Lemma 7. *Let Γ be an uncountable set. Then, for $1 \leq p \leq 2$, the space $l_p(\Gamma) \in \mathcal{C}_H$, while, for $2 < p \leq \infty, \quad l_p(\Gamma) \notin \mathcal{C}_H$.*

Corollary 17. *Let Γ be an uncountable set and let X be any of the spaces $l_p(\Gamma)$, $2 < p < \infty$, or $c_0(\Gamma)$. If P is a continuous 2-homogeneous polynomial on X, then* $\ker P'$ *is a closed linear subspace contained in* $\ker P$ *whose dual is not weak*-separable. Consequently, every maximal linear subspace contained in* $\ker P$ *has a dual which is not weak*-separable. If $X = l_\infty(\Gamma)$, then, for every $P \in \mathcal{P}(^2X)$,* $\ker P$ *contains a closed linear subspace Z such that $Z \notin \mathcal{W}^*$.*

We say that a space X is in class $\mathcal{C}'_H$ whenever, for any sequence $(u_j^*)_{j=1}^\infty$ in X^*, we have that $\bigcap_{j=1}^\infty \ker u_j^* \notin \mathcal{C}_H$.

Clearly, $\mathcal{C}_H$ and $\mathcal{C}'_H$ are disjoint classes and we show that for the elements of class $\mathcal{C}'_H$ the conjecture holds.

Proposition 11. *Let $X \in \mathcal{C}'_H$. If $P \in \mathcal{P}(^2X)$, then every maximal linear subspace contained in* $\ker P$ *is non-separable.*

Proof. Let Z be one of such maximal subspaces and suppose it is separable. Let $Y := P'(Z)_\perp$. Then, by the maximality of Z, we have that $\ker P \cap Y = Z$ and P does not change sign in Y (we shall assume that $P|_Y \geq 0$).

Since $Y^\perp = \overline{P'(Z)}^{w^*}$ is $\sigma(X^*, X)$-separable, after Lemma 5 we have that there is a sequence $(u_j^*)_{j=1}^\infty$ in X^* such that $Y = \cap_{j=1}^\infty \ker u_j^*$. Thus, since $X \in \mathcal{C}'_H$, it follows that $Y \notin \mathcal{C}_H$. Now, by defining

$$Q(x + Z) := P(x), \; x \in Y,$$

we obtain a polynomial $Q \in \mathcal{P}(^2(Y/Z))$ which is positive definite. This implies that $Y/Z \in \mathcal{C}_H$, but, Z being separable yields $Z \in \mathcal{W}^*$, and so, after Proposition 6, we have that $Y \in \mathcal{C}_H$, a contradiction. □

4. The Real Case

All results of this subsection was proved in [1].

Let E be a real Banach space. The author showed that either E admits a positive definite 2-homogeneous polynomial or every 2-homogeneous polynomial on E has an infinite dimensional subspace on which it is identically zero. Under addition assumptions, he showed that such subspaces are non-separable. He examined analogous results for nuclear and absolutely (1,2)-summing 2-homogeneous polynomials and give necessary and sufficient conditions on a compact set K so that $C(K)$ admits a positive definite 2-homogeneous polynomial or a positive definite nuclear 2-homogeneous polynomial.

The case of the polynomial $P : \mathbb{R}^n \to \mathbb{R}$, $\quad P(x) = \sum\limits_{j=1}^{n} x_j^2$ not with standing, it is exactly the zeros of real valued 2-homogeneous polynomials which will be of interest here, in the case when the domain $\mathbb{R}^n$ is replaced by an infinite dimensional real Banach space E. There are many "large" Banach spaces E for which there is no positive definite 2-homogeneous polynomial P. As we will see, for a real Banach space E, either E admits a positive definite 2-homogeneous polynomial or every 2-homogeneous polynomial on E is identically zero on an infinite dimensional subspace of E.

We recall that an n-homogeneous polynomial $P : E \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is, by definition, the restriction to the diagonal of a necessarily unique symmetric continuous n-linear form $\check{P} : E \times$

$\ldots \times E \to \mathbb{K}$; that is, $P(x) = \check{P}(x, ..., x)$ for every $x \in E$. The polynomial P is said to be *positive definite* if $P(x) \geqslant 0$ for every x and $P(x) = 0$ implies that $x = 0$.

An n-homogeneous polynomial P on E is *nuclear* if there is bounded sequence $(\phi_j)_{j=1}^{\infty} \subset E'$ and a point $(\lambda_j)_{j=1}^{\infty}$ in l_1 such that

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n$$

for every x in E. The space of all nuclear n-homogeneous polynomials on E is denoted by $\mathcal{P}_N(^n E)$. A sequence $(x_j)_j$ in E is said to be *weakly* $2-$ *summing* if

$$\sup_{\phi \in B_{E'}} \sum_{j=1}^{\infty} \phi(x_j)^2 < \infty.$$

An n-homogeneous polynomial P on E is said to be (absolutely) (1,2)-summing if P maps weakly 2-summing sequences into absolutely summable sequences; that is if $\sum_{j=1}^{\infty} ||P(x_j)|| < \infty$ for every weakly 2-summing sequence $(x_j)_j$. P is (1,2)-summing if and only if there is $C > 0$ so that for every positive integer m and every $x_1, \ldots, x_m$ in E we have

$$\sum_{j=1}^{m} |P(x_j)| \leqslant C \left(\sup_{\phi \in B_{E'}} \sum_{j=1}^{m} \phi(x_j)^2 \right)^{\frac{n}{2}}.$$

Proposition 12. *A polynomial $P \in \mathcal{P}(^2 E)$ is positive definite if and only if for every $x, y \in E$ such that $x \neq \pm y$,*

$$|\check{P}(x, y)| < \frac{1}{2}(P(x) + P(y)).$$

Consequently, if P is a positive definite 2-homogeneous polynomial on E, then $||P|| = ||\check{P}||$.

Proof. Assume that P is positive definite, and so $\check{P}$ is an inner product. Hence we may apply the Cauchy-Schwarz inequality: $|\check{P}(x, y)| \leq |P(x)P(y)|^{\frac{1}{2}}$, with equality if and only if $x = \pm y$. Next, by the arithmetic-geometric inequality, $|P(x)P(y)|^{\frac{1}{2}} \leq \frac{1}{2}(P(x) + P(y))$. The converse follows by taking an arbitrary $x \neq 0$ and $y = 0$ in the inequality. □

Proposition 13. *The following conditions on a Banach space E are equivalent:*

(i) E admits a positive 2-homogeneous polynomial.

(ii) There is a continuous linear injection from E into a Hilbert space.

(iii) The point 0 *is an exposed point of the convex cone of the subset $\{\delta_x \bigotimes \delta_x : x \in S_E\}$ of the symmetric tensor product $E \bigotimes_{\pi,s} E$, where S_E is the unit sphere of E.*

(iv) There is a 2-homogeneous polynomial P on E whose set of zeros is contained in a finite dimensional subspace of E.

Proof. (i) ⇒ (ii): Let $\check{P}$ be the symmetric positive definite bilinear form associated to the positive definite polynomial P, so that $(E, \check{P})$ is a pre-Hilbert space with completion, say, H with the induced pre-Hilbert norm. Then the injectionj : $E \to H$ is continuous since $||j(x)|| = |\check{P}(x,x)|^{\frac{1}{2}} = |P(x)|^{\frac{1}{2}} \leq ||P||^{\frac{1}{2}}||x||$.

(ii) ⇒ (iii): Note that the space of 2-homogeneous polynomials on E is the dual of $E\widehat{\bigotimes}_{\pi,s}E$. Also, recall that the convex cone of the set $\{\delta_x \bigotimes \delta_x : x \in S_E$ consists of all points of the form $\{\sum_{i=1}^{n} a_i\delta_{x_i} \bigotimes \delta_{x_i},$ where $x_i \in S_E$ and $a_i \geq 0\}$. Now, the polynomial $P(x) \equiv \langle j(x), j(x)\rangle$ is positive definite on E. If we regard P as an element of $(E\widehat{\bigotimes}_{\pi,s}E)'$, we see at $P(0) = 0$ while $P(\delta_x \bigotimes \bigotimes \delta_x) = P(x) > 0$ for all $x \in S_E$. Consequently, for any point $\sum\limits_{i=1}^{n} a_i\delta_{x_i} \bigotimes \delta_{x_i}$ in the convex cone, $P(\sum\limits_{i=1}^{n} a_i\delta_{x_i} \bigotimes \delta_{x_i}) = \sum\limits_{i=1}^{n} a_i P(x_i) \geq 0$, with equality if and only if all $a_i = 0$.

(iii) ⇒ (iv): Let $T \in (E\widehat{\bigotimes}_{\pi,s}E)'$ be such that $T(0) = 0$ and $T(b) > 0$ for all b in the convex cone. In particular, for all $x \in S_E$, $P(x) \equiv T(\delta_x \bigotimes \delta_x) > 0$, so that $\ker P = 0$.

(iv) ⇒ (i): We only consider the non-trivial situation, when $\dim E = \infty$. Suppose that P is a 2-homogeneous polynomial whose zero set is contained in the finite dimensional subspace V with basis, say, $\{v_1, \ldots, v_n\}$. We first observe that $P(x)$ is always positive or always negative, for all $x \in E\backslash V$. Otherwise, there would exist $x, y \in S_{E\backslash V}$ such that $P(x) < 0 < P(y)$. Let $\gamma : [0,1] \to E\backslash V$ be a curve linking x and y. Then $P \circ \gamma(t) = 0$ for some $t \in [0,1]$, which is a contradiction. So, without loss of generality, we assume that $P(x) \geq 0$ for all $x \in E$. Let $\Pi : E \to V$ be a projection, with $\Pi(x) = \sum_{i=1}^{n} a_i(\Pi(x))v_i$. Then, the 2-homogeneous polynomial Q defined by $Q(x) \equiv P(x) + \sum_{i=1}^{n} a_i(\Pi(x))^2$ is positive definite. □

Remark 1. Suppose that there is a normalized sequence $(\phi_j)_j \in E'$ such that if $x \in E, \phi_j(x) = 0$ for all j, then $x = 0$. Then the mapping $x \in E \mapsto (\frac{1}{j}\phi_j(x))$ defines an injection into l_2, and so Proposition 13 applies. In particular, any separable space and $C(K)$spaces, when K is compact and separable, admit a positive definite 2-homogeneous polynomial. On the other hand $E = c_0(\Gamma)$ and $E = l_p(\Gamma)$, where Γ is an uncountable index set and $p > 2$, do not admit positive definite 2-homogeneous polynomials.

We also note that if there is a continuous linear injection $j : E \to l_2$, $j(x) = (j_n(x))$, then the mapping $x \mapsto (\frac{j_n(x)}{2^n})$ is a *nuclear* injection between these spaces. We have proved $(ii) \Rightarrow (iii)$ of the following separable version of Proposition 13.

Proposition 14. *Let E be a real Banach space. The following conditions are equivalent:*

(i) E admits a positive definite 2-homogeneous nuclear polynomial.

(ii) E admits a continuous injection $j : E \to l_2$.

(iii) There is a nuclear injection $j : E \to l_2$ of the form $j(x) = \sum\limits_{n=1}^{\infty} \chi_n(x)e_n$ with $(||\chi_n||) \in l_1$.

Proof. (i) ⇒ (ii): If $P(x) = \sum_{n=1}^{\infty} \phi_n(x)^2$ is a positive definite nuclear polynomial on E, then $j(x) = \sum_{n=1}^{\infty} \phi_n(x)e_n$ will satisfy (ii).

(iii) $\Rightarrow$ (i): Let $j : E \to l_2$ be a nuclear injection, $j(x) = \sum_{n=1}^{\infty} \chi_n(x)e_n$, where $(||\chi_n||)_n \in l_1$. Since $\bigcap_{n=1}^{\infty} \ker \chi_n = \{0\}$, it follows that the 2-homogeneous polynomial $P : E \to \mathbb{R}$, $P(x) = \sum_{n=1}^{\infty} \chi_n^2(x)$ is positive definite. Finally, P is nuclear since $\sum_{n=1}^{\infty} ||\chi_n||^2 \le \sum_{n=1}^{\infty} ||\chi_n|| < \infty$. □

Theorem 4.1. *Let E be a real Banach space which does not admit a positive definite 2-homogeneous polynomial. Then, for every $P \in \mathcal{P}(^2E)$, there is an infinite dimensional subspace of E on which it is identically zero.*

Proof. Suppose E does not admit a positive definite 2-homogeneous polynomial and that $P \in \mathcal{P}(^2E)$. Let $\mathcal{S} = \{S : S$ is a subspace of E and $P|_S \equiv 0\}$. Order $\mathcal{S}$ by inclusion and use Zorn's Lemma to deduce the existence of a maximal element S of $\mathcal{S}$. Suppose that S is finite dimensional. $v_1, \ldots, v_n$ be a basis for S and let $T = \bigcap_{x\in S} \ker A_x = \bigcap_{i=1}^{n} \ker A_{v_i}$ where $A_x : E \to \mathbb{R}$ is the linear map which sends y in E to $\check{P}(x, y)$. We note that $S \subset T$. To see this suppose that $y \in S$. Then for every $s \in S$, $s + y$ is also in S. Since

$$0 = P(s + y) = P(s) + 2A_s(y) + P(y) = 2A_s(y)$$

for every $s \in S$ we see that $y \in T$.

Since S is finite dimensional we can write T as $T = S \bigoplus Y$ for some subspace Y of T. It is easy to see that all the zeros of $P|_T$ are contained in S. Therefore, either $P|_T$ or $-P|_T$ is positive definite on Y. Let us suppose, without loss of generality, that $P|_T$ is positive definite on Y. As S is n-dimensional we can find $\phi_1, \ldots, \phi_n$ so that $P + \sum_{i=1}^{n} \phi_i^2$ is positive definite on T. Note that T has finite codimension in E and hence is complemented. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. □

Theorem 4.2. *Let E be a real Banach space of type 2. Then either E admits a positive definite 2-homogeneous polynomial or every $P \in \mathcal{P}(^2E)$ has an non-separable subspace on which it is identically zero.*

Proof. Assume that E does not admit a positive definite 2-homogeneous polynomial and let $P \in \mathcal{P}(^2E)$. Let $S \subset E$ be a maximal subspace such that $P|_S \equiv 0$. If S is separable, the argument in Theorem 4.1 shows that the subspace $T \subset E$ can be written $T = S \bigoplus_a Y$, where Y is an algebraic complement of S in T and where, without loss of generality, $P|_T$ is positive definite on Y. Then for every $s \in S$ and $t \in T$:

$$P(s + t) = P(s) + 2\check{P}_s(t) + P(t) = P(t) \ge 0.$$

Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S, and hence $P + \sum_{i=1}^{\infty} \phi_i^2$ is positive definite on T. Hence we have a continuous linear injection i of T into some Hilbert space $L_2(I)$. Since E is type 2, Maurey's Extension Theorem ([8], Theorem 12.22) allows us to extend i to a (not necessarily injective) linear map $\widetilde{i}$ from E into $L_2(I)$. Finally, define a map j from E into $L_2(I) \bigoplus_{l_2} l_2$ by

$$j(x) = \left(\widetilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2||A_{v_i}||} e_i \right),$$

where e_i is the i^{th} basis vector in l_2. Since j is a continuous injection, E admits a positive definite polynomial, which is a contradiction. □

Theorem 4.3. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then for every 2-homogeneous polynomial P on E, there is a non-separable subspace of E on which P is identically zero.*

Theorem 4.4. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial, and let $(\psi_k)_{k=1}^{\infty}$ be a sequence in E'. Then for any countable family $(P_j)_{j=1}^{\infty} \subset \mathcal{P}(^2E)$, there is a non-separable subspace of $\bigcap_{k=1}^{\infty} \ker \psi_k$ on which each P_j is identically zero.*

Note that if E does not admit a positive definite 4-homogeneous polynomial, then it cannot admit a positive definite 2-homogeneous one either. An example of an E satisfying the hypotheses of Theorems 4.3 and 4.4 is $E = l_p(I)$, where I is an uncountable index set and $p > 4$.

Proof. **of Theorem 4.4**: The argument begins in a similar way to our earlier proofs. As before, let S be a maximal element of $\mathcal{S} = \{S : S \text{ is a subspace of } \bigcap_{k=1}^{\infty} \ker \psi_k \text{ and } P_j|_S \equiv 0, \text{all } j\}$. Suppose that S is separable, with countable dense set $(v_i)_{i=1}^{\infty}$. Let $\bigcap_{k=1}^{\infty} \ker \psi_k \cap \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \ker(A_j)_{v_i}$. As before, $S \subset T$. We can write T as $T = S \bigoplus_a Y$ for some subspace Y of T. Since all the common zeros of $P_j|_T$, $j \in \mathbb{N}$, are contained in S, $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2}$ is positive definite on Y. As S is separable we can find $(\phi_i)_{i=1}^{\infty}$ so that $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2} + \sum_{i=1}^{\infty} \phi_i^4$ is positive definite on T. Then

$$\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2} + \sum_{i=1}^{\infty} \phi_i^4 + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(A_j)_{v_i}^4}{i^2 j^2 ||(A_j)_{v_i}||^4} + \sum_{k=1}^{\infty} \frac{\psi_k^4}{k^2||\psi_K||^4}$$

is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. □

Corollary 18. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then every $P \in P(^3E)$ is identically zero on a non-separable subspace of E.*

Theorem 4.5. *Let E be a real Banach space which does not admit a positive definite homogeneous polynomial. Then, for every polynomial P on E such that $P(0) = 0$, there is a non-separable subspace of E on which P is identically zero.*

Lemma 8. *A real Banach space E admits a positive definite 2-homogeneous (1,2)-summing polynomial if and only if there is a continuous 2-summing injection from E into a Hilbert space.*

Corollary 19. *Let E be an $L_{\infty,\lambda}$ -space for some real λ. Then every positive definite polynomial on E is (1,2)-summing.*

Note, though, that there may well not exist any positive definite polynomials on an $L_{\infty,\lambda}$ space.

We next consider the question of the existence of positive definite 2-homogeneous polynomials in case E is a $C(K)$ space. We recall that a (Borel) measure μ on a compact set K is said to be *strictly positive* if $\mu(B) > 0$ for every non-empty open subset $B \subset K$.

Corollary 20. *Let $E = C(K)$ where K is a compact Hausdorff space. Then*
(i) $C(K)$ admits a positive definite 2-homogeneous polynomial if and only if K admits a strictly positive measure.
(ii) $C(K)$ admits a positive definite 2-homogeneous nuclear polynomial if and only if there is a sequence of finite Borel measures $(\mu_n)_{n=1}^{\infty}$ on K such that $\int_K f(x)d\mu_n(x) = 0$ for all n implies $f \equiv 0$.

Theorem 4.6. *Let E be a real Banach space.*
(i) Either E admits a positive definite 2-homogeneous nuclear polynomial or every $P \in \mathcal{P}_N(^2E)$ has a non-separable subspace on which it is identically zero.
(ii) Either E admits a positive definite 2-homogeneous (1,2)-summing polynomial or every (1,2)-summing has an non-separable subspace on which it is identically zero.

Proof. (i) We reason as before, supposing that E does not admit a positive 2-homogeneous nuclear polynomial and that $P \in \mathcal{P}_N(^2E)$. Let S be a maximal subspace of E on which P is identically 0, assume that $S = \overline{\{v_i : i \in \mathbb{N}\}}$, let $T = \bigcap_{i=1}^{\infty} \ker A_{v_i}$, and write $T = S \bigoplus_a Y$. Without loss of generality, we may assume that $P|_T$ is positive definite on Y, so that $P|_T \geq 0$. Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S and nuclear on E. Hence $P + \sum_{i=1}^{\infty} \phi_i^2$ is positive definite and nuclear on T. We therefore have a continuous linear nuclear injection i of T into l_2. We can extend i to a nuclear linear map $\widetilde{i}$ from E into l_2.

Define a map $j : E \to l_2 \bigoplus_2 l_2$ by

$$j(x) = \left(\widetilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2 ||A_{v_i}||} e_i \right).$$

Since j is a nuclear injection, E admits a positive definite nuclear polynomial, which is a contradiction.

(ii) The argument given above works in the (1,2)-summing case, the only significant change being an appeal to the $\prod_2$ Extension Theorem to prove the existence of a 2-summing extension mapping $\widetilde{i} : E \to L_2(I) \bigoplus_2 l_2$, for a sufficiently large index set I. □

Even if we know that an $L_{\infty,\lambda}$-space admits a positive definite (1,2)-summing polynomial, it is nevertheless possible to conclude something about the zeros of those 2-homogeneous polynomials which are not (1,2)-summing.

Theorem 4.7. *Let E be a real $L_{\infty,\lambda}$-space. Then every $P \in \mathcal{P}(^2E)$ which is not (1,2)- summing has an infinite dimensional subspace on which it is identically zero.*

Proof. Suppose $P \in \mathcal{P}(^2E)$ is not (1,2)-summing. Suppose that a maximal subspace S on which P vanishes is only finite dimensional, with basis $\{v_1, \ldots, v_n\}$. Let $T = \bigcap_{i=1}^{n} \ker A_{v_i}$, and write $T = S \bigoplus Y$, for some complemented subspace $Y \subset T$. Without loss of generality, $P|_T$ is positive definite on Y and, since S is finite dimensional, we can find $\phi_1, \ldots, \phi_n$ so that $P + \sum_{i=1}^{n} \phi_i^2$ is positive definite on T. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is positive definite on E. But E is an $L_{\infty,\lambda}$-space and so by Corollary 14, $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is (1,2)-summing implying that $P|_T$ and hence P itself is (1,2)-summing, a contradiction. □

4.1. Zeroes of Real Polynomials on $C(K)$ Spaces

All results of this subsection was proved in [13].

By $\mathcal{C}_H$, and $\mathcal{W}^*$, we shall denote the class formed by those Banach spaces which can be injected (i.e., there is a continuous one-to-one linear map) into a Hilbert space, and the subclass formed by those that can be injected into a separable Hilbert space, respectively. Notice that X in $\mathcal{W}^*$ is equivalent to say that X^* is weak*-separable. If Y is a closed linear subspace of X

such that Y is in $\mathcal{W}^*$ and X/Y is in $\mathcal{C}_H$, then X is in $\mathcal{C}_H$. Thus obtaining that being in $\mathcal{W}^*$ is a three-space property.

If X is not in $\mathcal{C}_H$, then every element of $\mathcal{P}(^2X)$ admits an infinite-dimensional linear subspace where it vanishes, and the following conjecture is stated:

Conjecture. *If X is a real Banach space such that $X \notin \mathcal{C}_H$, then the zero-set of every quadratic polynomial, i.e., an element of $\mathcal{P}(^2X)$, contains a non-separable linear zero subspace.*

This conjecture is proved to be correct for spaces having the Controlled Separable Projection Property (CSPP)([7]), a class which contains the Weakly Countably Determined spaces. Let us recall that X has the CSPP whenever, if (x_j) and (x_j^*) are sequences in X and X^*, respectively, there exists a norm-one projection on X with separable range containing (x_j) and such that the range of its conjugate contains (x_j^*).

Let us recall that the $(n-1)$-derivative of the polynomial $P \in \mathcal{P}(^nX)$ is given by the continuous and linear map $P^{(n-1)} : X \to \mathcal{L}_s(X^{n-1})$ such that

$$P^{(n-1)}(x) = n!\check{P}(x, \cdot, \overset{(n-1)}{\dots}, \cdot),\ x \in X,$$

where $\mathcal{L}_s(X^{n-1})$ is the space of continuous symmetric $(n-1)$-linear functionals on X and $\check{P}$ is the n-linear functional provided by the polarization formula.

Proposition 15. *Given $n \in \mathbb{N}$, if $P \in \mathcal{P}(^nc_0(\Gamma))$, then* $\ker P^{(n-1)}$ *contains an isometric copy of $c_0(\Gamma)$.*

Proof. After ([10], Exercise 1.72, p. 68), we know that $\mathcal{P}_w(^nc_0(\Gamma))$ coincides with $\mathcal{P}(^nc_0(\Gamma))$. Hence, if $P \in \mathcal{P}(^nc_0(\Gamma))$, again using ([10], Proposition 2.6, p. 88), we have that the linear map $P^{(n-1)}$ is weak-to-norm continuous on bounded sets from $c_0(\Gamma)$ into $\mathcal{L}_s(c_0(\Gamma)^{n-1})$.

For each $m \in \mathbb{N}$, we consider the set

$$\Gamma_m := \{\gamma \in \Gamma : ||P^{(n-1)}(e_\gamma)|| \geq 1/m\},$$

where e_γ stands for the unit vector in $c_0(\Gamma)$ corresponding to γ. We claim that Γ_m is finite, otherwise there would be an infinite sequence $(\gamma_j)_{j=1}^\infty$ contained in Γ_m; but, since $P^{(n-1)}$ is weak-to-norm continuous on bounded sets, and the sequence $(e_{\gamma_j})_{j=1}^\infty$ is weakly null in $c_0(\Gamma)$, this would yield

$$\lim_j ||P^{(n-1)}(e_{\gamma_j})|| = 0,$$

a contradiction. Consequently, the set

$$\Gamma_0 := \{\gamma \in \Gamma : P^{(n-1)}(e_\gamma) = 0\} \bigcup_{m=1}^\infty \Gamma_m$$

is countable. Thus, if E denotes the closed linear span of $\{e_\gamma : \gamma \in \Gamma \backslash \Gamma_0\}$ in $c_0(\Gamma)$, it clearly follows that E is isometric to $c_0(\Gamma)$. Besides, if $\gamma \in \Gamma\backslash\Gamma_0$, we have $P^{(n-1)}(e_\gamma) = 0$, from where we deduce that, since $P^{(n-1)}$ is linear, $E \subset \ker P^{(n-1)}$. □

In the coming result it is convenient to observe that, for $P \in \mathcal{P}(^nX)$, the linear subspace $\ker P^{(n-1)}$ is always contained in the zero-set $\ker P$.

Corollary 21. *Let Γ be an uncountable set. If $P \in \mathcal{P}(c_0(\Gamma))$, then there is a closed linear subspace E of $c_0(\Gamma)$ such that $P|_E = P(0)$ and E is isometric to $c_0(\Gamma)$.*

Corollary 22. *Every real-valued analytic function on $c_0(\Gamma)$ admits a closed linear subspace isometric to $c_0(\Gamma)$ where it has constant value.*

Notice that Proposition 15, as well as the two previous corollaries, could also be obtained from the well-known fact that any continuous polynomial on $c_0(\Gamma)$ factors through $c_0(\Gamma')$ with Γ' countable.

Proposition 16. *Let K be a compact Hausdorff topological space. The following conditions are equivalent:*
(i) $C(K)$ contains a non-separable weakly compact subset.
(ii) K does not satisfy the CCC.
(iii) $C(K)$ contains an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.
(iv) There is an uncountable set Γ such that there is a one-to-one bounded linear map from $c_0(\Gamma)$ into $C(K)$.

Proof. (i) $\Rightarrow$ (ii). Let W be a weakly compact non-separable subset of $C(K)$, which we may assume to be absolutely convex. By a result of Corson, see [15], W contains a subset which is homeomorphic, in its weak topology, to the one-point compactification of an uncountable discrete set and we may thus find an uncountable subset $W_0 \subset W\setminus\{0\}$ such that every sequence of distinct elements of W_0 is weakly-null. There is clearly some $\delta > 0$ such that the set $W_1 := \{x \in W_0 : ||x|| > \delta\}$ is uncountable. For each $x \in W_1$, let $V_x := \{t \in K : |x(t)| > \delta/2\}$. Then, if $(x_j)_{j=1}^{\infty}$ is a sequence of distinct elements of W_1, it follows that $\bigcap_{j=1}^{\infty} V_{x_j} = \varnothing$, otherwise, since $x_j \to 0$ weakly, this would imply that $\lim_j x_j(t) = 0$, for all $t \in K$, in particular, if $t \in \bigcap_{j=1}^{\infty} V_{x_j}$, this would lead to a contradiction. Hence, we have an uncountable collection $\{V_x : x \in W_1\}$ of non-empty open subsets of K such that for all sequences $(V_{x_j})_{j=1}^{\infty}$ of distinct terms the intersection of its members is empty; this is a sufficient condition for K not to satisfy the CCC.

(ii)$\Rightarrow$(iii). Let $(V_\gamma)_{\gamma\in\Gamma}$ be an uncountable collection of pairwise disjoint non-empty open subsets of K. For each $\gamma \in \Gamma$, we find a function $x_\gamma \in C(K)$ such that $||x_\gamma|| = 1$ and $x_\gamma(t) = 0$, $t \in K\setminus V_\gamma$. Thus, if E denotes the closed linear span of $\{x_\gamma : \gamma \in \Gamma\}$ in $C(K)$, it is clear that E is isometric to $c_0(\Gamma)$.

(iii) $\Rightarrow$ (iv) Being obvious, we see that (iv) $\Rightarrow$ (i).

Let Γ be an uncountable set and $T : c_0(\Gamma) \to C(K)$ a one-to-one bounded linear map. Then, it is clear that the set

$$\{Te_\gamma : \gamma \in \Gamma\} \cup \{0\}$$

is weakly compact and non-separable in $C(K)$. □

Corollary 23. *Let K be a compact space not satisfying the CCC. For any positive integer n, every continuous n-homogeneous real-valued polynomial on $C(K)$ vanishes in an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.*

Corollary 24. *If K does not satisfy the CCC, then every analytic real-valued function on $C(K)$ has constant value in an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.*

l_∞/c_0 is isometric to $C(\beta\mathbb{N} \setminus \mathbb{N})$. Since it is well known that there is a family, with the continuum cardinality, of infinite subsets of $\mathbb{N}$ such that any two distinct members meet only in a finite set, it follows that $\beta\mathbb{N} \setminus \mathbb{N}$ does not have the CCC, so the next result obtains.

Corollary 25. *Every analytic real-valued function on l_∞/c_0 is constant in an isometric copy of $c_0(\Gamma)$, Γ having the continuum cardinality.*

Corollary 26. *For every positive integer n, if $P \in \mathcal{P}(^n l_\infty)$ is such that* $\ker P^{(n-1)}$ *contains c_0, then there is a closed linear subspace Z of l_∞ such that $c_0 \subset Z \subset$* $\ker P$ *and Z/c_0 is isometric to $c_0(\Gamma)$, Γ with the continuum cardinality.*

Lemma 9. *For an uncountable set Γ, the spaces $c_0(\Gamma)$, $l_p(\Gamma)$, $2 < p \le \infty$, do not belong to the class $\mathcal{C}_H$.*

Proposition 17. *If X is a Banach space such that, for some uncountable Γ, either $c_0(\Gamma)$, or $l_p(\Gamma)$, $2 < p < \infty$, is injected into X, then X belongs to the class $\mathcal{C}'_H$.*

Proof. Let $T : c_0(\Gamma) \to X$ be a one-to-one bounded linear map (an analogous proof works for the case of $l_p(\Gamma)$ injected into X). Let $(u_j^*)_{j=1}^\infty$ be a sequence in X^* and let $Y := \bigcap_{j=1}^\infty \ker u_j^*$. As we have already seen before in similar situations, it can be seen that, for each positive integer m, and $j \ge 1$, the set

$$\Gamma_{m,j} := \{\gamma \in \Gamma : |\langle u_j^*, Te_\gamma\rangle| > 1/m\}$$

is finite, and so, for each $j \ge 1$, the set

$$\Gamma_{0,j} := \{\gamma \in \Gamma : \langle u_j^*, Te_\gamma\rangle \neq 0$$

is countable. Hence, the set $\Gamma_0 := \Gamma \setminus \bigcup_{j>0} \Gamma_{0,j}$ has the same cardinality as Γ and we have that, $Te_\gamma \in Y, \gamma \in \Gamma_0$. Denoting by E the closed linear span of $\{e_\gamma : \gamma \in \Gamma_0\}$, we obtain an isometric copy of $c_0(\Gamma)$ which is injected into Y. After the previous lemma, this implies that Y cannot be in $\mathcal{C}_H$. □

From Propositions 16 and 17 the coming result obtains.

Corollary 27. *If K does not have the CCC, then $C(K)$ belongs to the class $\mathcal{C}'_H$.*

Lemma 10. *The following statements are equivalent :*

(i) $X \in \mathcal{C}_H$.

(ii) There is a positive definite 2-homogeneous continuous polynomial on X.

(iii) If $X = C(K)$, K carries a strictly positive measure (a non-negative regular finite Borel measure which has positive value on every non-empty open subset).

Proposition 18. *Let X be in class $\mathcal{C}'_H$. Then, if $P \in \mathcal{P}(^2X)$, every maximal linear subspace contained in* $\ker P$ *is non-separable.*

Proof. Let $P \in \mathcal{P}(^2X)$. Let Z be a maximal linear subspace contained in $\ker P$, whose existence is guaranteed by Zorn's Lemma. We show that Z is non-separable. If this were not so, since the Frechet derivative $P' : X \to X^*$ is a bounded linear map, setting $Y := \{x \in X : \langle P'(z), x\rangle = 0, z \in Z\}$, it follows that $(X/Y)^* = Y^\perp = \overline{P'(Z)}^{w^*}$ is weak*-separable, i.e., Y is a countable intersection of closed hyperplanes. Hence, $X \in \mathcal{C}'_H$ implies that $Y \notin \mathcal{C}_H$. But, from the maximality of Z, it is easy to see that $Y \cap \ker P = Z$ and that P does not change sign in Y; thus, defining $Q(y + Z) := P(y), \quad y \in Y$, we obtain a quadratic polynomial in Y/Z such that either Q, or $-Q$, is positive definite. From the above lemma, this implies that $Y/Z \in \mathcal{C}_H$, and, since $Z \in W^*$, after the 3-space result it follows that $Y \in \mathcal{C}_H$, a contradiction. □

If A is a subset of the compact space K, then by $C_A(K)$ we denote the closed linear subspace of $C(K)$ formed by those functions which vanish in A.

Proposition 19. *For a compact Hausdorff topological space K, if Y is a closed linear subspace of $C(K)$ such that $C(K)/Y \in \mathcal{W}^*$ and $Y \in \mathcal{C}_H$, then $C(K) \in \mathcal{C}_H$.*

Proof. Let Y be a closed linear subspace of $C(K)$ such that it satisfies the conditions of our statement. Since $(C(K)/Y)^* = Y^\perp$ is weak*-separable, there is a sequence $(\mu_j)_{j=1}^\infty$ contained in $C(K)^*$, which, after Riesz's theorem, we identify with $M(K)$, the space of regular finite Borel measures in K, such that $Y = \bigcap_{j=1}^\infty \ker \mu_j$. In light of Jordan's decomposition theorem, there is no loss of generality in assuming that those measures are probabilities on K. We show first that K must have the Countable Chain Condition (CCC, for short). Otherwise, after Proposition 16, $C(K)$ would contain a copy (isometrically indeed) of $c_0(\Gamma)$, for some uncountable set Γ. Thus, let $S : c_0(\Gamma) \to C(K)$ be such isometry (what we need really is that S is weakly continuous and one-to-one). Then setting, for each pair of positive integers j, m,

$$\Gamma_{jm} := \{\gamma \in \Gamma : |\langle \mu_j, Se_\gamma \rangle| > 1/m\},$$

where e_γ stands for the corresponding unit vector of $c_0(\Gamma)$, it is clear that Γ_{jm} must be a finite set. Hence, the set $\Gamma_0 := \bigcup_{j,m} \Gamma_{jm}$ is countable and the closed linear span of $\{Se_\gamma : \gamma \in \Gamma \backslash \Gamma_0\}$ is contained in Y. This implies that a copy of $c_0(\Gamma\backslash\Gamma_0)$ would be injected into Y, a contradiction, since, after Lemma 9, $c_0(\Gamma\backslash\Gamma_0) \notin \mathcal{C}_H$.

Let $K_0 := \overline{\bigcup_{j=1}^\infty \operatorname{supp}\mu_j}$. Then, it is easy to see that K_0 carries a strictly positive measure and so, after Lemma 10, $C(K_0) \in \mathcal{C}_H$. Let H_1 be a Hilbert space and T_1 be a one-to-one bounded linear map from $C(K_0)$ into H_1. Having in mind that the family of cozero sets, i.e., the complements of zero-sets of elements of $C(K)$, is a base for the open sets in K, Zorn's Lemma guarantees the existence of a maximal collection of pairwise disjoint cozero sets contained in $K\backslash K_0$ (we assume that $K\backslash K_0 \neq 0$, otherwise $C(K) \in \mathcal{C}_H$). Now, the CCC forces this collection to be a countable one, so let $(V_j)_{j=1}^\infty$ represent this maximal collection. Clearly, if we set $V := \bigcup_{j=1}^\infty V_j$, then

$$V \subset K\backslash K_0 \subset \overline{V}.$$

Since V is also a cozero set, let φ be a continuous real-valued function such that $\varphi^{-1}(0) = K\backslash V$. Observing that $C_{K\backslash V}(K) \subset C_{K_0}(K) \subset Y$, we have that $C_{K\backslash V}(K) \in \mathcal{C}_H$. And so, there is a Hilbert space H_2 and a one-to-one bounded linear map T_2 from $C_{K\backslash V}(K)$ into H_2. Let H be the Hilbert space given by the product $H_1 \times H_2$. We define the map $T : C(K) \to H$ as

$$Tx := (T_1(x_{|_{K_0}}), T_2(x\varphi)).$$

Then, it can be easily verified that T is well defined, as well as that it is linear and bounded. We see that it is one-to-one: If $Tx = 0$, then, since $\ker T_1 = \{0\}$, we have that x vanishes in K_0; also, $\ker T_2 = \{0\}$ implies that $x\varphi = 0$ and so x must also vanish in $\overline{V} \supset K\backslash K_0$; hence, $x = 0$. Therefore, $C(K) \in \mathcal{C}_H$. □

Corollary 28. *For a compact Hausdorff topological space K, the following statements are equivalent:*

(i) K does not carry a strictly positive measure.

(ii) $C(K)$ is not injected into a Hilbert space.

(iii) For every closed linear subspace Y of $C(K)$ such that $C(K)/Y \in \mathcal{W}^*$, it follows that $Y \notin \mathcal{C}_H$, i.e., $C(K) \in \mathcal{C}'_H$.

(iv) For every continuous 2-homogeneous polynomial on C(K), its zero-set contains a non-separable linear subspace.

4.2. Zero Sets of Polynomials in Several Variables

All results of this subsection was proved in [4].

Let $k, n \in \mathbb{N}$ where n is odd. Let us denote by $\{e_i\}_{i=1}^k$ the canonical basis of $\mathbb{R}^k$. Given $l \in \mathbb{N}$ a partition $\mathcal{A} = (A_1, \ldots, A_k)$ of $\{1, \ldots l\}$ is called an *ordered partition of* $\{1, \ldots l\}$ *of rank* $|\mathcal{A}| = k$.

We set $N(k,n) = \begin{pmatrix} k+n-1 \\ k-1 \end{pmatrix}$. A set $S(k,n)$ of cardinality $N(k,n)$ in $\mathbb{R}^k$ is called a *basic set of nodes* if $P|_{S(k,n)} \equiv 0$ implies $P \equiv 0$ whenever P is an n-homogeneous polynomial on $\mathbb{R}^k$.

Lemma 11. *Given $k, n \in \mathbb{N}$, there exists a set $S(k,n) = \{v^i\}_{i=1}^{N(k,n)} \subset \mathbb{R}^k$ such that $e_j \in S(k,n)$, $1 \le j \le k$, with the property that for every n-homogeneous polynomial $Q(x)$ on $\mathbb{R}^k$, if $Q(v^i) = 0$, $1 \le i \le N(k,n)$ then $Q \equiv 0$ on $\mathbb{R}^k$.*

Lemma 12. *Given $k, n \in \mathbb{N}$, $\ k \le l$ there exist $p(k,l) \in \mathbb{N}$, $p(k,l) \le k!(\log_2(l))^k$, and a system $\{\mathcal{A}_I\}_{I=1}^{p(k,l)}$ of ordered partitions of $\{1, \ldots, l\}$ of rank k, such that for every $B \subset \{1, \ldots, l\}$, $|B| = k$, there exists $\mathcal{A}_I = (A_1, \ldots, A_k)$ for which $B \cap A_i \ne 0$ for every $1 \le i \le k$.*

Given an (abstract) n-homogeneous nonzero polynomial $Q(x)$ on a k-dimensional Banach space X, by a suitable choice of the basis $\{\widetilde{e_1}, \ldots, \widetilde{e_k}\}$ of X we can easily achieve that in the formula

$$Q(\sum_{i=1}^{k} y_i \widetilde{e_i}) = \sum_{|\alpha|=n} b^\alpha y^\alpha,$$

we have $b^{(n,0,\ldots,0)} \ne 0$. Indeed, it is enough to choose a direction $\widetilde{e_1}$ in which Q is nonzero. It is easily verified that a change of variables $y_1 \to C(\sum\limits_{i=n}^{k} x_i)$, $y_2 \to x_2, \ldots, y_k \to x_k$, where C is sufficiently large, will lead to a transformed algebraic formula for the same abstract polynomial Q on X:

$$Q((x_1, \ldots, x_k)) = \sum_{|\alpha|=n} a^\alpha x^\alpha,$$

in which $a^{(n,0,\ldots,0)}, a^{(0,n,0,\ldots,0)}, \ldots, a^{(0,0,\ldots,0,n)}$ are all nonzero. To summarize, we have the following.

Lemma 13. *Let $Q(x)$ be an (abstract) n-homogeneous nonzero polynomial on a k-dimensional Banach space X. Then there exists a basis $\{e_1, \ldots, e_k\}$ in X such that in the formula*

$$Q(\sum_{i=1}^{k} x_i e_i) = Q((x_1, \ldots, x_k)) = \sum_{|\alpha|=n} a^\alpha x^\alpha,$$

all the constants $a^{(n,0,\ldots,0)}, a^{(0,n,0,\ldots,0)}, \ldots, a^{(0,0,\ldots,0,n)}$ are nonzero.

Let us introduce the following notation. Let $A \subset \{1, 2, \ldots, l\}$.

We put $P_A : \mathbb{R}^l \to \mathbb{R}^l$, $P_A(\sum\limits_{i=1}^{k} x_i e_i) = \sum\limits_{j \in A} x_i e_i$.

Given $v = (v_1, \dots, v_k) \in \mathbb{R}^k$ and an ordered partition $\mathcal{A} = (A_1, \dots, A_k)$ of $\{1, \dots, l\}$, $|\mathcal{A}| = k$, we define $v.\mathcal{A} : \mathbb{R}^l \to \mathbb{R}^l$ as

$$v.\mathcal{A}(x) = \sum_{j=1}^{k} v_j P_{A_j}(x).$$

Theorem 4.8. *Let $n \in \mathbb{N}$, n be odd, and let $Q(x)$ be n-homogeneous polynomial on $\mathbb{R}^N$. Provided $N > k!(\log_2(N))^k \begin{pmatrix} k+n-1 \\ k-1 \end{pmatrix}$, there exists a linear subspace $X \hookrightarrow \mathbb{R}^N$, $\dim X = k$ such that $Q \equiv 0$ on X.*

Proof. By Lemma 13, we may assume that the basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ is chosen so that all the monomials in the formula for Q have nonzero coefficients. Consider the system $\{\mathcal{A}_I\}_{I=1}^{p(k,N)}$ from Lemma 12 and the set $S(k,n) = \{v^J\}_{J=1}^{N(k,n)}$ from Lemma 11. Fix a basis $e_{I,J}$, $1 \le I \le p(k,N)$, $1 \le J \le N(k,n)$ in $\mathbb{R}^{p(k,N)N(k,n)}$. Form an n-homogeneous polynomial $\widetilde{Q} : \mathbb{R}^N \to \mathbb{R}^{p(k,N)N(k,n)}$

$$\widetilde{Q}(x) = \sum_I \sum_J Q(v^J.\mathcal{A}_I(x)) e_{I,J}.$$

By assumption, $N + 1 \ge p(k,N)N(k,n)$ and the mapping $\widetilde{Q}(x)$ is odd ($\widetilde{Q}(-x) = -\widetilde{Q}(x)$). By the Borsuk antipodal theorem ([16]) there exists a nonzero $x^0 = (x_1^0, \dots, x_N^0) \in \mathbb{R}^N$, such that $\widetilde{Q}(x^0) = 0$. Denote $B = \mathrm{supp}(x^0)$.

We first claim that $|B| > k$. Indeed, otherwise there exists some $\mathcal{A}_I = (A_1, \dots, A_k)$ such that $|B \cap A_i| \le 1$, whenever $1 \le i \le \Delta k$, and $|B \cap A_j| = 1$ for some j. Pick J such that $v^J = (0, \dots, 0, 1, 0, \dots, 0) = e_m \in S(k,n)$, where $\{m\} = B \cap A_j$. Clearly, $v_J.\mathcal{A}_I(x^0) = (0, \dots, 0, x_m^0, 0, \dots, 0)$ as all monomials in the formula for Q are nonzero, $Q(v^J.\mathcal{A}_I(x^0)) \neq 0$, a contradiction to $\widetilde{Q}(x^0) = 0$.

Thus $|B| > k$ and we can find $\mathcal{A}_I = (A_1, \dots, A_k)$ such that $|B \cap A_i| \ge 1$, $1 \le i \le k$, which means that $x^i = P_{A_i}(x^0) \neq 0$. Next define a polynomial R on $\mathbb{R}^k$

$$R((t_1, \dots, t_k)) = Q(\sum_{i=1}^{K} t_i x^i).$$

Since $\widetilde{Q}(x_0) = 0$, it is immediate that $R(v^J) = Q(v^J.\mathcal{A}_I(x^0)) = 0$, $1 \le J \le N(k,n)$. Thus $R \equiv 0$ on $\mathbb{R}^k$. Consequently, it suffices to choose $X = \mathrm{span}\{x^i\}_{i=1}^k$, in order to obtain $Q \equiv 0$ on X.

□

Note that in Theorem 4.8, $N = (\log_2 N)^k \to \infty$ as $N \to \infty$. In order to give an explicit formula for the asymptotic dependence of N on the values of k and n, let us note that $N \ge (k+n)^{3k}$ satisfies the requirements of Theorem 4.8, provided $k + n \ge 2^4$.

As the following corollary shows, the fact that Theorem 4.8 was stated for homogeneous polynomials is not a real restriction.

Corollary 29. *Given k and $m \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that every odd polynomial $Q(x)$ of degree $2m+1$ on $\mathbb{R}^N$ vanishes on a subspace of dimension k.*

In order to motivate the last part of our note, which deals with even degree polynomials, let us recall the statement of the fundamental theorem of Dvoretzky on almost spherical sections of unit balls of finite dimensional Banach spaces.

Theorem 4.9. (Dvoretzky) *Let $(X, ||\cdot||)$ be an N-dimensional Banach space, $\varepsilon > 0, k \in \mathbb{N}$. There exists a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that provided $k \leq \eta(\varepsilon)\log(N)$ there exists a linear operator $T : l_2^k \to X$, such that $||T||||T^{-1}|| \leq 1+\varepsilon$.*

4.3. Odd Degree Polynomials on Real Banach Spaces

All results of this subsection was proved in [5].

A classical result of Birch claims that for given k, n integers, n-odd there exists some $N = N(k,n)$ such that for arbitrary n-homogeneous polynomial P on $\mathbb{R}^N$, there exists a linear subspace $Y \hookrightarrow \mathbb{R}$ of dimension at least k, where the restriction of P is identically zero (we say that Y is a null space for P).

Given $n > 1$ odd, and arbitrary real separable Banach space X (or more generally a space with w^*-separable dual X^*), we construct a n-homogeneous polynomial P with the property that for every point $0 \neq x \in X$ there exists some $k \in \mathbb{N}$ such that every null space containing x has a dimension at most k. In particular, P has no infinite dimensional null space. For a given n odd and a cardinal τ, we obtain a cardinal $N = N(\tau, n) = \exp^{n+1}\tau$ such that every n-homogeneous polynomial on a real Banach space X of density N has a null space of density τ.

In every real separable Banach space X (or more generally every real Banach space with w^*-separable dual X^*), of a n-homogeneous polynomial $P(n > 1$ arbitrary odd integer) which has no n-finite dimensional null space.

We say that the dual X^* has w^* density character $w^*\text{-dens}(X^*) = \Gamma$, if there exists a set $S \subset X^*$ of cardinality Γ, such that $\overline{S}^{w^*} = X^*$, and moreover Γ is the minimal cardinal with this property. Recall the following well-known fact.

Fact 1. Let X be a Banach space, then $w^*\text{-dens}(X^*)$ iff there exists a bounded linear injection $T : X \to l_\infty(\Gamma)$.

Theorem 4.10. *Let X be an infinite dimensional real Banach space with $w^*\text{-dens}X^* = \omega$, $n > 1$ an odd integer. Then there exists a n-homogeneous polynomial $P : X \to \mathbb{R}$ without any infinite dimensional null space. More precisely, given any $0 \neq x \in X$, $P(x) = 0$, there exists a $N \in \mathbb{N}$ such that every null space $x \in Y \hookrightarrow X$ has $\dim Y \leq N$.*

Proof. Suppose that we have already proven the statement of the theorem for $X = c_0$ and $n = 3$. Let $P : c_0 \to \mathbb{R}$ be the polynomial. Given any Banach space X with $w^*\text{-dens}X^* = \omega$, and $n = 3+2l$, we can construct the desired n-homogeneous polynomial $Q : X \to \mathbb{R}$ as follows. Fix any bounded linear injection $T : X \to c_0$ (put for example $T(x) = \left(\frac{f_i(x)}{i}\right)_{i=1}^\infty$, where $\{f_i\}_{i=1}^\infty \subset B_{X^*}$ is a separating set of functionals), and put $Q(x) = P \circ T(X) \cdot \left(\sum_{i=1}^\infty \frac{1}{2^i} f_i(x)^{2l}\right)$. It is easy to verify that a linear subspace of X where Q vanishes translates via T into a linear subspace (of the same dimension) of c_0 where P vanishes, which concludes the implication. It remains to produce P on c_0. We put

$$P((x_i)) = \sum_{k=1}^\infty x_k \sum_{i=k+1}^\infty \alpha_k^i x_i^2,$$

where $\alpha_k^i > 0$, together with the auxiliary system $\tau_{k,i}^j > 0$, are chosen satisfying conditions (0)-(3) below.

(0) $\sum_{k=1}^{\infty}\sum_{i=k+1}^{\infty}|\alpha_k^i|<\infty$.

(1) $\frac{1}{i}\alpha_k^i > \sum_{j=k+1}^{\infty}\alpha_j^i$.

(2) $\frac{1}{2i}\alpha_k^i \geq \sum_{j=1}^{\infty}\tau_{k,i}^j$.

(3) $(\alpha_p^r)^2 \leq \frac{1}{16}\tau_{r,p}^q\tau_{r,q}^p$ whenever $r<p<q$.

To construct such a system of coefficients α_k^i (and auxiliary system $\tau_{k,i}^j>0$) is rather straightforward, proceeding inductively by the infinite rows of the matrix $\{\alpha_i^k\}$. Indeed, the additional conditions always require that elements of a certain row are small enough depending on the elements of the previous rows. Note that our choice guarantees that the formula for P converges absolutely for every $x\in c_0$.

Claim. *Given any $0\neq x\in c_0$, $P(x)=0$, there exists $N\in\mathbb{N}$ such that for every null space $x\in Y\hookrightarrow c_0$ we have that* $\dim Y\leq N$.

We may assume that $||x||_\infty\leq 1$. Consider a (nonhomogeneous) 3-rd degree polynomial $R(y)=P(x+y)$.

$$R((y_i))=\sum_{k=1}^{\infty}(x_k+y_k)\sum_{i=k+1}^{\infty}\alpha_i^k(x_i+y_i)^2.$$

Writing $R=R_0+R_1+R_2+R_3$, where R_m is the m-homogeneous part of R, we obtain in particular:

$$R_2((y_i))=\sum_{k=1}^{\infty}x_k\sum_{i=k+1}^{\infty}\alpha_k^i y_i^2+\sum_{k=1}^{\infty}y_k\sum_{i=k+1}^{\infty}2\alpha_k^i x_i y_i.$$

Thus $R_2((yi))=\sum_{s=1}^{\infty}\sum_{l=s}^{\infty}\beta_s^l y_s y_l$, where $\beta_s^s=\sum_{k=1}^{s-1}x_k\alpha_k^s$, $\beta_l^s=2x_l\alpha_s^l$.

To prove the claim it suffices to find $N\in\mathbb{N}$, such that R_2, restricted to $Z=[e_i:i>N]\hookrightarrow c_0$ (Z has codimension N) is strictly positive outside the origin. Indeed, if so, then $R(\lambda z)=\sum_{m=0}^{3}\lambda^m R_m(z)$ is a nontrivial 3-rd degree polynomial in λ, for every $z\in Z$, and in particular for every $z\in Z$ there exists some $\lambda\in\mathbb{R}$ such that $P(x+\lambda z)=R(\lambda z)\neq 0$. Now if $x\in Y\hookrightarrow c_0$ is a null space, then $Z\cap Y=\{0\}$, and so $\dim Y\leq N$, as stated.

Let us without lost of generality assume that $x_r>0$, where $r=\min\{i:x_i\neq 0\}$. We choose $N>r$ large enough, so that the following are satisfied.

(i) $\beta_s^s=\sum_{j=r}^{s-1}x_j\alpha_j^s\geq\frac{1}{2}x_r\alpha_r^s$ for every $s\geq N+1$.

There exists a decomposition $\beta_s^s\geq\sum_{i=N+1}^{\infty}\delta_s^i$, $\delta_s^i>0$ such that

(ii) $(\alpha_p^q)^2\leq\frac{1}{16}\delta_p^q\delta_q^p$ whenever $N<p<q$.

To see that such a choice on N is possible, we estimate using property (1), whenever $s>\frac{3}{x_r}$

$$\beta_s^s\geq x_r\alpha_r^s-\sum_{j=r+1}^{s-1}x_j\alpha_j^s\geq x_r\alpha_r^s-\sum_{j=r+1}^{s-1}\alpha_j^s\alpha>\frac{1}{2}x_r\alpha_r^s.$$

Thus $N > \frac{3}{x_r}$ guarantees that (i) is satisfied. To see (ii), for N large enough, and $s > N$, $\frac{1}{2}x_r > \frac{1}{2N} > \frac{1}{2s}$, so we have $\beta_s^s \frac{1}{2s}\alpha_r^s$. So putting $\delta_s^i = \tau_{r,s}^i$ suffices using properties (2) and (3).

The conditions are set up so that R_2 restricted to $Z = [e_i : i > N]$ satisfies

$$R_2((y_i)) \geq \sum_{p=N+1}^{\infty} \sum_{q=p+1}^{\infty} (\delta_p^q y_p^2 + \delta_q^p y_q^2 + 2\alpha_p^q x_q y_p y_q).$$

However, condition (ii) implies that

$$\begin{aligned} \delta_p^q y_p^2 + \delta_q^p y_q^2 + 2\alpha_p^q x_q y_p y_q &\geq \delta_p^q y_p^2 + \delta_q^p y_q^2 - 2\alpha_p^q |y_p y_q| \geq \frac{3}{4}(\delta_p^q y_p^2 + \delta_q^p y_q^2) \\ &+ (\frac{1}{2}\sqrt{\delta_p^q |y_p|} - \frac{1}{2}\sqrt{\delta_q^p |y_q|})^2 > \frac{1}{2}(\delta_p^q y_p^2 + \delta_q^p y_q^2). \end{aligned}$$

The last expression is clearly a positive quadratic form in variables y_p, y_q, which concludes the claim that

$$R_2((y_i)) \geq \sum p = N + 1^{\infty} \sum_{q=p+1}^{\infty} (\frac{1}{2}\delta_p^q y_p^2 + \frac{1}{2}\delta_q^p y_q^2) > 0$$

for every $0 \neq (y_i) \in Z$. □

The statement of the theorem applies to all separable Banach spaces, $l_\infty, C(K)$, where K is separable (not necessarily metrizable). It is inherited by the subspaces, so since $l_1(c) \hookrightarrow l_\infty$, it applies also to l_1.

Our objective now is to obtain some estimate on the size of $card(\Gamma)$, such that every n-homogeneous odd polynomial on $l_1(\Gamma)$ has large null sets. Given and ordinal Γ, we say that a polynomial $P : l_1(\Gamma) \to \mathbb{R}$ is subsymmetric if $P\left(\sum_{i=1}^{l} x_i e_{\gamma_i}\right) = P\left(\sum_{i=1}^{l} x_i e_{\beta_i}\right)$ whenever we have $\gamma_1 < \gamma_2 < \ldots < \gamma_l$, $\beta_1 < \ldots < \beta_l$, for arbitrary $x_i \in \mathbb{R}$.

Lemma 14. *Let $P : l_1(\Gamma) \to \mathbb{R}$ be a subsymmetric n-homogeneous polynomial, n odd. Then P has a null set of density Γ.*

Denote by $\exp\alpha = 2^\alpha$, $\exp^{n+1}\alpha = \exp(\exp^n\alpha)$, where α is a cardinal. For a set S, let $[S]^n = \{X \subset S : \mathrm{card}X = n\}$. We will use the following result, which in the language of partition relations claims that $(exp^{n-1}\alpha)^+ \to (\alpha^+)\alpha_\alpha^n$.

Theorem 4.11. (Erdos, Rado) *Let α be an infinite cardinal, $n \in \mathbb{N}, \kappa = (exp^{n-1}\alpha)^+$ and $\{G_\gamma\}_{\gamma<\alpha}$ be a partition of $[\kappa]^n$. Then there exist $M \subset \kappa$, $cardM = \alpha^+$ and $[M]^n \subset G_\gamma$ for some $\gamma < \alpha$.*

Proof. Let P be an n-homogeneous polynomial, suppose Γ is an ordinal. We partition the set $[\Gamma]^n$ using continuum many sets $\{G_{\{a_{i_1,\ldots,i_n}:1\leq i_1\leq\ldots\leq i_n\leq n\}} : a_{i_1,\ldots,i_n} \in \mathbb{R}\}$ as follows.

We put $[\gamma_1, \ldots, \gamma_n] \in G_{\{a_{i_1,\ldots,i_n}:1\leq i_1\leq\ldots\leq i_n\leq n\}}$ iff $\{a_{i_1,\ldots,i_n} : 1 \leq i_1 \leq \ldots \leq i_n \leq n\}$ coincides with the set of coefficients of P, when restricted to the n-dimensional space with coordinate vectors $e_{\beta_1}, \ldots, e_{\beta_n}$ where $\{\beta_i\}$ is an increasingly reordered set $\{\gamma_i\}$ (in the order coming from Γ). Applying the Erdos-Rado theorem 4.11 yields a subset $S \subset \Gamma$ of the desired cardinality, such that the restriction of P to $l_1(S)$ is a subsymmetric polynomial. □

Theorem 4.12. *Suppose $card\Gamma \leq \exp^n\alpha$, n odd. Then every n-homogeneous polynomial on $l_1(\Gamma)$ has a null space of density at least α^+.*

Theorem 4.13. *Let X be a real Banach space of $dens(X) \geq \exp^{n+1}\alpha$, where α is a cardinal, n odd integer. Then every n-homogeneous polynomial on X has a null space of density at least α^+.*

Proof. Let $\Gamma = \exp^n \alpha$. We construct a continuous injection $T : l_1(\Gamma) \to X$ inductively as follows. Having chosen $T(e_i) \in B_X$ for all $i < \beta < \Gamma$ together with functionals $f_i \in B_{X^*}$, $f_i(T(e_i)) \geq \frac{1}{2}$, we choose $T(e_\beta) \in \bigcap_{i<\beta} \ker f_i$. The last set is nonempty, since $\mathrm{card}X \leq 2^{w^*-\mathrm{dens}X^*}$, so w^*-$\mathrm{dens}X^* \geq \exp^n \alpha$ and we can continue the inductive process. Now it remains to note that $P \circ T$ is an n-homogeneous polynomial on $l_1(\Gamma)$, its null subspaces carry right into X, and the previous theorem applies. □

Proposition 20. *Let Γ be an infinite cardinal, $P : c_0(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial. Then P has a null space of separable codimension in $c_0(\Gamma)$.*

Proof. Since P is wsc, it mapps in particular ω-null sequences to sequences convergent to $0 \in \mathbb{R}$. Using a standard argument we see that P depends only on a countable set of coordinates $S \subset \Gamma$, and so P restricted to $\Gamma \backslash S$ is identically zero. □

A similar proof based on wsc property for polynomials of degree less than p on l_p spaces gives.

Proposition 21. *Let Γ be an infinite cardinal, $P : l_p(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial of degree less than p. Then P has a null space of separable codimension in $l_p(\Gamma)$.*

In order to investigate polynomials of degree higher than p on $l_p(\Gamma)$ spaces, we need the following lemma.

Lemma 15. *Let P be a polynomial of n-th degree on $l_p(\Gamma)$, $\quad \Gamma > w$, $\quad n < 2[p]$. Then there exists a subset $\Gamma' \subset \Gamma$, linearly ordered, such that the restriction of P to Γ' has the form*

$$P((x_i)) = \sum_{j\in\Gamma',[p]\leq m\leq n} \quad \sum_{i_1\leq\ldots\leq i_l\leq j} \alpha^m_{i_1,\ldots,i_l,j} x_{i_i} \ldots x_{i_l} x_j^m$$

The previous proposition may be further generalized to arbitrary degree polynomial. The resulting formula will contain only those mixed terms whose last power is of degree at least $[p]$.

Proposition 22. *Let P be a n-homogeneous polynomial on $l_p(w_1^+)$, $\quad n < 2[p]$. Then P has an infinite dimensional (block) null space.*

Proof. Consider the P in the above form. Since for every j, the set of nonzero $a^m_{i_1,\ldots,i_l,j}$ is at most countable. We proceed inductively as follows. Pick the first ω_1 elements of $\Gamma = \omega_1^+$. It follows that there is some $k_0 \in \Gamma$, and a set Γ_1, $\min \Gamma_1 > k_0$, of cardinality ω_1^+ such that $a^m_{i_1,\ldots,i_l,j} = 0$, whenever $k_0 \in \{i_1, \ldots, i_l\}$, for all $j \in \Gamma_1$. Since ω_1^+ is a regular cardinal, we can in the next step choose the initial ω_1-interval of Γ_1, and k_1 in there, such that for some $\Gamma_2 \subset \Gamma_1$, min $\Gamma_2 > k_1$ of cardinality ω_1^+ we have that $a^m_{i_1,\ldots,i_l,j} = 0$, whenever $k_1 \in \{i_1, \ldots, i_l\}$, for all $j \in \Gamma_2$.

We proceed inductively along ω. The final set $\{k_j\}_{j=0}^\infty$ clearly defines a splitting of P restricted to this index set. □

Proposition 23. *Let P be a 3rd degree polynomial on $l_2(w_1)$. Then P has an infinite dimensional null (block) space.*

Proof. Without lost of generality, P has the formula

$$P((x_i)) = \sum_{j<\omega_1} \sum_{i \leq j} a_{i,j} x_i x_j^2.$$

We are going to construct a block sequence $\{u_k\}_{k=1}^{\infty}$ inductively as follows. First step. If there exists some i such that $\Gamma_1 = \{j : i < j, a_{i,j} = 0\}$ is uncountable, then we choose $u_1 = e_i$. Clearly, P restricted to $[u_1, e_i : i \in \Gamma_1]$ splits with respect to the decomposition $\{i\}, \Gamma_1$.

Otherwise, for every i there exists $\varepsilon_i > 0$ such that $\Delta_i = \{j : j > l, |a_{l,j}| > \varepsilon_i\}$ is uncountable. Fix $i = 1$ and still using the previous assumption, pick an $l > 1$ such that the set $\Gamma_1 = \{j : j \in \Delta_1, j > i,\ |a_{i,j}| < \frac{\varepsilon_1}{2}\}$ is uncountable. Here we are using the property of the ground space l_2, namely if such a choice were not possible, we would have some j for which the set $\{i : i < j,\ |a_{i,j}| \geq \frac{\varepsilon_1}{2}\}$ is infinite. This is a contradiction with the continuity of the linear term in the shifted polynomial $Q(x) = P(e_j + x)$. Assume, without lost of generality, that there exists some $\delta > 0, \quad a = \varepsilon_1 > a - 3\delta > \frac{\varepsilon_1}{2} > b > b - 3\delta > c \geq 0$, and a disjoint decomposition of Γ_1 into uncountable subsets Γ_1^1, Γ_1^2 such that $|a_{1,j} - a| < \delta$ for all $j \in \Gamma_1$, $|a_{l,j} - b| < \delta$ for all $j \in \Gamma_1^1$ and $|a_{l,j} - c| < \delta$ for all $j \in \Gamma_1^2$. Put $u_1 = e_l - \frac{b=c}{2a} e_1$. Consider now the polynomial P restricted to the subspace generated by the basic long sequence $\{e_i^1 : i < \omega_1\} = \{u_1, e_j : j \in \Gamma_1\}$. Its formula has the canonical form $P((x_i)) = \sum\limits_{j<\omega_1} \sum\limits_{i \leq j} a_{i,j}^1 x_i x_j^2$, where moreover $|a_{1,i}^1| > \delta$ for all $i > 1$, and both sets $A = \{i : i > 1, a_{1,i}^1 > \delta\}$ and $B = \{i : i > 1, a_{1,i}^1 < -\delta\}$ are uncountable. Blocking once more, this time using a bijection $\phi : A \to B$ and suitable coefficients $c_i, i \in A$ we obtain the disjoint blocks $v_i = e_i + c_i e_{\phi(i)}$, $i \in A$, such that in the restriction of P to $[e_1^1, v_i]$ splits with respect to e_1 and $[v_i]$. The inductive step consists of repeating the previous argument, for the polynomial P restricted to the last index set defining the previous splitting. This leads to a sequence $\{u_k\}_{k=1}^{\infty}$, where each u_k lies in the block subsequent to blocks containing $u_i, i < k$, and defining a splitting of P. Thus P splits with respect to disjoint block vectors $\{u_k\}_{k=1}^{\infty}$, and the result follows. ☐

Remark 2. The assumption that τ is uncountable cannot be dropped. Indeed, consider the subspace of l_p generated by vectors $v_n = \sum\limits_{i=k_n}^{\infty} a_i^n e_i$ for some fast decreasing sequence $a_i^n \searrow 0$, and fast increasing $k_n \to \infty$. We have $\{v_n\} \sim \{e_n\}$ the canonical basis. The coordinates of $v_j(i), j \leq n$ in the intervals $i \in [k_n, k_{n+1})$ are chosen so that for every pair of nonzero vectors $x = \sum\limits_{j=1}^{n} b_j v_j$, $y = \sum\limits_{j=1}^{n} c_j v_j$ there exists some $i \in [k_n, k_{n+1})$ for which $x(i), y(i) \neq 0$. This can be obtained by a simple compactness argument. It follows, that $[v_n : n \in \mathbb{N}]$ contains no two nonzero disjoint blocks.

Given $n - 2 < p \leq n$, where n is odd, we define a polynomial operator $Q_p : l_p(c) \to l_1(c)$ by $Q((x_i)) = (x_i^n)$. Clearly, Q is n-homogeneous and injective. Let P be the 3-homogeneous polynomial on $l_1(c)$ without any infinite dimensional null space.

Lemma 16. *$R = P \circ Q$ is a 3n-homogeneous polynomial on $l_p(c)$, which has no infinite dimensional block null space. In particular, it has no nonseparable null space. Moreover, for every $l \geq 4n + 1$ odd, there exists an l-homogeneous polynomial on $l_p(c)$ without a nonseparable null space.*

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Ключові слова: поліноми, лінійні підпростори, ядра поліномів на банахових просторах.