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*Vasyl Stefanyk Carpathian National University*

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O.M. Romanchuk, V.E. Sakhnyuk

# Influence of nonmagnetic impurities on the magnetic properties of tunnel superconducting junctions with a nonsinusoidal current-phase relation

*Lesya Ukrainka Volyn National University, Lutsk, Ukraine, Sakhnyuk.Vasyl@vnu.edu.ua*

We investigate the influence of nonmagnetic impurities and pair-breaking effects on the magnetic properties of tunnel superconducting junctions at temperatures close to the critical one. We show that as the transparency of the dielectric layer increases, the current-phase relation (CPR) strongly deviates from the classical sinusoidal form. An analytical expression for the magnetic field dependence of the critical current is derived, which is valid for an arbitrary impurity concentration. We analyze the role of the electron mean free path (impurity concentration) in the formation of the diffraction pattern. It is demonstrated that an increase in the barrier transparency and a change in the junction purity parameter lead to pronounced CPR anharmonicity. This anharmonicity results in a significant suppression of the side lobes in the magnetic diffraction pattern of the supercurrent. Asymptotic analysis confirms that in the limit of low barrier transparency, the diffraction pattern reduces to the classical Fraunhofer distribution.

**Keywords:** tunnel superconducting junction, current-phase relation, nonmagnetic impurities, magnetic properties, barrier transparency, oscillatory behavior.

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## Introduction

The study of equilibrium current states in superconducting junctions of various geometries remains a fundamental problem in the modern microscopic theory of superconductivity. A special place among such systems is occupied by SIS (superconductor-insulator-superconductor) tunnel junctions, where the supercurrent is carried by the coherent tunneling of Cooper pairs through a potential barrier [1]. The stationary Josephson effect in these junctions and their response to an external magnetic field are well established for the case of a sinusoidal current-phase relation (CPR) [2,3]. However, as modern theoretical studies indicate [4-13], the standard sinusoidal relation is merely an asymptotic approximation, valid only for junctions with low barrier transparency.

In structures with arbitrary barrier transparency, underlying physical mechanisms arise that cause a substantial deviation of the CPR from a simple sinusoid [4-6]. Moving away from the low-transparency limit, the intense supercurrent flow induces pair-breaking processes near the junction plane, leading to nonlinear phase shifts and pronounced anharmonicity. This anharmonicity plays a crucial role in determining the macroscopic quantum properties of tunnel junctions [4,14,15]. Therefore, a rigorous analysis of SIS junctions in an external magnetic field cannot rely on the sinusoidal approximation; instead, it must employ a generalized nonsinusoidal CPR [8] that consistently accounts for both pair-breaking and nonmagnetic impurity scattering.

The presence of nonmagnetic impurities in the superconducting banks introduces additional complexity [5,8]. Impurities reduce the electron mean free path, modifying the characteristic coherence length and altering the electron transmission through the barrier. Particularly in the dirty limit, where the mean free path is much smaller than the pristine coherence length, an appropriate mathematical formalism – namely, the Ginzburg-Landau theory with renormalized coefficients – is required [1].

In this paper, we aim to theoretically investigate the influence of pair-breaking effects on the magnetic properties of SIS junctions with arbitrary concentrations of nonmagnetic impurities. Our goal is to derive an analytical expression for the critical supercurrent as a function of the magnetic flux, based on the microscopic derivation of the CPR for a tunnel junction in the presence of nonmagnetic impurities [5]. Particular attention is given to analyzing how the interplay between impurity concentration and barrier transparency affects the junction's sensitivity to an external magnetic field.

# I. Model and main equations

Let us consider a planar superconducting junction formed by two bulk superconductors occupying the half-spaces $z > 0$ and $z < 0$, separated by a thin dielectric layer at $z = 0$. The supercurrent flows along the $Oz$ axis. The dielectric film is modeled as a $\delta$-function potential barrier with an electron transmission coefficient $D$ ranging from 0 to 1. The model incorporates the effect of nonmagnetic impurities on the electron mean free path $l$. The purity of the superconductor is characterized by the dimensionless parameter $\lambda = l/\xi_0$, where $\xi_0$ is the coherence length. In this notation, $\lambda \ll 1$ corresponds to the dirty limit, while $\lambda \gg 1$ represents the clean limit. Since we consider temperatures close to $T_c$, the theoretical description is based on the Ginzburg-Landau equations.

The microscopic derivation of the CPR for such a system, accounting for arbitrary impurity concentrations and pair-breaking effects, was performed in Ref. [5]. The resulting analytical expression for the current is:

$$I(\varphi) = \frac{\chi_0(\lambda)}{2\tau q_\infty}\left(1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\right)\frac{\sin\varphi}{1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2+1}}\cos\varphi}. \quad (1)$$

In Eq. (1), we use the following relations:

$$\tau^2 = \frac{12}{7\zeta(3)}\frac{1}{\chi_0(\lambda)}\left(1 - \frac{T}{T_c}\right),$$

$$q_\infty = \frac{3\chi_1(\lambda)}{\chi(\lambda)}\int_0^1 x^3 R(x)dx + \frac{3\chi(\lambda)}{S_2}\left(\int_0^1 x\, D(x)dx\right)^{-1}\left(\int_0^1 x^2 R(x)dx\right)^2, \quad (2)$$

where

$$\chi(\lambda) = \sum_{n=-\infty}^{\infty}\frac{1}{|2n+1|^2\left(|2n+1| + \frac{1}{\lambda}\right)}, \chi_0(\lambda) = \frac{4}{7\zeta(3)}\chi(\lambda),$$

$\zeta(3)$ is the Riemann zeta function, and

$$\chi_1(\lambda) = \sum_{n=-\infty}^{\infty}\frac{1}{|2n+1|^2\left(|2n+1| + \frac{1}{\lambda}\right)^2}, S_2 = \sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2}.$$

In Eq. (2), the reflection and transmission coefficients for electrons traversing the dielectric film are given by:

$$D(x) = \frac{x^2}{x^2 + \alpha^2}, \quad R(x) = 1 - D(x),$$

where $x = \cos\theta$, $\theta$ is the angle of incidence, and $\alpha^2 = R(1)/D(1)$ is the ratio of reflection to transmission for normal incidence.

Equation (1) reveals that the parameters governing the CPR shape and its deviation from a sinusoid are $\tau$ and $q_\infty$. As the product $q_\infty\tau$ increases, the coefficient preceding $\cos\varphi$ in the denominator decreases. In the limit of large $q_\infty\tau$, the CPR asymptotically recovers a purely sinusoidal form.

Figure 1 illustrates the evolution of the CPR under varying system parameters. As seen in Fig. 1(a), increasing the relative mean free path from the dirty limit ($\lambda = 0.1$) to the clean limit ($\lambda \gg 1$) enhances the critical current amplitude by approximately an order of magnitude. Concurrently, the CPR shape becomes progressively more sinusoidal. Conversely, increasing the

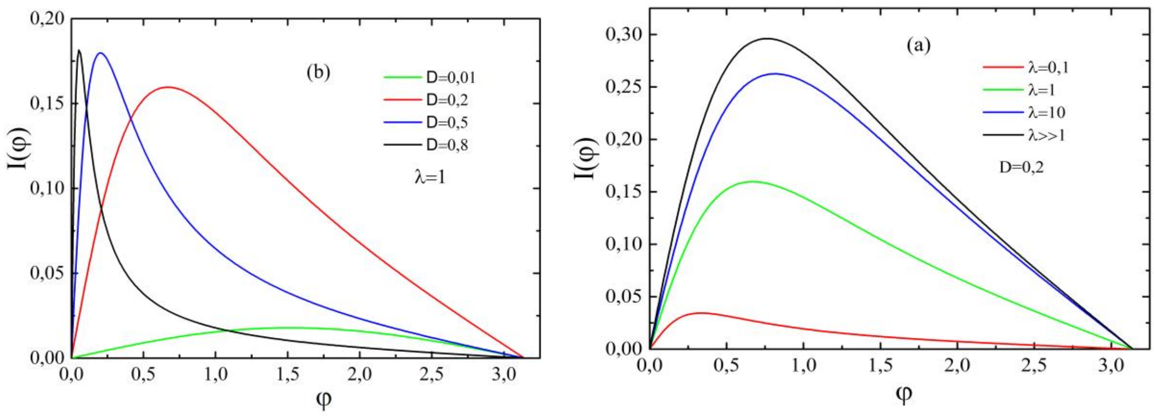


**Fig. 1.** The current-phase relation $I(\varphi)$: (a) for different values of the mean free path $\lambda$ at a fixed transmission coefficient; (b) for different values of the barrier transparency $D$ at a fixed mean free path [5].

barrier transparency (from $D = 0.01$ to $D = 0.8$) leads to a sharp increase in the supercurrent amplitude (Fig. 1(b)). Most importantly, the position of the maximum shifts significantly toward lower phase values $\varphi$, indicating strong anharmonicity and a profound deviation from the standard sinusoidal Josephson relation.

## II. Critical current of the junction in a magnetic field

We now consider the effect of an external magnetic field on the critical current. Assuming the junction dimensions are small compared to the Josephson penetration depth, we can neglect the self-field of the supercurrent. For a magnetic field $\vec{H} = (0, H, 0)$ applied parallel to the barrier plane, the spatial variation of the phase difference along the $Ox$ axis is strictly linear [2,8]:

$$\varphi(x) = 2edHx + \varphi_0. \tag{3}$$

The phase is uniform along the $Oy$ axis. Here, $d$ is the effective magnetic thickness of the junction, and $e$ is the elementary charge.

This external field induces a spatially non-uniform supercurrent distribution. Substituting Eq. (3) into Eq. (1), the local current density becomes:

$$I(H, x, \varphi_0) = \frac{\chi_0(\lambda)}{2\tau q_\infty}\left(1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\right)\frac{\sin(2edHx + \varphi_0)}{1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos(2edHx + \varphi_0)}. \tag{4}$$

By integrating Eq. (4) over the junction area (with length $l$ along $Ox$ and unit width along $Oy$), we obtain the total supercurrent as a function of the magnetic field and the integration constant $\varphi_0$:

$$I(H, \varphi_0) = \int_0^l d\,x \int_0^1 d\,y I(H, x, \varphi_0) = \chi_0(\lambda)\frac{\sqrt{2\tau^2 q_\infty^2 + 1} - 1}{4\tau q_\infty edH}\ln\left(\frac{1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos(2edHl + \varphi_0)}{1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos(\varphi_0)}\right) \tag{5}$$

To find the critical current, we maximize $I(H, \varphi_0)$ with respect to $\varphi_0$ by applying the extremum condition:

$$\frac{\partial I(H, \varphi_0)}{\partial \varphi_0} = 0.$$

This yields the trigonometric equation:

$$\sin(2edHl + \varphi_0)\left(1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos\varphi_0\right) - \left(1 - \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos(2edHl + \varphi_0)\right)\sin\varphi_0 = 0 \tag{6}$$

By rearranging terms and utilizing basic trigonometric identities, Eq. (6) simplifies to:

$$\cos(edHl + \varphi_0) = \frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos(edHl).$$

Selecting the physically relevant positive branch for $\varphi_0 \in [0, 2\pi]$, we find:

$$\varphi_0 = -\frac{\pi\Phi}{\Phi_0} + \arccos\left[\frac{1}{\sqrt{2\tau^2 q_\infty^2 + 1}}\cos\left(\frac{\pi\Phi}{\Phi_0}\right)\right] \tag{7}$$

Here, we have introduced the total magnetic flux $\Phi = dHl$ and the flux quantum $\Phi_0 = \pi/e$, which gives $edHl = \pi\Phi/\Phi_0$.

The behavior of the optimal initial phase $\varphi_0$ as a function of the normalized magnetic flux $\Phi/\Phi_0$, calculated according to Eq. (7), is illustrated in Fig. 2. It can be observed that the evolution of $\varphi_0$ exhibits a nonlinear, step-like character, which is particularly pronounced for junctions in the dirty limit $\lambda = 0.1$. This nonlinearity originates directly from the anharmonicity of the underlying current-phase relation. The term inside the arccosine function in Eq. (7) is scaled by the parameter $(2\tau^2 q_\infty^2 + 1)^{-1/2}$, which links the spatial phase adjustment to the microscopic scattering processes. As the purity parameter $\lambda$ increases toward the clean limit, these sharp phase variations are gradually smoothed out. This smoothing reflects the transition of the current-phase relation back toward a more classical sinusoidal form. Understanding this dynamic phase adjustment is crucial, as it physically dictates the spatial configuration of the supercurrent required to maximize the total measurable critical current across the junction.

Substituting Eq. (7) back into Eq. (5) eliminates $\varphi_0$, yielding the exact analytical expression for the critical current diffraction pattern:

$$I_c(\Phi) = \chi_0(\lambda)\frac{\left(\sqrt{2\tau^2 q_\infty^2+1}-1\right)}{4\pi\tau q_\infty\frac{\Phi}{\Phi_0}}\ln\left(\frac{\sqrt{2\tau^2 q_\infty^2+\sin^2\left(\pi\frac{\Phi}{\Phi_0}\right)}+\sin\left(\pi\frac{\Phi}{\Phi_0}\right)}{\sqrt{2\tau^2 q_\infty^2+\sin^2\left(\pi\frac{\Phi}{\Phi_0}\right)}-\sin\left(\pi\frac{\Phi}{\Phi_0}\right)}\right). \quad (8)$$

This result is valid near $T_c$ for arbitrary nonmagnetic impurity concentrations and a wide range of barrier transparencies.

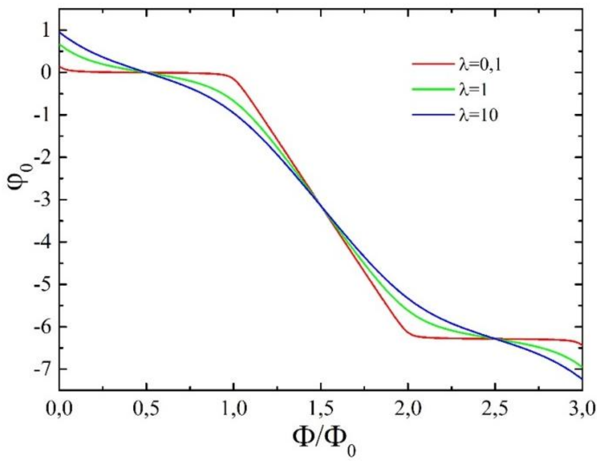


**Fig. 2.** Dependence of the phase $\varphi_0$ on the normalized external magnetic flux $\Phi/\Phi_0$ for different values of the purity parameter $\lambda$.

Figure 3 illustrates the magnetic field dependence of the critical current $I_c(\Phi)$. The curves exhibit the standard oscillatory behavior inherent to macroscopic quantum interference. In cleaner junctions (larger $\lambda$), the absolute amplitude of the supercurrent is higher (Fig. 3(a)), consistent with the underlying CPR. To compare the shapes of the diffraction envelopes, Fig. 3(b) plots the current normalized to its zero-field value, $I_c(0)$. It is evident that as $\lambda$ decreases and the CPR deviates from a sinusoid, the relative amplitude of the side lobes is significantly suppressed.

Increasing the barrier transparency $D$ from 0.01 to 0.5 dramatically enhances the absolute critical current (Fig. 4(a)) and simultaneously increases its sensitivity to the magnetic flux. Specifically, at $\Phi = 0.5\Phi_0$, the critical current drops to $0.65I_c(0)$, $0.54I_c(0)$, and $0.3I_c(0)$ for $D = 0.01$, 0.2, and 0.5, respectively. Figure 4(b) displays the corresponding normalized diffraction patterns. The strong CPR anharmonicity associated with high transparency leads to a pronounced suppression of the side lobes. In the tunneling limit ($D = 0.01$), where the CPR is nearly sinusoidal, the envelope perfectly recovers the classical Fraunhofer pattern [2].

To formalize this limiting case, we consider the regime where $\tau q_\infty \gg 1$. We set $x = \tau q_\infty$ and expand Eq. (8) asymptotically. The prefactor simplifies to:

$$\frac{\sqrt{2x^2+1}-1}{4\pi x\frac{\Phi}{\Phi_0}} \approx \frac{\sqrt{2}}{4\pi\frac{\Phi}{\Phi_0}}. \quad (9)$$

Expanding the square roots in the logarithmic argument gives: $\sqrt{2x^2+\sin^2\left(\pi\frac{\Phi}{\Phi_0}\right)} \approx x\sqrt{2}$, which leads to the approximation: $\frac{x\sqrt{2}+\sin\left(\pi\frac{\Phi}{\Phi_0}\right)}{x\sqrt{2}-\sin\left(\pi\frac{\Phi}{\Phi_0}\right)} \approx 1+\frac{\sqrt{2}}{x}\sin\left(\pi\frac{\Phi}{\Phi_0}\right)$. By applying $\ln(1+\epsilon) \approx \epsilon$ for small $\epsilon$, the logarithmic term becomes:

$$\ln\left(1+\frac{\sqrt{2}}{x}\sin\left(\pi\frac{\Phi}{\Phi_0}\right)\right) \approx \frac{\sqrt{2}}{x}\sin\left(\pi\frac{\Phi}{\Phi_0}\right). \quad (10)$$

Multiplying Eqs. (9) and (10), we arrive at the asymptotic critical current:

$$I_c(\Phi) \approx \frac{1}{2\tau q_\infty}\left|\frac{\sin\left(\pi\frac{\Phi}{\Phi_0}\right)}{\pi\frac{\Phi}{\Phi_0}}\right|. \quad (11)$$

In this limit, the magnetic diffraction strictly reduces to the standard Fraunhofer pattern. The physical origin is straightforward: for $\tau q_\infty \gg 1$, the CPR recovers a pure sinusoidal shape with $I_c(0) = (2\tau q_\infty)^{-1}$, directly yielding the classical spatial interference encapsulated by Eq. (11).

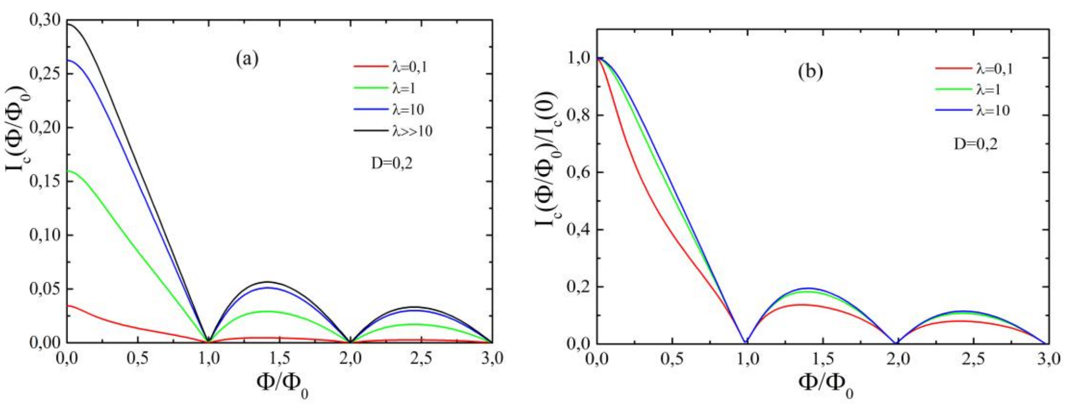


**Fig. 3.** (a) Critical current $I_c$ versus normalized external magnetic flux $\Phi/\Phi_0$, calculated via Eq. (8). (b) The same dependencies normalized to the zero-field critical current $I_c(0)$, highlighting the deformation of the envelope.

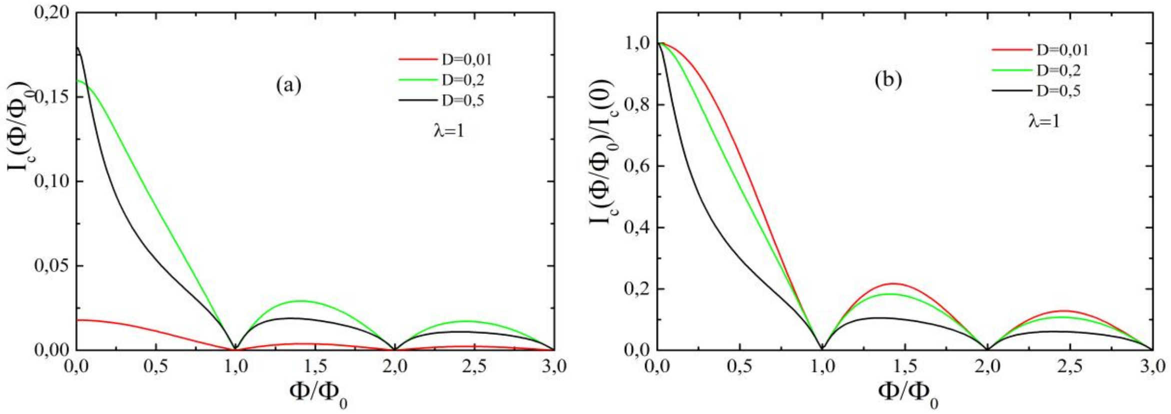


**Fig. 4.** (a) Absolute critical current $I_c$ versus normalized magnetic flux $\Phi/\Phi_0$ for different barrier transparencies $D$ at a fixed purity parameter $\lambda = 1$. (b) Normalized diffraction patterns demonstrating the suppression of side lobes at high transparencies.

## Conclusions

In summary, we have theoretically investigated the impact of depairing effects and nonmagnetic impurities on the magnetic properties of planar tunnel superconducting junctions. Using a microscopic approach, we derived an exact analytical expression for the magnetic diffraction pattern of the critical current. This result holds near the critical temperature for arbitrary concentrations of nonmagnetic impurities and spanning the full range of barrier transparencies.

We demonstrate that varying the system parameters – namely, the electron mean free path ($\lambda$) and the barrier transparency ($D$) – significantly modifies the CPR. Specifically, higher transparency shifts the maximum of the supercurrent toward lower phase differences, manifesting as strong anharmonicity and a breakdown of the standard sinusoidal Josephson relation. This anharmonicity profoundly affects the junction's macroscopic magnetic properties. As $\lambda$ is varied and $D$ increases, the diffraction pattern deviates from the ideal Fraunhofer behavior, evidenced by a marked suppression of the relative amplitude of the side lobes.

Finally, our asymptotic analysis confirms that in the low-transparency tunneling limit (or equivalently, $q_\infty \to \infty$), the CPR becomes purely sinusoidal, and the spatial supercurrent distribution in an applied magnetic field perfectly reproduces the classical Fraunhofer diffraction pattern [2].

***Romanchuk O.M.*** – PhD student;
***Sakhnyuk V.E.*** – Candidate of Physical and Mathematical Sciences, Associate Professor, Head of the A.V. Svidzynsky Department of Theoretical and Computer Physics.

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О.М. Романчук, В.Є. Сахнюк

# Вплив немагнітних домішок на магнітні властивості тунельних надпровідних контактів із несинусоїдною струм-фазовою залежністю

*Волинський національний університет імені Лесі Українки, Sakhnyuk.Vasyl@vnu.edu.ua*

Досліджено вплив немагнітних домішок та ефектів розпаровування на магнітні властивості тунельних надпровідних контактів при температурах, близьких до критичної. Показано, що зі збільшенням прозорості діелектричного шару залежність струму від фази суттєво відхиляється від класичної синусоїдальної форми. Отримано аналітичний вираз для залежності критичного струму від магнітного поля, який є справедливим для довільної концентрації домішок. Проаналізовано роль довжини вільного пробігу електронів (концентрації домішок) у формуванні дифракційної картини. Продемонстровано, що зростання прозорості бар'єру та зміна параметра чистоти контакту приводять до вираженої ангармонійності в залежності струму від різниці фаз. Ця ангармонійність спричиняє суттєве зменшення бічних максимумів на магнітній дифракційній картині надпровідного струму. Асимптотичний аналіз підтверджує, що у граничному випадку низької прозорості бар'єру дифракційна картина зводиться до класичного розподілу Фраунгофера.

**Ключові слова:** тунельний надпровідний контакт, струм-фазова залежність, немагнітні домішки, магнітні властивості, прозорість бар'єра, осцилююча поведінка.